

On the Use of Conjunctors With a Neutral Element in the Modus Ponens Inequality

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ABSTRACT

The inference rule of Modus Ponens has been extensively investigated in the framework of approximate reasoning, especially for the case of t-norms. Recently, more general kinds of conjunctors have also been considered, like semi-copulas, copulas, and conjunctive uninorms. A common feature of all these kinds of conjunctors is the fact that they have a neutral element $e \in]0, 1[$. This paper is devoted to the study of Modus Ponens for conjunctors with a neutral element with no additional conditions. Many properties are proved to be necessary for a fuzzy implication function I to satisfy the Modus Ponens with respect to a conjunctor with neutral element $e \in]0, 1[$. Although the most usual families of fuzzy implication functions do not satisfy all these properties, other possibilities for I are presented showing many new examples and generalizing some already known results on this topic. Moreover, all fuzzy implication functions satisfying the Modus Ponens with respect to the least (and with respect to the greatest) conjunctor with neutral element $e \in]0, 1[$ are characterized. The particular case of $e = 1$, that provides semi-copulas, is studied separately, retrieving many known results that can be easily derived from the current study.

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1. INTRODUCTION

The *Modus Ponens inequality* [1–5] is a well-known functional inequality that comes out when using the classical Modus Ponens rule in order to implement forward fuzzy inference processes. The latter are approximate reasoning schemes that allow to infer a conclusion of the form “ y is Q^* ” from two premises: a fuzzy proposition “ x is P^* ” and a fuzzy conditional statement “If x is P , then y is Q ”. The inequality, which involves a *fuzzy implication function* [1,6–9] $I : [0, 1]^2 \rightarrow [0, 1]$ (used to model the fuzzy conditional) and a bivariate *aggregation function* [10–13] $C : [0, 1]^2 \rightarrow [0, 1]$ (needed to aggregate the premises), is written as $C(a, I(a, b)) \leq b$ for any $a, b \in [0, 1]$.

The aggregation of the premises in these inference processes has traditionally been performed by means of *triangular norms* [4,14,15], even though lately other functions such as overlap functions [16,17] and conjunctive uninorms [18,19] have also been investigated for this purpose. Another recent paper, Ref. [20], shows that *conjunctors* (aggregation functions having zero as annihilator element) are the only aggregation functions that may be able to solve the Non-Contradiction principle and hence also the Modus Ponens inequality, since the satisfaction of the former (with respect to the

natural negation of the fuzzy implication function) is a necessary condition for the satisfaction of the latter.

This paper explores the generalization of the aggregation function used in the Modus Ponens inequality to the class of *conjunctors with a neutral element* $e \in]0, 1[$ a broad family of aggregation functions that includes semi-copulas (when the neutral element is equal to one) such as triangular norms, copulas or representable aggregation functions, as well as conjunctive uninorms or continuous generated functions (otherwise). Our main goal is to investigate the relationships that exist between the properties of conjunctors with a neutral element and those of the fuzzy implication functions that may satisfy the Modus Ponens inequality with respect to them. We will then apply such relationships to the most important families of fuzzy implication functions in order to study whether they satisfy or not the Modus Ponens inequality with respect to this kind of conjunctors, allowing many new examples and general results.

The paper is organized as follows. Section 2 recalls the main issues related to negation, aggregation, and fuzzy implication functions, as well as the most important available results regarding the Modus Ponens inequality. Section 3 provides several necessary and/or sufficient conditions for the satisfaction of the Modus Ponens

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¹Conjunctors have annihilator zero and hence can not have zero as neutral element.

inequality with respect to a conjunctor with a neutral element. Next, Sections 4 and 5 focus, respectively, on the specific cases of conjunctors with a neutral element $e \neq 1$ and those with neutral element $e = 1$ (semi-copulas). Finally, Section 6 ends with some conclusions and pointers to future work.

2. PRELIMINARIES

In this section we recall the definitions and main properties of negation functions, conjunctors with a neutral element and fuzzy implication functions, as well as the main results related to the Modus Ponens inequality.

2.1. Negation Functions

Negation functions constitute a well-known tool allowing to represent fuzzy negations.

Definition 1. [21–23] A function $N : [0, 1] \rightarrow [0, 1]$ is called a *negation function* if it is decreasing and satisfies $N(0) = 1$ and $N(1) = 0$. Moreover,

- N is *vanishing* when $N(x) = 0$ for some $x \neq 1$,
- N is *filling* when $N(x) = 1$ for some $x \neq 0$.

Typical examples of negation functions are the classical negation N_c , defined as $N_c(x) = 1 - x$ for all $x \in [0, 1]$, as well as the *least* and the *greatest* negation functions, given, respectively, by

$$N_{\perp}(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise,} \end{cases} \quad N_{\top}(x) = \begin{cases} 0 & \text{if } x = 1, \\ 1 & \text{otherwise.} \end{cases}$$

2.2. Conjunctors With a Neutral Element

Conjunctors with a neutral element constitute a special class of aggregation functions (see Refs. [10–13]). In this paper we will only be interested in the bivariate case (where aggregation functions are increasing functions $A : [0, 1]^2 \rightarrow [0, 1]$ verifying $A(0, 0) = 0$ and $A(1, 1) = 1$) and in the following properties.

Definition 2. [10–13] Let A be an aggregation function and let N be a negation function.

- A is a *conjunctor* when it satisfies $A(1, 0) = A(0, 1) = 0$.
- A has a *neutral element* $e \in [0, 1]$ (**NE(e)**) when $A(x, e) = A(e, x) = x$ for all $x \in [0, 1]$.
- A has *zero divisors* (**0Div**) when there exist $a, b \in]0, 1]$ such that $A(a, b) = 0$.
- A satisfies the *Non-Contradiction principle* with respect to N (**NC(N)**) when $A(x, N(x)) = 0$ for all $x \in [0, 1]$.

Note that when A is a conjunctor, due to its increasingness, A has zero as annihilator element.

Conjunctors with a neutral element may be classified into the two following categories:

1. Conjunctors with neutral element $e = 1$, also known as *semi-copulas*.² Semi-copulas are conjunctive ($C \leq \min$) and may or may not have zero divisors. Some distinguished families of semi-copulas are the following:

- *Triangular norms (t-norms for short)* [21,22], which are associative and commutative semi-copulas.
- *Copulas* [27], which are semi-copulas C satisfying $C(0, x) = C(x, 0) = 0$ for all $x \in [0, 1]$ and the so-called 2-increasing property: $C(x_1, y_1) - C(x_1, y_2) - C(x_2, y_1) + C(x_2, y_2) \geq 0$ for all $x_1, y_1, x_2, y_2 \in [0, 1]$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$.

The least and the greatest semi-copulas are, respectively, the *drastic product* t-norm T_D (given by $T_D(x, y) = 0$ if $(x, y) \in [0, 1]^2$ and $T_D(x, y) = \min(x, y)$ otherwise) and the *minimum* t-norm T_M ($T_M(x, y) = \min(x, y)$). Other well-known families of t-norms are the so-called *strict* t-norms (or t-norms in the Product family), defined as $(T_P)_\varphi(x, y) = \varphi^{-1}(\varphi(x)\varphi(y))$, and *nilpotent* t-norms (or t-norms in the Łukasiewicz family), $(T_{LK})_\varphi(x, y) = \varphi^{-1}(\max(0, \varphi(x) + \varphi(y) - 1))$, where $\varphi : [0, 1] \rightarrow [0, 1]$ is an increasing bijection [21,22].

2. Conjunctors with a neutral element $e \in]0, 1[$, which are conjunctive ($C \leq \min$) on the square $[0, e]^2$, disjunctive ($C \geq \max$) on $[e, 1]^2$ and averaging ($\min \leq C \leq \max$) elsewhere. These conjunctors are averaging on the whole unit square only when the limiting functions \min and \max are chosen in the regions $[0, e]^2$ and $[e, 1]^2$, respectively. They have zero divisors if and only if $C_{|[0, e]^2}$ has zero divisors. One distinguished family of aggregation functions in this category is the class of *conjunctive uninorms*, which are associative and commutative aggregation functions $U : [0, 1]^2 \rightarrow [0, 1]$ with a neutral element $e \in]0, 1[$ satisfying $U(0, 1) = 0$ that behave as scaled t-norms/t-conorms on $[0, e]^2 / [e, 1]^2$ (see Ref. [28] for a recent survey on these functions).

Figure 1 depicts the structure of $C_{e\perp}$ and $C_{e\top}$, the least and the greatest conjunctors with a neutral element $e \neq 1$ (note that the first one is also the least uninorm and belongs to the class \mathcal{U}_{\min}).

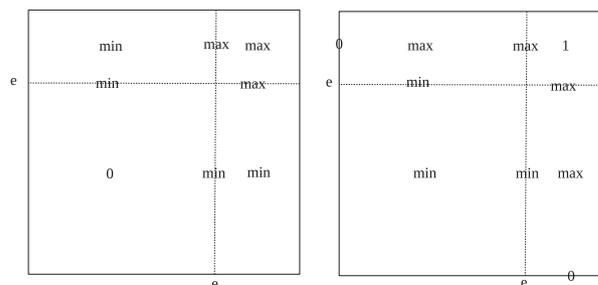


Figure 1 | The least ($C_{e\perp}$, left) and the greatest ($C_{e\top}$, right) conjunctors with a neutral element $e \in]0, 1[$.

²Note that the terms semi-copula and conjunctor are sometimes used interchangeably (e.g. Ref. [12]), whereas in this paper, as well as in Refs. [20,24–26], semi-copulas constitute a proper subclass of conjunctors.

2.3. Fuzzy Implication Functions

The most accepted definition of fuzzy implication function is the following.

Definition 3. [1,9,29] A fuzzy implication function is a function $I : [0, 1]^2 \rightarrow [0, 1]$ verifying the following properties:

1. I is decreasing in the first variable.
2. I is increasing in the second variable.
3. $I(0, 0) = I(1, 1) = 1$ and $I(1, 0) = 0$.

The papers in Refs. [8,30,31] provide recent compilations regarding these functions. The *least* and the *greatest* implication functions are given, respectively, by

$$I_{\perp}(x, y) = \begin{cases} 1 & \text{if } x = 0 \text{ or } y = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$I_{\top}(x, y) = \begin{cases} 0 & \text{if } x = 1, y = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Other popular fuzzy implication functions are recalled below:

- The Gödel implication

$$I_{GD}(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise.} \end{cases}$$

- The Goguen implication

$$I_{GG}(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ y/x & \text{otherwise.} \end{cases}$$

- The Łukasiewicz implication

$$I_{LK}(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ 1 - x + y & \text{otherwise.} \end{cases}$$

- The Weber implication

$$I_{WB}(x, y) = \begin{cases} 1 & \text{if } x \neq 1, \\ y & \text{otherwise.} \end{cases}$$

Recall also (see e.g., Ref. [1]) that, similarly to what happens with aggregation functions, any fuzzy implication function I may be transformed by means of an automorphism $\varphi : [0, 1] \rightarrow [0, 1]$ into a new fuzzy implication function denoted as I_{φ} and given by $I_{\varphi}(x, y) = \varphi^{-1}(I(\varphi(x), \varphi(y)))$. Remember finally that if I is a fuzzy implication function, then $N_I : [0, 1] \rightarrow [0, 1]$, defined as $N_I(x) = I(x, 0)$ for any $x \in [0, 1]$, is a negation function, called the *natural negation* [1] of the fuzzy implication function. Besides, the following additional properties of fuzzy implication functions will be needed in the remainder of this paper.

Definition 4. [1,32] Let I be a fuzzy implication function. Then:

- I satisfies the *left neutrality* principle (NP) when

$$\forall y \in [0, 1] : I(1, y) = y \tag{NP}$$

- I satisfies the *identity principle* when

$$\forall x \in [0, 1] : I(x, x) = 1 \tag{IP}$$

- satisfies the *ordering property* when

$$\forall x, y \in [0, 1] : x \leq y \Leftrightarrow I(x, y) = 1 \tag{OP}$$

- I satisfies the *consequent boundary* property when

$$\forall x, y \in [0, 1] : I(x, y) \geq y \tag{CB}$$

Recall that on the one hand, (OP) implies (IP) and on the other hand, (NP) implies (CB) (see e.g., Ref. [32]).

2.4. The Modus Ponens Inequality

The *Modus Ponens* scheme, a well-known classical inference rule allowing to perform forward reasoning, has usually been translated to the fuzzy framework as follows:

Definition 5. [1,9,29] Let I be a fuzzy implication function and let C be a conjunctor. Then I satisfies the *Modus Ponens inequality* with respect to C when

$$\forall x, y \in [0, 1] : C(x, I(x, y)) \leq y \tag{MP(C)}$$

As it has been mentioned in the introduction, the Modus Ponens inequality (MP(C)) has mostly been studied when the conjunctor C is taken as a t-norm (usually a continuous one) and the fuzzy implication function belongs to one of the most important families of implications: see e.g., Ref. [1] Section 7.4. for the main results related to (S,N)-implications, R-implications, and QL-implications, Ref. [15] for RU and (U,N)-implications and Ref. [14] for probabilistic implications and S-implications. Recently, some authors [18,19] have also dealt with the use of conjunctive uninorms instead of t-norms, studying the case of RU-implications, and others (see Ref. [16]) have considered the use of overlap functions, dealing with the so-called *O*-conditionality. Since t-norms and conjunctive uninorms are special cases of conjunctors with a neutral element, in the current paper we will recover and generalize some of these results.

Remark 1. Recall also (see e.g., Proposition 7.4.3. in Ref. [1] for the case of t-norms) that when studying the Modus Ponens inequality, the monotonicity of the conjunctor clearly allows one to focus on the greatest functions, since:

- If I satisfies (MP(C)), then I satisfies (MP(C*)) for any conjunctor C^* such that $C^* \leq C$.
- If I satisfies (MP(C)), then any fuzzy implication function I^* such that $I^* \leq I$ does also satisfy (MP(C)).

An interesting necessary condition for the satisfaction of the Modus Ponens inequality (which is in some cases also sufficient) comes from considering *residuated functions*, which are functions obtained from the residuation scheme $p \rightarrow q \equiv \forall \{t : p \wedge t \leq q\}$ as

$$I_C(x, y) = \sup\{t \in [0, 1] : C(x, t) \leq y\}, \quad \forall x, y \in [0, 1],$$

where C is a binary function (see e.g., Refs. [33,34]). Depending on the properties of the underlying function C , residuated functions may qualify as implication functions (according to Definition 3)

and/or may satisfy the so-called *residuation property* (see e.g., Refs. [24,35]), given by

$$\forall x, y, z \in [0, 1] : C(x, y) \leq z \Leftrightarrow y \leq I_C(x, z) \quad (\mathbf{RP})$$

The study of the properties of the residuated function I_C and the satisfaction of **(RP)** was first undertaken for t-norms but was later on generalized to other functions. In the case of conjunctors the following can be stated.

Proposition 1. [24,25,35] *Let C be a conjunctor and let I_C be its corresponding residuated function.*

- i. I_C is a fuzzy implication function if and only if $C(1, t) \neq 0$ for all $t \in]0, 1]$.
- ii. If C is left-continuous, then **(RP)** is satisfied, and the supremum of I_C can be replaced by a maximum.

Note that condition *i* in the previous proposition is satisfied, in particular, if C has a neutral element e , since $C(1, t) \geq C(e, t) = t$. On the other hand, as noted for example in Ref. [4], the residuation property may be used to study the satisfaction of the Modus Ponens inequality. Indeed, the following result was proved in Ref. [4] when dealing with t-norms (and in Ref. [36] for conjunctive uninorms), but it can be adapted for conjunctors in general.

Proposition 2. [4,36] *Let C be a conjunctor, let I_C be its corresponding residuated function and let I be a fuzzy implication function.*

- If I satisfies **(MP(C))**, then $I \leq I_C$.
- If C is left-continuous, then I satisfies **(MP(C))** if and only if $I \leq I_C$.

Proof. The first item is direct from the definition of I_C . To prove the second one, taking $x = a, y = I(a, b)$, and $z = b$ in Equation **(RP)** we obtain

$$C(a, I(a, b)) \leq b \Leftrightarrow I(a, b) \leq I_C(a, b) \quad \forall a, b \in [0, 1],$$

which ends the proof.

3. SOME GENERAL CONDITIONS FOR THE SATISFACTION OF THE MODUS PONENS INEQUALITY WITH RESPECT TO CONJUNCTORS WITH A NEUTRAL ELEMENT

In this section we analyze the satisfaction of the Modus Ponens inequality with respect to conjunctors with a neutral element $e \in]0, 1]$, studying the relationships that exist between the properties of the conjunctor and those of the fuzzy implication function involved.

We begin by recalling that the paper in Ref. [20], which deals with the Non-Contradiction principle **(NC(N))** (see Subsection 2.2), points out that the fulfillment of this principle with respect to the natural negation of a fuzzy implication function is a necessary condition for the satisfaction of the Modus Ponens inequality: indeed, it suffices to take the value $y = 0$ in **(MP(C))** to obtain $C(x, I(x, 0)) = C(x, N_I(x)) = 0$. As a consequence, all the results presented in Ref. [20] become necessary conditions for the satisfaction of the

Modus Ponens inequality. The following Proposition encompasses the main ones related to conjunctors with a neutral element (Propositions 13, 15, 17, and 18 in Ref. [20]).

Proposition 3. [20] *Let C be a conjunctor with a neutral element $e \in]0, 1]$, let I be an implication function satisfying the Modus Ponens inequality with respect to C and let N_I be the natural negation of the fuzzy implication function ($N_I(x) = I(x, 0)$). Then:*

1. $C(x, y) = 0$ for all $x, y \in [0, 1]$ such that $y \leq N_I(x)$, which means that C ($C_{|[0,e]^2}$ when $e \neq 1$) must satisfy **(ODiv)** unless it is $N_I = N_\perp$.
2. $[\forall x \geq e : N_I(x) = 0]$ and $[\forall x < e, x \neq 0 : N_I(x) < e]$ (note that this implies, in particular, that N_I is non-filling and, whenever $e \neq 1$, N_I is in addition vanishing and non-continuous at least on $x = 0$).
3. If $e = 1, C = \left(\langle a_j, b_j, T_j \rangle \right)_{j \in J}$ is a non-trivial ordinal sum t-norm and $N_I \neq N_\perp$, then:
 - (a) There exists $i \in J$ such that $a_i = 0$ and $b_i \neq 1$.
 - (b) $[\forall x \geq b_i : N_I(x) = 0]$ and $[\forall x < b_i, x \neq 0 : N_I(x) < b_i]$ (note that this implies that N_I must be vanishing and non-continuous at least on $x = 0$).
 - (c) $T_i(x/b_i, y/b_i) = 0$ for all $x \in]0, b_i[, y \in [0, N_I(x)]$.

In the following we present some additional necessary conditions for fuzzy implication functions satisfying the Modus Ponens inequality with respect to conjunctors with a neutral element, and we analyze their consequences.

Proposition 4. *Let C be a conjunctor with a neutral element $e \in]0, 1]$ and let I be an implication function satisfying the Modus Ponens inequality with respect to C . Then*

- i. $I(x, y) < e \quad \forall x, y \in [0, 1]$ such that $x > y$.
- ii. $I(x, y) \leq y \quad \forall x, y \in [0, 1]$ such that $x \geq e$.

Proof. We prove the items step by step.

- Suppose there exist $x_0, y_0 \in [0, 1]$ such that $x_0 > y_0$ and $I(x_0, y_0) \geq e$. In this case, from the monotonicity of C and the fact that e is the neutral element of C , we deduce

$$C(x_0, I(x_0, y_0)) \geq C(x_0, e) = x_0 > y_0,$$

obtaining a contradiction.

- Note first that choosing $x = e$ in **(MP(C))** provides $C(e, I(e, y)) = I(e, y) \leq y$ for any $y \in [0, 1]$. Thus, by monotonicity of I we have

$$I(x, y) \leq I(e, y) \leq y$$

for any $x, y \in [0, 1]$ such that $x \geq e$.

Remark 2. When $e = 1$ the first item in the proposition above may be written as $I(x, y) \neq 1$ for all $x > y$ and hence, it is a generalization of Proposition 7.4.2. in Ref. [1]. Note also that, if $C = \min$, the stronger condition $I(x, y) \leq y$ when $x > y$ is obtained. Similarly,

the second item when $e = 1$ can be written as $I(1, y) \leq y$ for all $y \in [0, 1]$ and hence, it is a generalization of the result given in Ref. [4] for continuous t-norms.

On the other hand, when $e \neq 1$, Proposition 4 recovers some weaker statements related to uninorms proved in Ref. [19].

Next result involves several additional properties of fuzzy implication functions.

Proposition 5. *Let C be a conjunctive with a neutral element $e \in]0, 1[$ and let I be an implication function satisfying the Modus Ponens inequality with respect to C . Then:*

1. $N_I(x) < e$ for any $x \neq 0$ and $N_I(x) = 0$ for any $x \geq e$.
2. If I satisfies (IP), then it necessarily satisfies (OP).
3. If I satisfies (CB), then necessarily $e = 1$ and I satisfies (NP).
4. If $e \neq 1$, then
 - $I(x, y) < y$ for any $x > y \geq e$, in particular $I(1, y) < y$ for any $1 > y \geq e$ (i.e., I does not satisfy (CB) either (NP)).
 - I does not satisfy (IP) either (OP).

Proof. Again we give the proof step by step.

1. It suffices to choose $y = 0$ in Proposition 4 (note that this is also a consequence of item 2 in Proposition 3).
2. Directly comes from the fact that (OP) is equivalent to (IP) and $[I(x, y) \neq 1 \text{ if } x > y]$.
3. Suppose on the contrary that I satisfies (CB) and $e \neq 1$. Then choosing $x > y \geq e$ we get $I(x, y) < e \leq y$ by Proposition 4. This contradicts (CB) and hence $e = 1$. Now (NP) follows directly by taking $x = 1$ in the second item of Proposition 4.
4. The first part of this item is a direct consequence of the previous one. To prove the second part just take $x = y \geq e$ with $x \neq 1$, then Proposition 4 provides $I(x, x) \leq x \neq 1$.

We finally deal with some sufficient conditions that apply to a broad class of conjunctors that includes, as a particular case, conjunctors with a neutral element.

Proposition 6. *Let C be a conjunctive, let $e \in]0, 1[$ and let I be a fuzzy implication. Then:*

1. Suppose $C(x, e) \leq x$ for any $x \in [0, 1]$ and $I(x, y) \leq e$ for any $x, y \in [0, 1]$ such that $x \leq y, x \neq 0, y \neq 1$. If $C(x, I(x, y)) \leq y$ for all $x, y \in [0, 1]$ such that $x > y$ then I satisfies (MP(C)).
2. Suppose $C(e, y) \leq y$ for any $y \in [0, 1]$ and $I(x, y) \leq y$ for any $x, y \in [0, 1]$ such that $x \leq e, x \neq 0$. If $C(x, I(x, y)) \leq y$ for all $x, y \in [0, 1]$ such that $x > e$ then I satisfies (MP(C)).

Proof.

1. It suffices to check that (MP(C)) is true for any $x, y \in [0, 1]$ such that $x \leq y$. This is obvious if $x = 0$ or $y = 1$, and otherwise we have, by monotonicity of C , $C(x, I(x, y)) \leq C(x, e) \leq x \leq y$.

2. We now have to prove that (MP(C)) is true for any $x, y \in [0, 1]$ such that $x \leq e$. If $x = 0$, $C(x, I(x, y)) = 0 \leq y$ for any $y \in [0, 1]$. Otherwise, using the monotonicity of C it is $C(x, I(x, y)) \leq C(e, I(x, y)) \leq I(x, y) \leq y$.

In the limit case where $e = 1$ we get the following results, which are valid, in particular, for any conjunctive aggregation function and hence for any semi-copula:

Corollary 7. *Let C be a conjunctive and let I be a fuzzy implication function.*

1. Suppose $C(x, 1) \leq x$ for any $x \in [0, 1]$. If $C(x, I(x, y)) \leq y$ for all $x, y \in [0, 1]$ such that $x > y$ then I satisfies (MP(C)) (i.e., the values of I when $x \leq y$ are indifferent).
2. Suppose $C(1, y) \leq y$ for any $y \in [0, 1]$ and $I(x, y) \leq y$ for any $x, y \in [0, 1]$ such that $x \neq 0$. Then I satisfies (MP(C)).
3. Suppose $C \leq \min$ (i.e., C is conjunctive) and I satisfies $I(x, y) \leq y$ for any $x, y \in [0, 1]$ such that $x > y$. Then I satisfies (MP(C)).

Proof. The two first items come directly from Proposition 6, and the third one is a combination of them (indeed, because of monotonicity, the conditions $C(x, 1) \leq x$ and $C(1, y) \leq y$ are equivalent to $C \leq \min$).

4. THE MODUS PONENS INEQUALITY WITH RESPECT TO CONJUNCTORS WITH A NEUTRAL ELEMENT $e \in]0, 1[$

The present section is specifically devoted to the use of conjunctors with a neutral element $e \in]0, 1[$. First of all, we summarize in Figure 2 some of the conditions found in Section 3 when they are restricted to this kind of conjunctors. Note that the fact that I can not satisfy (NP) allows to discard many important families of fuzzy implication functions.

Proposition 8. *Let C be a conjunctive with a neutral element $e \in]0, 1[$ and let I be a fuzzy implication function. If I is an (S, N)-implication, an R-implication, a QL-implication, an f, g, h or h_g -generated implication, a probabilistic implication or an S-implication, then I does not satisfy (MP(C)).*

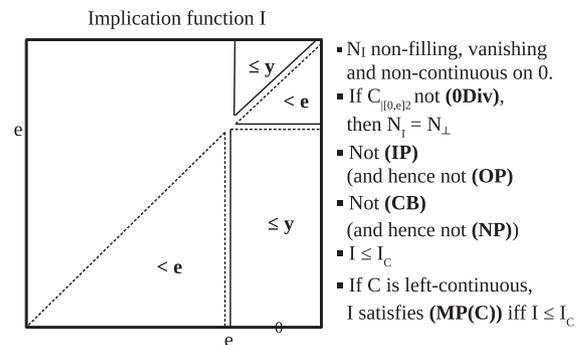


Figure 2 | Some conditions for the satisfaction of (MP(C)) when C is a conjunctive with a neutral element $e \in]0, 1[$ (conditions do not apply to dashed lines).

Proof. All the abovementioned families of fuzzy implication functions satisfy **(NP)** (see e.g., Refs. [8,14,30,31]) and hence Proposition 5 item 4 can be applied.

Nevertheless, there are still many fuzzy implication functions that could satisfy the Modus Ponens inequality with respect to a conjunctor with a neutral element $e \in]0, 1[$. Observe, as it was recalled in Section 2.2, that the behavior of these conjunctors depends on the region of the unit square which is considered, since they are below min on $[0, e]^2$, above max on $[e, 1]^2$ and averaging otherwise. The next result analyzes the consequences of choosing the least possible values on each of these regions.

Proposition 9. *Let C be a conjunctor with neutral element $e \in]0, 1[$ and let I be an implication function.*

1. *Suppose $C_{|[0,e]^2} = 0$ and let $R_1 = \{(x, y) \in [0, 1]^2 : x > y, x < e\}$. Then I satisfies **(MP(C))** on R_1 if and only if $I(x, y) < e$ for any $(x, y) \in R_1$.*
2. *Suppose $C_{|[e,1]^2} = \max$ and let $R_2 = \{(x, y) \in [0, 1]^2 : x \leq y < 1, x \geq e\}$. Then I satisfies **(MP(C))** on R_2 if and only if $I(x, y) \leq y$ for any $(x, y) \in R_2$.*
3. *Suppose $C_{|[e,1] \times [0,e]} = \min$ and let $R_3 = \{(x, y) \in [0, 1]^2 : x > y, x \geq e\}$. Then I satisfies **(MP(C))** on R_3 if and only if $I(x, y) < e$ for any $(x, y) \in R_3$ such that $y \geq e$ and $I(x, y) \leq y$ for any $(x, y) \in R_3$ such that $y < e$.*
4. *Suppose $C_{|[0,e] \times [e,1]} = \min$ and let $R_4 = \{(x, y) \in [0, 1]^2 : 0 < x \leq y < 1, x < e\}$. Then I satisfies **(MP(C))** on R_4 .*

Proof. The necessity part of any of the three first items is given by Proposition 4. The sufficiency is proved below:

1. If $(x, y) \in R_1$ and $I(x, y) < e$ then we have $C(x, I(x, y)) = 0 \leq y$.
2. Suppose that $(x, y) \in R_2$ and $I(x, y) \leq y$. If it were $I(x, y) \leq e$ then Proposition 6 proves the conclusion. Otherwise, if $I(x, y) > e$ then $C(x, I(x, y)) = \max(x, I(x, y)) \leq y$ since both x and $I(x, y)$ are smaller than or equal to y .
3. Suppose now that $(x, y) \in R_3$ and we distinguish two cases. If $y \geq e$ and $I(x, y) < e$ we have $(x, I(x, y)) \in [e, 1] \times [0, e[$ and consequently

$$C(x, I(x, y)) = \min(x, I(x, y)) = I(x, y) < e \leq y.$$

On the other hand, if $y < e$ and $I(x, y) \leq y$ we again have $(x, I(x, y)) \in [e, 1] \times [0, e[$ and

$$C(x, I(x, y)) = \min(x, I(x, y)) = I(x, y) \leq y.$$

4. Finally, if $(x, y) \in R_4$ we again distinguish two cases. If it were $I(x, y) < e$ this would obviously entail $C(x, I(x, y)) \leq C(x, e) = x \leq y$, and otherwise it would be $(x, I(x, y)) \in [0, e[\times [e, 1]$ and then $C(x, I(x, y)) = \min(x, I(x, y)) = x \leq y$.

Note that the combination of the results given in Proposition 9 allows us to characterize the capacity of the least conjunctor with a neutral element $e \neq 1$ (the least uninorm with neutral element e , depicted on the left part of Figure 1) for solving the Modus Ponens inequality.

Proposition 10. *Let $C_{e\perp}$ be the least conjunctor with a neutral element $e \in]0, 1[$ and let I be a fuzzy implication function. Then I satisfies **(MP(C))** with $C = C_{e\perp}$ if and only if the two following conditions hold:*

- $I(x, y) \leq y$ for any $x, y \in [0, 1]$ such that $x \geq e$ and either $y \leq e$ or $x \leq y$.
- $I(x, y) < e$ for any $x, y \in [0, 1]$ such that $x > y$ and either $y \geq e$ or $x < e$.

Remark 3. The two following remarks regarding the above characterization are worth noting:

- I satisfies **(MP($C_{e\perp}$))** is not equivalent to $I \leq I_{C_{e\perp}}$, where $I_{C_{e\perp}}$ is the residuated implication built from $C_{e\perp}$ and given (see e.g., Ref. [37]) by

$$I_{C_{e\perp}}(x, y) = \begin{cases} 1 & \text{if } x \leq y, x < e, \\ e & \text{if } x > y, (x < e \text{ or } y \geq e), \\ y & \text{otherwise.} \end{cases}$$

Indeed, as stated in Proposition 2, if I satisfies **(MP($C_{e\perp}$))** then clearly $I \leq I_{C_{e\perp}}$, but the converse is not necessarily true if C is not left-continuous, as it happens with $C_{e\perp}$. Note in particular that $I_{C_{e\perp}}$ itself does not satisfy **(MP($C_{e\perp}$))**, since taking e.g., (x_0, y_0) such that $x_0 > y_0, x_0 < e$ provides $C_{e\perp}(x_0, I_{C_{e\perp}}(x_0, y_0)) = C_{e\perp}(x_0, e) = x_0 > y_0$.

- Proposition 10 is actually a characterization of fuzzy implication functions satisfying **(MP(C))** with respect to at least some conjunctor with a neutral element $e \in]0, 1[$, since $C_{e\perp}$ is the least of them.

The regions R_1 to R_4 considered in Proposition 9 are depicted in the top of Figure 3, whereas the possible values of fuzzy implication functions I satisfying **(MP(C))** with $C = C_{e\perp}$ proved in Proposition 10 can be viewed in the bottom of the figure.

Example 1. Fixed some $e \in]0, 1[$, let us show that there are many examples of fuzzy implication functions satisfying conditions in Proposition 10 and, consequently, satisfying **(MP($C_{e\perp}$))**.

- i. The parameterized family of fuzzy implication functions given by

$$I_{a,e}(x, y) = \begin{cases} 1 & \text{if } x \leq y, x < e, \\ a & \text{if } a < y < x, \\ y & \text{otherwise,} \end{cases}$$

satisfies **(MP($C_{e\perp}$))**, for any a such that $0 \leq a < e$.

- ii. There are also examples among the residual implications I_U derived from uninorms, or RU -implications for short. For instance, it can be easily viewed from Proposition 10 that all RU -implications derived from uninorms in some of the following classes³ satisfy **(MP($C_{e\perp}$))**:

³See Ref. [28] for the different classes of uninorms and Ref. [1] for the corresponding RU -implications.

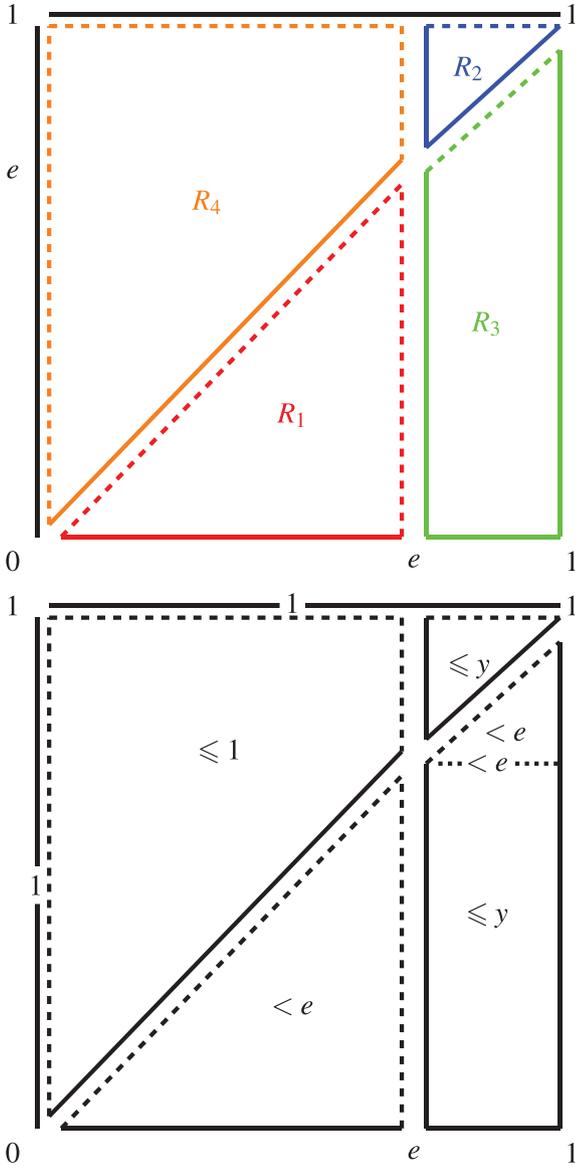


Figure 3 | Regions R_1 to R_4 considered in Proposition 9 can be viewed in the top of the figure. Dashed segments do not belong to the corresponding region. The bottom of the figure corresponds to the possible values of the fuzzy implication functions satisfying $(MP(C_{e\perp}))$ characterized in Proposition 10.

1. Uninorms in \mathcal{U}_{\min} with neutral element $e' < e$ whose underlying t-conorm is given by the maximum (this can be derived also from Proposition 7 in Ref. [19]).
2. All representable uninorms with neutral element e (this can be also derived from Theorem 5 in Ref. [19]).
3. All idempotent uninorms with $g(e) = 0$ where g is the *Id*-symmetrical function associated to the uninorm (see also Proposition 10 in Ref. [19]).

The next Proposition deals with the cases where the greatest values are chosen in each of the four different regions R_1 to R_4 (note that in this case there are some small differences in the location of the border values to each region) of the unit square.

Proposition 11. Let C be a conjunctor with neutral element $e \in]0, 1[$ and let I be an implication function.

1. Suppose $C_{|[0,e]^2} = \min$ and let $R_1 = \{(x, y) \in [0, 1]^2 : x > y, x \leq e\}$. Then I satisfies $(MP(C))$ on R_1 if and only if $I(x, y) \leq y$ for any $(x, y) \in R_1$.
2. Suppose $C_{|[e,1]^2} = 1$ and let $R_2 = \{(x, y) \in [0, 1]^2 : x \leq y, x > e, y \neq 1\}$. Then I satisfies $(MP(C))$ on R_2 if and only if $I(x, y) \leq e$ for any $(x, y) \in R_2$.
3. Suppose $C_{|[e,1] \times]0,e]} = \max$ and let $R_3 = \{(x, y) \in [0, 1]^2 : x > y, x > e\}$. Then I satisfies $(MP(C))$ on R_3 if and only if $I(x, y) = 0$ for any $(x, y) \in R_3$.
4. Suppose $C_{|[0,e] \times]e,1]} = \max$ and let $R_4 = \{(x, y) \in [0, 1]^2 : x \leq y, x \leq e, x \neq 0, y \neq 1\}$. Then I satisfies $(MP(C))$ on R_4 if and only if $I(x, y) \leq \max(e, y)$ for any $(x, y) \in R_4$.

Proof. The sufficiency part of items 1, 2, and 4 is given by Proposition 6, whereas the sufficiency of item 3 is obvious since $I(x, y) = 0$ implies $C(x, I(x, y)) = 0 \leq y$. The necessity is proved below:

1. Take $(x, y) \in R_1$. Since $x > y$ we have $I(x, y) < e$ by Proposition 4, and consequently

$$C(x, I(x, y)) = \min(x, I(x, y)).$$

Now, if it were $I(x, y) > y$ we would deduce $C(x, I(x, y)) > y$ obtaining a contradiction with the Modus Ponens, thus it must be $I(x, y) \leq y$.

2. Consider now $(x, y) \in R_2$ and suppose that $I(x, y) > e$. Since $x > e, y \neq 1$ and $C_{|[e,1]^2} = 1$ we obtain $C(x, I(x, y)) = 1 \not\leq y$ contradicting again the Modus Ponens. Thus, we have $I(x, y) \leq e$ in this case.
3. Let $(x, y) \in R_3$ and suppose that $I(x, y) \neq 0$. In this case, Proposition 4 provides $I(x, y) < e$, and since $x > e$ we get the contradiction

$$C(x, I(x, y)) = \max(x, I(x, y)) = x > y.$$

4. Finally, consider $(x, y) \in R_4$ and let us distinguish two cases. If $I(x, y) \leq e$ then obviously $I(x, y) \leq \max(e, y)$. On the other hand, if $I(x, y) > e$ and taking into account that $x \leq e$ we obtain by the Modus Ponens that

$$I(x, y) = \max(x, I(x, y)) = C(x, I(x, y)) \leq y,$$

and hence $I(x, y) \leq \max(e, y)$.

Note that the first item in Proposition 11 forces implications functions to verify $N_I = N_{\perp}$ (see Proposition 3, item 1) and it may be applied, in particular, to the greatest conjunctor with a neutral element $e \neq 1, C_{e\top}$. Moreover, the combination of the four items of this Proposition characterizes the family of implications functions that satisfy the Modus Ponens inequality with respect to $C_{e\top}$.

Proposition 12. Let $C_{e\top}$ be the greatest conjunctor with a neutral element $e \in]0, 1[$ and let I be a fuzzy implication function. Then

I satisfies $(\mathbf{MP}(\mathbf{C}))$ with $C = C_{e\top}$ if and only if the three following conditions hold:

- $I(x, y) = 0$ for any $x, y \in [0, 1]$ such that $x > y, x > e$.
- $I(x, y) \leq y$ for any $x, y \in [0, 1]$ such that $(x > y, x \leq e)$ or $(x \leq e, y \geq e, x \neq 0)$.
- $I(x, y) \leq e$ for any $x, y \in [0, 1]$ such that $(x \leq y, x > e, y \neq 1)$ or $(x \leq y, y < e, x \neq 0)$.

The regions R_1 to R_4 slightly modified in Proposition 11 are depicted in the top of Figure 4, whereas the possible values of implications I satisfying $(\mathbf{MP}(\mathbf{C}))$ with $C = C_{e\top}$ proved in Proposition 12 can be viewed in the bottom of the figure.

Remark 4. Since $C_{e\top}$ is left-continuous, Proposition 2 shows that the characterization of the satisfaction of $(\mathbf{MP}(C_{e\top}))$, given in the previous proposition, is equivalent to the inequality $I \leq I_{C_{e\top}}$, where the latter is the residuated implication of $C_{e\top}$, given by

$$I_{C_{e\top}}(x, y) = \begin{cases} 1 & \text{if } x = 0 \text{ or } y = 1, \\ 0 & \text{if } x > y, x > e, \\ y & \text{if } (y < x \leq e) \text{ or } (x \leq e \leq y, x \neq 0), \\ e & \text{otherwise.} \end{cases}$$

Observe finally that the fact that $C_{e\top}$ is the greatest conjunctive with neutral element e , along with Remarks 1 and 4 and Proposition 12, allows for the following result.

Proposition 13. Let I be a fuzzy implication function and let $C_{e\top}$ be the greatest conjunctive with a neutral element $e \in]0, 1[$. The following items are equivalent:

1. I satisfies $(\mathbf{MP}(C_{e\top}))$.
2. I satisfies $(\mathbf{MP}(\mathbf{C}))$ with respect to any conjunctive C such that $C \leq C_{e\top}$, in particular any conjunctive with a neutral element $e \in]0, 1[$.
3. I satisfies the three conditions given in Proposition 12 (bottom of Figure 4).
4. $I \leq I_{C_{e\top}}$, where $I_{C_{e\top}}$ is given in Remark 4.

Example 2. Fixed some $e \in]0, 1[$, let us give some examples of fuzzy implication functions satisfying conditions in Proposition 12, i.e., satisfying $(\mathbf{MP}(C_{e\top}))$, and consequently $(\mathbf{MP}(\mathbf{C}))$ with respect to any conjunctive C with neutral element e . Of course $I_{C_{e\top}}$ itself is one of them, as it has been proved before. The following parameterized family of fuzzy implication functions presents other possibilities.

$$I_{a,b}(x, y) = \begin{cases} 1 & \text{if } x = 0 \text{ or } y = 1, \\ 0 & \text{if } x > y, x > e, \\ a & \text{if } (a \leq y \leq e) \text{ and } (0 < x \leq e), \\ b & \text{if } (b \leq y < 1) \text{ and } (0 < x \leq e), \\ e & \text{if } e < x \leq y < 1, \\ y & \text{otherwise,} \end{cases}$$

where a, b are such that $0 \leq a \leq e \leq b \leq 1$.

Results in Propositions 9 and 11 can be also used to characterize those fuzzy implication functions satisfying $(\mathbf{MP}(\mathbf{C}))$ with $C = U_{e\perp}$,

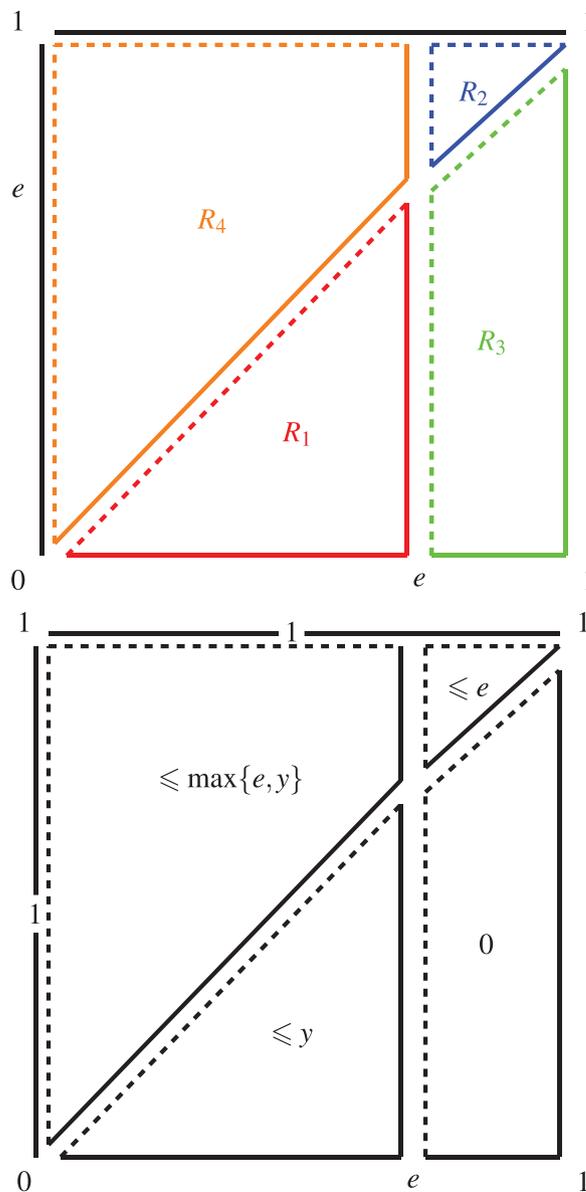


Figure 4 Regions R_1 to R_4 considered in Proposition 11 can be viewed in the top of the figure. Dashed segments do not belong to the corresponding region. The bottom of the figure corresponds to the possible values of the fuzzy implication functions satisfying $(\mathbf{MP}(C_{e\top}))$ characterized in Proposition 12.

where $U_{e\perp}$ is the least idempotent uninorm with neutral element e and with $C = U_{e\top}$, where $U_{e\top}$ is the greatest conjunctive idempotent uninorm with neutral element e . These uninorms are respectively given by (see for instance Ref. [28]):

$$U_{e\perp}(x, y) = \begin{cases} \max(x, y) & \text{if } x, y \geq e, \\ \min(x, y) & \text{otherwise,} \end{cases}$$

and

$$U_{e\top}(x, y) = \begin{cases} 0 & \text{if } x = 0 \text{ or } y = 1, \\ \min(x, y) & \text{if } 0 < x, y \leq e, \\ \max(x, y) & \text{otherwise,} \end{cases}$$

and the results are presented in the next proposition.

Proposition 14. Let $U_{e\perp}$ be the least idempotent uninorm with neutral element e , $U_{e\top}$ the greatest conjunctive idempotent uninorm with neutral element e and let I be a fuzzy implication function. Then

1. I satisfies **(MP(C))** with $C = U_{e\perp}$ if and only if the following two conditions hold:
 - $I(x, y) \leq y$ for any $x, y \in [0, 1]$ such that $[x > y, y < e]$ or $[e \leq x \leq y < 1]$.
 - $I(x, y) < e$ for any $x, y \in [0, 1]$ such that $e \leq y < x$.
2. I satisfies **(MP(C))** with $C = U_{e\top}$ if and only if the following three conditions hold:
 - $I(x, y) = 0$ for any $x, y \in [0, 1]$ such that $e, y < x$.
 - $I(x, y) \leq y$ for any $x, y \in [0, 1]$ such that $[y < x \leq e]$ or $[0 < x \leq y, y \geq e]$.
 - $I(x, y) \leq e$ for any $x, y \in [0, 1]$ such that $x \leq y < e$.

Note that $U_{e\perp}$ is an example of a uninorm in \mathcal{U}_{\min} which is not left-continuous and whose residual implication $I_{U_{e\perp}}$ does not satisfy the Modus Ponens with $U_{e\perp}$. On the contrary, $U_{e\top}$ is a left-continuous uninorm and consequently the characterization given in the second item of the previous proposition is equivalent to say $I \leq I_{U_{e\top}}$.

5. THE MODUS PONENS INEQUALITY WITH RESPECT TO SEMI-COPULAS

This section analyzes the satisfaction of **(MP(C))** when C is a conjunctive with neutral element $e = 1$, i.e., a semi-copula. Figure 5 encompasses the conditions mentioned in Section 3 when they are specifically applied to semi-copulas.

The least and the greatest semi-copulas are two well-known t-norms, T_D and T_M , whose definition was recalled in Section 2.2, along with other important families of t-norms. The next Proposition characterizes their use in the Modus Ponens inequality.

Proposition 15. Let I be a fuzzy implication function. The following statements are true:

- I satisfies **(MP(T_D))** if and only if $I(1, y) \leq y$ for any $y \in [0, 1]$ and $I(x, y) \neq 1$ for all $x, y \in [0, 1]$ such that $x > y$.

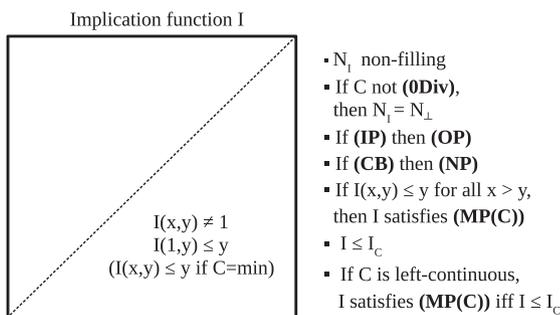


Figure 5 | Some conditions for the satisfaction of **(MP(C))** when C is a semi-copula (conditions do not apply to the dashed line).

- I satisfies **(MP($(T_{LK})_\varphi$))** if and only if $I(x, y) \leq \varphi^{-1}(\varphi(y) + 1 - \varphi(x))$ for all $x, y \in [0, 1]$ such that $x > y$, i.e., $I \leq I_{LK\varphi}$.
- I satisfies **(MP($(T_P)_\varphi$))** if and only if $I(x, y) \leq \varphi^{-1}(\varphi(y)/\varphi(x))$ for all $x, y \in [0, 1]$ such that $x > y$, i.e., $I \leq I_{GG\varphi}$.
- I satisfies **(MP(T_M))** if and only if $I(x, y) \leq y$ for all $x, y \in [0, 1]$ such that $x > y$, i.e., $I \leq I_{GD}$.

Proof. The first item is a matter of calculation (note that T_D is not left-continuous and the satisfaction of **(MP(T_D))** implies but is not equivalent to $I \leq I_{WB}$ - the Weber implication is the residuated implication associated to T_D). The following three items are obtained from Proposition 2 since it is well-known (see e.g., Ref. [1]) that I_{LK}, I_{GG} , and I_{GD} are, respectively, the residuated implication functions associated to the continuous t-norms T_{LK}, T_P , and T_M .

Similarly to what was noticed in the previous section, the facts that T_D and T_M are, respectively, the least and the greatest semi-copulas, along with Proposition 15 and Remark 1, allow for the following equivalences:

Proposition 16. A fuzzy implication function I satisfies **(MP(C))** with respect to at least some semi-copula C if and only if $I(1, y) \leq y$ for any $y \in [0, 1]$ and $I(x, y) \neq 1$ for all $x, y \in [0, 1]$ such that $x > y$.

Proposition 17. Let I be a fuzzy implication function. The following items are equivalent:

1. I satisfies **(MP(T_M))**.
2. I satisfies **(MP(C))** with respect to any C such that $C \leq T_M$ (in particular any semi-copula).
3. I satisfies $I(x, y) \leq y$ for all $x, y \in [0, 1]$ such that $x > y$.
4. $I \leq I_{GD}$.

It is worth noting that the above results allow to easily recover some already existing results concerning the satisfaction of the Modus Ponens, as shown in the example below.

Example 3. Probabilistic implications and probabilistic S-implications [38] are two recently introduced classes of fuzzy implication functions which are defined, respectively, as

$$I_C(x, y) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{C(x, y)}{x} & \text{if } x \neq 0, \end{cases}$$

and

$$\tilde{I}_C(x, y) = C(x, y) - x + 1,$$

where C is a copula.⁴ The paper in Ref. [14] investigates, among other properties, the T -conditionality (Modus Ponens with respect to t-norms) of these functions, with the following results:

1. “A probabilistic implication I_C satisfies **(MP(T_M))**, and consequently **(MP(T))** for any t-norm T , if and only if $C(x, y) \leq xy$

⁴In the first case, to guarantee that I_C is a fuzzy implication function, the copula C must satisfy additionally the condition $C(x_1, y)x_2 \geq C(x_2, y)x_1$ for all $x_1 \leq x_2$.

for all $x, y \in [0, 1]$ such that $x > y$ ” (see Ref. [14], Theorem 4.2). This result could have been directly obtained from Proposition 17.

2. “Any probabilistic implication satisfies **(MP(C))** with $C = T_p$ (and with any weaker t-norm, in particular $C = T_{LK}$)” (see Ref. [14], Proposition 4.3). This may be proved using Proposition 15 taking into account that $I_C \leq I_{GG}$ (see Ref. [39]).
3. “No probabilistic S-implication satisfies **(MP(C))** with $C = T_M$ ” (see Ref. [14], Proposition 4.5). This result may be directly obtained from Proposition 3, item 1, since clearly $N_{I_C} = N_C \neq N_{\perp}$ and T_M does not have zero divisors.
4. “Each probabilistic S-implication satisfies **(MP(C))** with $C = T_{LK}$ ” (see Ref. [14], Proposition 4.6). Indeed, this may be obtained from Proposition 15 by just noting that since any copula C satisfies $C \leq T_M$, any probabilistic S-implication satisfies $I_C \leq I_{T_M}$, and the latter is nothing but I_{LK} .
5. “A probabilistic S-implication I_C satisfies **(MP(C))** with $C = T_p$ if and only if $I_C \leq I_{GG}$ ” (See Ref. [14], Proposition 4.7). Again, this can be obtained directly from Proposition 15. Note nevertheless that the condition $I_C \leq I_{GG}$ is never true, and hence this sentence could be better expressed as “No probabilistic S-implication satisfies **(MP(C))** with $C = (T_p)_{\varphi}$ ”. This fact could have been directly obtained from Proposition 3, item 1, taking into account that $N_{I_C} \neq N_{\perp}$ and that $(T_p)_{\varphi}$ does not satisfy **(0Div)**.

Finally, let us note that the above examples could be easily adapted to the classes of the so-called *survival implications* and *survival S-implications*, since it was proved in Refs. [39,40] that these classes respectively coincide with the classes of probabilistic implications and probabilistic S-implications.

6. CONCLUSIONS AND FUTURE WORK

The inference rule of Modus Ponens is a property of paramount importance in approximate reasoning which is not studied only for the case of t-norms any more. Indeed, several more general classes of conjunctors have been considered recently, most of them having a neutral element $e \in]0, 1[$. Therefore, in this paper, many new results are proved in which the necessary properties for the fuzzy implication function I to satisfy the Modus Ponens with respect to a conjunctive with neutral element $e \in]0, 1[$ are determined. Although these properties are not fulfilled by well-known families such as (S,N), R, or QL-implications, in this paper many examples of admissible fuzzy implication functions are presented in which the common feature is that they do not satisfy the left neutrality principle, (NP). Particularly interesting are the characterizations of all fuzzy implication functions satisfying the Modus Ponens with respect to the least and the greatest conjunctors with neutral element $e \in]0, 1[$.

As future work, we want to study with more detail the Modus Ponens property with respect to some of the families of conjunctors with neutral element $e \in]0, 1[$ such as ordinal sum t-norms, representable aggregation functions [41–43], continuous generated functions [44], or conjunctive uninorms. The study of the latter has been made fixing the class of RU-implications [19] and

some preliminary results are available for (U,N)-implications [18]. However, these results must be expanded and many other classes of fuzzy implication functions derived from uninorms have not been studied yet.

CONFLICT OF INTEREST

None of the authors has any conflict of interest.

AUTHORS' CONTRIBUTIONS

All authors had a similar contribution to both the research and the manuscript preparation and have agreed to the final version.

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