Algebraic Properties of the Multistate Population Matrix Model

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***Abstract*—Discrete time population growth is often modeled by a matrix. Many growth parameters such as growth rate, reproduction rate, as well as the movement of the population are easily included in a matrix model. This paper will discuss a matrix model that describes the dynamics of a population having some stages of life and occupying some different patches. The matrix, which is the product of two matrices S and D, is often called SD matrix. The matrix S is a diagonal block matrix in which its block is a sub-stochastic column matrix. The matrix S represents the movement of a population between locations (patches). On the other hand, the matrix D is a block matrix in which its block is a nonnegative real diagonal matrix. The matrix D describes the population growth in specific patches. The paper will focus on the properties of the SD matrix from the algebraic point of view, particularly the spectral radius of the matrix. It will be shown that the spectral radius of the SD matrix is less than the spectral radius of D meanwhile the condition does not hold for the block matrices SD and D.**

Keywords: matrix, population, spectral radius

I. INTRODUCTION

One way to build a multistate population model is to use the **SD** matrix model.The matrix **S** is a diagonal block matrix in which its block is a sub-stochastic column matrix.

Let A_j be square matrix with nonnegative entries. The matrix $A_1 \oplus A_2 \oplus \dots \oplus A_n$ defined as a block diagonal matrix with diagonal blocks $\mathrm{A_1, A_2, \cdots, A_n}$ or can be written as follow

$$A_1 \oplus A_2 \oplus \dots \oplus A_n = \begin{pmatrix} A_1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_n \end{pmatrix} \qquad (1)$$

An $n \times n$ matrix S with non-negative entries is column sub stochastic if all its columns sum are less than or equal to +1, or can be written as follows [4]

$$\begin{pmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ s_{21} & s_{22} & \cdots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n1} & s_{n2} & \cdots & s_{nn} \end{pmatrix} \qquad (2)$$

where $\sum_{\{i=1\}}^{n} \left|\mathrm{s_{\{ij\}}}\right| \le 1, \forall \mathrm{j} = 1, \dots, \mathrm{n}$

The matrix $\boldsymbol{S}$ describes the movement of a population between patches and can be written as follow

$$\mathbf{S} = \mathrm{S_1} \oplus \mathrm{S_2} \oplus \cdots \oplus \mathrm{S_n} \qquad (3)$$

where $\mathrm{S_1, S_2, \dots, S_n}$ is substochastic matrices [4]. On the other hand, let P be a permutation matrix and A_j be an $m \times m$ nonnegative matrix representing the growth dynamic in location j, then the matrix $\boldsymbol{D}$ is a block matrix in which its block is a diagonal matrix with nonnegative real entries and can be written as follow

$$\mathbf{D} = \mathrm{P^t(A_1 \oplus A_2 \oplus \cdots \oplus A_n)P}. \qquad (4)$$

The multiplication of $\boldsymbol{S}$ and $\boldsymbol{D}$ is also a matrix model that represent a biological model that account for spatial structure. The discussion in this paper is focus on the study of spectral radius properties of the $\boldsymbol{SD}$ matrix model.

Let M be a square matrix, the spectral radius of M is the non-negative number $\rho(M)$ which is defined as follow [3]

$$\rho(\mathrm{M}) = \max_{1 \le i \le n} |\lambda_i|$$

where λ_i is eigenvalues for M.

Another term that is used in this paper is the Column Sum Norm. The Column Sum Norm of a square matrix M is defined as follow [4]

$$\|M\|_c = \max\{\sum_{\{i=1\}}^{n} \left|\mathrm{a_{\{ij\}}}\right| : 1 \le \mathrm{j} \le \mathrm{n}\} \qquad (5)$$

.

II. RESULTS

The spectral radius of a matrix is used to determine the rate of growth of a population that described in a matrix model. In this section it will be shown that the spectral radius of SD matrix model (where S is a column substocasthic matrix and D is a diagonal nonnegative matrx) is less than or equal to the spectral radius of D. However, before we proceed it will be explained first some theorem that will be used to proof that properties.

Theorem 1 [3] The largest eigenvalue of a substochastic matrix is 1 or in the other words is that the radius spectral of a substochastic matrix is equal to 1.

Lemma 2 [4] Let A be a square matrix, then $\rho\ (A) \leq \|A\|$.

Proof.
Let A be an arbitrary $n \times n$ matrix. Let e_i be eigenvector of A such that

$$Ae_i = \lambda_i e_i$$

where λ_i is the eigenvalue of A that correspond with e_i and $e_i \neq 0$, then we have

$$\|Ae_i\| = \|\lambda_i e_i\|.$$

Since λ_i is scalar, then based on norm properties, we obtain

$$\|Ae_i\| = |\lambda_i|\|e_i\| \leq \|A\|\|e_i\|$$

and then we have

$$|\lambda_i|\|e_i\| \leq \|A\|\|e_i\|$$

Hence

$$|\lambda_i| \leq \|A\|$$

is true for all i thus

$$\rho(A) = \max\{|\lambda_i|\} \leq \|A\|$$

The Lemma Above implies the following Proposition.

Proposition 3 [3] Let D be a diagonal nonnegative matrix and S a column substochastic matrix. Then

$$\rho(\mathbf{SD}) \leq \rho(\mathbf{D})$$

Proof. Based on the lemma above we have

$$\rho(SD) \leq \|SD\| \leq \|S\|\|D\|$$

Since the S is column sub stochastic matrix, then all of the columns sum are less than or equal to +1. Furthermore, the spectral radius of D is equal to its column sum norm. Thus, we have

$$\rho(SD) \leq \|D\| = \rho(D).$$

It proves the lemma.

Propositions 3 has some important meanings, especially for biological models that account for spatial structure [3]. One of them is, from that fact one can know whether population growth occurs before dispersal or population growth occurs after dispersal.

However, the conditions cannot always apply for $\boldsymbol{SD}$ matrix model. Suppose $S = S_1 \oplus \dots \oplus S_n$ where $S_1, \dots, S_n$ is substochastic matrices then for every block matrix $\boldsymbol{D}$ we can not always have condition of $\rho(\mathbf{SD}) \leq \rho(\mathbf{D})$.

For example let $A_1 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Then, from (4) we obtain

$$\boldsymbol{D} = P^t(A_1 \oplus A_2)P$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^t \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

and we also have $\rho(\boldsymbol{D}) = 0$.
Meanwhile, Let $S_1 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ and $S_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, then

$$\boldsymbol{S} = S_1 \oplus S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Multiplying $\boldsymbol{S}$ and $\boldsymbol{D}$ we have

$$\boldsymbol{SD} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and we have $\rho(\boldsymbol{SD}) = \sqrt{2} > 0$. In other words $\rho(\boldsymbol{SD}) > \rho(\boldsymbol{D})$. Hence, in general we can't always have $\rho(\boldsymbol{SD}) \leq \rho(\boldsymbol{D})$.

III. Conclusion

The spectral radius from SD matrix model is less than or equal to the spectral radius of the matrix D. That relation has some important meanings, especially for biological model that account for spatial structure [4]. One of them is that one can know whether population growth occurs before dispersal or after dispersal. However for $\boldsymbol{SD}$ matrix model, the condition does not always apply.

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