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Chebyshev Polynomials, Rhodonea Curves and Pseudo-Chebyshev Functions. A Survey

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ABSTRACT

In recent works, starting from the complex Bernoulli spiral and the Grandi roses, sets of irrational functions have been introduced and studied that extend to the fractional degree the polynomials of Chebyshev of the first, second, third and fourth kind. The functions thus obtained are therefore called pseudo-Chebyshev. This article presents a review of the elementary properties of these functions, with the aim of making the topic accessible to a wider audience of readers. The subject is presented as follows. In Section 2 a review of spiral curves is given. In Section 3 the main properties of the classical Chebyshev polynomials are recalled. The Grandi (Rhodonea) curves and possible extensions are introduced in Section 4, and a method for deriving new curves, changing cartesian into polar coordinates, is touched on. The possibility to consider the Grandi curves even for rational indexes allows to introduce in Section 5 the pseudo-Chebyshev functions, which are derived from the Chebyshev polynomials assuming rational values for their degree. The main properties of these functions are shown, including recursions and differential equations. In particular, the case of half-integer degree is examined in Section 6 since, in this case, the pseudo-Chebyshev functions verify even the orthogonality property. As a consequence, new system of irrational orthogonal functions are introduced.

1. INTRODUCTION

In writing an article on Growth and Shape, one cannot help but link the treatment to geometrical entities that translate those concepts into mathematical terms. They are the logarithmic spiral of Bernoulli, the curves of Lamé, the roses of Grandi, the lemniscate of Bernoulli and their generalizations.

The spiral has always been associated with growth phenomena, starting with that of the shell Nautilus widely studied in the book by Thompson [1] and in many subsequent works.

The Lamé's curves have been generalized by J. Gielis in the 2D and 3D case in works [2,3] that have had wide international resonance [4–7]. Grandi's roses (also called Rhodoneas) and Bernoulli's lemniscate have polar equations that lend themselves to being generalized, as is done here in Section 4.1. All these curves (or surfaces) of the plane (of space) lend themselves to creating mathematical forms that model natural forms [3].

In this article, starting from the spiral of Bernoulli, in the complex form, we make the obvious connection with the first and second kind Chebyshev polynomials, and with the roses of Grandi.

After that, having observed that roses also exist for rational index values, extensions of that polynomials are introduced in the case of fractional degree. Thus, irrational functions are found which are called pseudo-Chebyshev of first and second kind, because they continue to verify many of the properties of the corresponding Chebyshev polynomials.

Subsequently, using the links with Chebyshev polynomials of third and fourth kind and the good work [8], the pseudo-Chebyshev functions of third and fourth kind are also introduced and studied.

Particular importance is given, in Section 6, to the case of the half-integer degree, because, in this case, the pseudo-Chebyshev functions verify not only the corresponding recurrence relations and differential equations, but also the orthogonality properties, in the interval $[-1, 1]$, with respect to the same weights of the classical polynomials.

In this survey we limited ourselves to considering only the most elementary properties of the pseudo-Chebyshev functions, which can be proven starting from trigonometric identities, that are known to secondary school students, so as to make the treatment usable to a wide audience. Moreover, the use of higher tools seems to be unessential in the context of this study, which deals with functions of elementary nature, connected in a simple way to trigonometric functions.

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2. SPIRALS

The spiral symbol is found in every ancient culture, all over the world (see e.g. Figures 1, 2). The spiral is a sacred symbol, possibly reminding us the evolution of our life.

The first attempt to describe a spiral is due to Theodore of Cyrene, a mathematician from the school of Pythagoras, in the 5th century BC.

By the mathematical point of view spirals are described by polar equations.

Many information on this subject can be found in Lockwood [9] and in Thompson [1], where applications to natural shapes (see e.g. Figure 3) are deeply analyzed. In a recent article [10] a Bernoulli spiral in complex form has been related to the Grandi (Rhodonea) curves and Chebyshev polynomials.

Figure 1 | Ancient Crete island vases.

Figure 2 | A a well of Nazca culture.

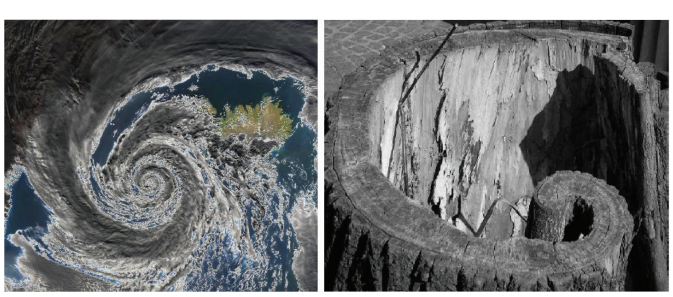


Figure 3 | Spirals - natural shapes [29].

Connection with curvature can be found in Gielis et al. [11].

2.1. Archimedes vs Bernoulli Spiral

The Archimedes (Figure 4) spiral [12] (Figure 5) has the polar equation:

$$\rho = a\theta, \qquad (a > 0, \quad \theta \in \mathbb{R}^+). \tag{1}$$

If $\theta > 0$ the spiral turns counter-clockwise, if $\theta < 0$ the spiral turns clockwise. Bernoulli's (logarithmic) spiral [13] (Figure 5) has the polar equation

$$\rho = ab^{\theta}, \qquad (a, b \in \mathbb{R}^+), \qquad \theta = \log_b\left(\frac{\rho}{a}\right). \tag{2}$$

Varying the parameters a and b one gets different types of spirals.

The size of the spiral depends on a, while the term b controls the verse of rotation and how it is "narrow".

Being a and b positive costants, there are some interesting cases. The most popular logarithmic spiral is the *harmonic spiral*, in which the distance between the spires is in harmonic progression, with ratio $\phi = \frac{\sqrt{5}-1}{2}$, that is the "Golden ratio" of the unit segment.

The logarithmic spiral was discovered by René Descartes in 1638, and studied by Jakob Bernoulli (1654–1705) (Figure 6).

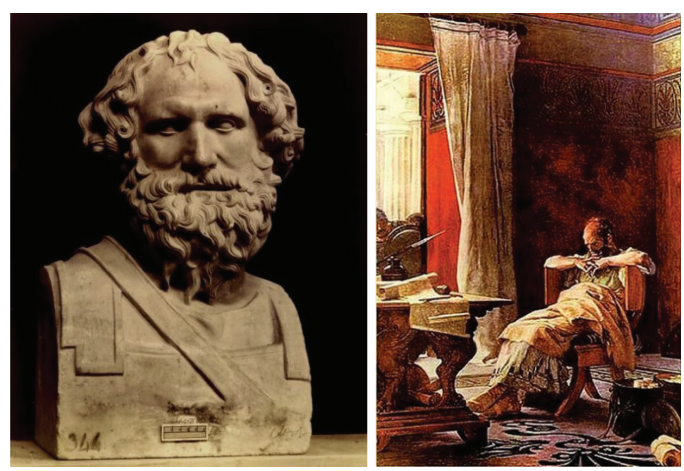


Figure 4 | Archimedes (traditional) and his death by N. Barabino.

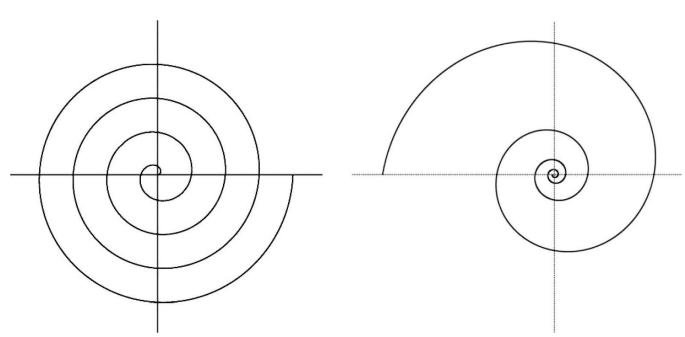


Figure 5 | Archimedes vs Bernoulli spiral.

Pierre Varignon (1654–1722) called it *spiral equiangular*, because:

1. There is a constant angle between the tangent at a given point and the polar radius passing through the same point.
2. The inclination angle with respect to concentric circles centered at the origin is also constant.

It is a first example of a fractal. As it is written on J. Bernoulli's tomb: *Eadem Mutata Resurgo* (but the spiral represented there is of Archimedes type).

2.2. Fermat Spiral, Fibonacci and Other Types of Spirals

The Fermat (parabolic) spiral (Figure 7) has polar equation:

$$\rho = \pm a\theta^{1/2}. \tag{3}$$

Fermat's spiral suggests the possibility of introducing other kind of spiral graphs.

In fact there is a straightforward correspondence, between cartesian and polar systems of coordinates, which transforms $y = f(x)$ functions of the (x, y) plane into polar curves $\rho = f(\theta)$ of the (ρ, θ) plane.

In this planar transformation, the Archimedes spiral $\rho = a\theta$ corresponds to the straight line $y = ax$, the Bernoulli spiral $\rho = ab^{\theta}$ to the exponential function $y = ab^{x}$, and the Fermat spiral to the parabolic function $y = a\sqrt{x}$.

Figure 6 | René Descartes by F. Hals and Jakob Bernoulli.

Then, putting:

$$\rho = a\theta^{m/n}, \qquad (m, n \text{ positive integers}, n \neq 0), \tag{4}$$

one gets a parametric family of spirals, at varying m and n.

Notice that, if $m > n$, so that the exponent is >1, the coils of spiral are widening (Figure 7), while if $m < n$ being the exponent <1, the coils of spiral are shrinking (as in Fermat's case).

Other possibilities are:

1. To assume $\theta^{m/n}$ with $m/n < 0$; in this case the coils are wrapped around the origin.
2. To use a graph with horizontal asymptotes, in order to get an asymptotic spiral (Figure 8).

In what follows, we consider a "canonical form" of the Bernoulli spirals assuming $a = 1$, $b = e^{n}$, that is, the simplified polar equation:

$$\rho = e^{n\theta}, \qquad (n \in \mathrm{N}). \tag{5}$$

2.3. The Complex Bernoulli Spiral

We now introduce the complex case, putting

$$\rho = \Re\rho + i\Im\rho, \tag{6}$$

and considering a Bernoulli spiral of the type:

$$\rho = e^{in\theta} = \cos n\theta + i\sin n\theta. \tag{7}$$

Therefore, we have:

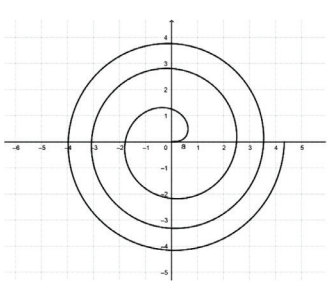
$$\rho_1 = \Re\rho = \cos n\theta, \qquad \rho_2 = \Im\rho = \sin n\theta. \tag{8}$$

The curves with polar equation:

$$\rho = \cos(n\theta) \tag{9}$$

are known as Rhodonea or Grandi curves, in honour of G. G. Grandi (1671–1742) (Figure 12), who communicated his discovery to G. W. Leibniz (1646–1716), in 1713.

Curves with polar equation: $\rho = \sin(n\theta)$ are equivalent to the preceding ones, up to a rotation of $\pi/(2n)$ radians.



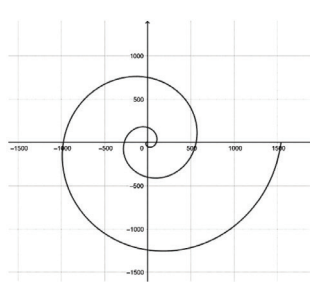
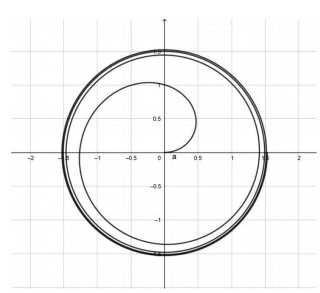


Figure 7 | Fermat spiral $\rho = \theta^{1/2}$, Spiral $\rho = \theta^{3/2}$.



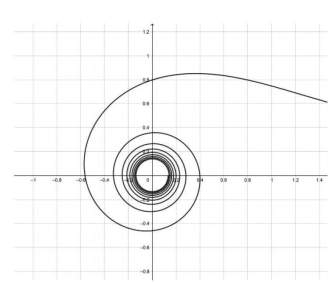


Figure 8 | Spiral $\rho = \theta^{-1/2}$, Asymptotic spiral $\rho = \arctan(\theta)$.

The Grandi roses display

- n petals, if n is odd.
- $2n$ petals, if n is even.

By using Eq. (9) it is impossible to obtain, roses with $4n + 2$ ($n \in \mathrm{N} \cup \{0\}$;) petals.

Roses with $4n + 2$ petals can be obtained by using the Bernoulli lemniscate and its extensions. More precisely,

- for $n = 0$, a two petals rose comes from the equation $\rho = \cos^{1/2}(2\theta)$ (the Bernoulli lemniscate),
- for $n \geq 1$, a $4n + 2$ petals rose comes from the equation $\rho = \cos^{1/2}[(4n + 2)\theta]$.

Further very general extensions of the Bernoulli lemniscate are presented in Section 4.1.

3. CHEBYSHEV POLYNOMIALS

P. Butzer and F. Jongmans, in their biography of Chebyshev [14], assert that Pafnuty Lvovich Chebyshev (Figure 9) was the creator in St. Petersburg of the greatest Russian mathematical school before the revolution.

Starting from the equations:

$$(e^{it})^n = e^{int}, \qquad (\cos t + i\sin t)^n = \cos(nt) + i\sin(nt), \tag{10}$$

and using the binomial expansion, we find:

$$\sum_{k=0}^{n} i^k \binom{n}{k} \cos^{n-k} t \sin^k t = \sum_{h=0}^{\left[\frac{n}{2}\right]} (-1)^h \binom{n}{2h} \cos^{n-2h} t \sin^{2h} t + i \sum_{h=0}^{\left[\frac{n-1}{2}\right]} (-1)^h \binom{n}{2h+1} \cos^{n-2h-1} t \sin^{2h+1} t = \sum_{h=0}^{\left[\frac{n}{2}\right]} (-1)^h \binom{n}{2h} \cos^{n-2h} t (1-\cos^2 t)^h + i \sin t \sum_{h=0}^{\left[\frac{n-1}{2}\right]} (-1)^h \binom{n}{2h+1} \cos^{n-2h-1} t (1-\cos^2 t)^h.$$

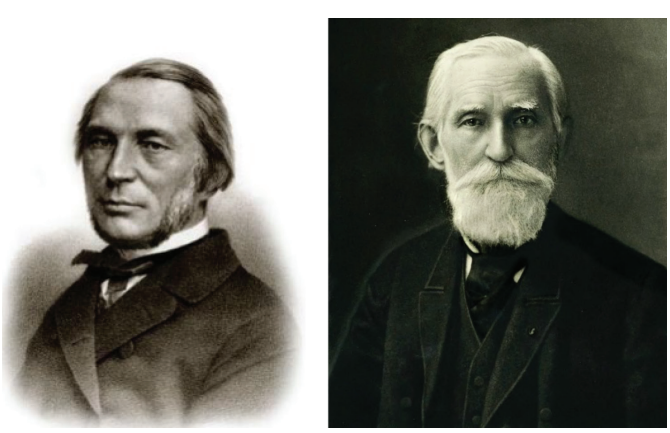


Figure 9 | Pafnuty Lvovich Chebyshev (1821–1894).

By comparing these equations with (10), we find:

$$\cos(nt) = \sum_{h=0}^{\left[\frac{n}{2}\right]} (-1)^h \binom{n}{2h} \cos^{n-2h} t \, (1-\cos^2 t)^h \tag{11}$$

and

$$\frac{\sin(nt)}{\sin t} = \sum_{h=0}^{\left[\frac{n-1}{2}\right]} (-1)^h \binom{n}{2h+1} \cos^{n-2h-1} t \, (1-\cos^2 t)^h. \tag{12}$$

Putting $x = \cos t$, in Eqs. (11) and (12) we find two polynomials, in the x variable, of degrees respectively n and $n - 1$, which are the first and second kind Chebyshev polynomials [15,16]:

$$T_n(x) := \cos(n \arccos x) = \sum_{h=0}^{\left[\frac{n}{2}\right]} (-1)^h \binom{n}{2h} x^{n-2h} (1-x^2)^h, \tag{13}$$

$$U_{n-1}(x) := \frac{\sin(n \arccos x)}{\sin(\arccos x)} = \sum_{h=0}^{\left[\frac{n-1}{2}\right]} (-1)^h \binom{n}{2h+1} x^{n-2h-1} (1-x^2)^h. \tag{14}$$

3.1. Basic Properties of the Chebyshev Polynomials of the First Kind

The trigonometric equation

$$\cos((n+1)t) + \cos((n-1)t) = 2\cos t \cos(nt)$$

gives the recurrence relation:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x). \tag{15}$$

By using the initial values:

$$T_0(x) = 1, \qquad T_1(x) = x,$$

the subsequent polynomials are found:

$$\begin{aligned}
T_2(x) &= 2x^2 - 1 \\
T_3(x) &= 4x^3 - 3x \\
T_4(x) &= 8x^4 - 8x^2 + 1 \\
T_5(x) &= 16x^5 - 20x^3 + 5x \\
T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1 \\
T_7(x) &= 64x^7 - 112x^5 + 56x^3 - 7x \\
T_8(x) &= 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1 \\
&\ldots
\end{aligned}$$

Note that:

- The leading coefficient of $T_n(x)$ is equal to 2^{n-1}.
- The polynomials $T_{2n}(x)$ are even functions and the $T_{2n+1}(x)$ are odd functions.
- $\forall\, n \in \mathrm{N}, \quad T_n(1) = 1, \quad T_n(-1) = (-1)^n$.
- The n zeros of $T_n(x)$ are real, distinct and internal to the interval $[-1, 1]$.

More precisely, they are given by:

$$x_{n,k} = \cos\left(\frac{(2k+1)}{n}\frac{\pi}{2}\right), \qquad (k = 0, 1, \ldots, n-1), \tag{16}$$

In fact, we have:

$$|T_n(x_{n,k})| = |\cos(n \arccos x_{n,k}| = \left|\cos\left((2k+1)\frac{\pi}{2}\right)\right| = 0.$$

- The polynomials $\{T_n(x)\}$ are orthogonal in the interval $[-1, 1]$, with respect to the weight: $(1 - x^2)^{-1/2}$.

In fact, from the orthogonality of the cosine functions:

$$\int_0^{\pi} \cos(nt)\cos(mt)\, dt = 0, \qquad (\text{se } m \neq n),$$

by the change of variable: $t = \arccos x$, we find:

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}\, dx = 0, \qquad (\text{if } m \neq n). \tag{17}$$

We furthermore have:

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx = \pi \tag{18}$$

$$\int_{-1}^{1} \frac{T_n^2(x)}{\sqrt{1-x^2}}\, dx = \int_0^{\pi} \cos^2(nt)\, dt = \frac{\pi}{2}, \qquad (n \in \mathrm{N}). \tag{19}$$

3.2. Basic Properties of the Chebyshev Polynomials of the Second Kind

In a similar way, the same recurrence relation holds for the second kind polynomials:

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x). \tag{20}$$

By using the initial values:

$$U_0(x) = 1, U_1(x) = 2x,$$

the subsequent polynomials are found:

$$\begin{aligned}
U_2(x) &= 4x^2 - 1\\
U_3(x) &= 8x^3 - 4x\\
U_4(x) &= 16x^4 - 12x^2 - 1\\
U_5(x) &= 32x^5 - 32x^3 - 6x\\
U_6(x) &= 64x^6 - 80x^4 - 24x^2 - 1\\
U_7(x) &= 128x^7 - 192x^5 + 80x^3 - 8x\\
U_8(x) &= 256x^8 - 448x^6 + 240x^4 - 40x^2 + 1\\
&\cdots
\end{aligned}$$

The polynomials $\{U_n(x)\}$ are orthogonal in the interval $[-1, 1]$, with respect to the weight: $(1 - x^2)^{1/2}$:

$$\int_{-1}^{1} U_n(x)U_m(x)\sqrt{1-x^2}\, dx = 0, \qquad (\text{if } m \neq n). \tag{21}$$

We furthermore have:

$$\int_{-1}^{1} U_n^2(x)\sqrt{1-x^2}\, dx = \frac{\pi}{2}, \qquad (n \in \mathrm{N}). \tag{22}$$

Connections with the polynomials of the first kind

$$\begin{aligned}
(1-x^2)U_{n-1}(x) &= xT_n(x) - T_{n+1}(x),\\
T_n(x) &= U_n(x) - xU_{n-1}(x).
\end{aligned} \tag{23}$$

The second kind Chebyshev polynomials play an important role in representing the powers of a 2×2 non singular matrix [17,18]. Extension of this polynomial family to the multivariate case has been considered for representing the powers of a $r \times r$ ($r \geq 3$) non-singular matrix (see [18,19]).

Remark 3.1. Chebyshev polynomials are a particular case of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, which are orthogonal in the interval $[-1, 1]$ with respect to the weight $(1 - x)^{\alpha}(1 + x)^{\beta}$. More precisely, the following equations hold:

$$T_n(x) = P_n^{(-1/2,-1/2)}(x), \qquad U_n(x) = P_n^{(1/2,1/2)}(x).$$

Therefore, properties of the Chebyshev polynomials could be deduced in a more general framework of the hypergeometric functions. However, in this approach the connection with trigonomeric function disappears. In this article, dealing with elementary functions, we use only elementary methods, in order to make the topic accessible to a wider audience of readers. In such a way, we avoid to shoot flies with cannons.

Remark 3.2. In connection with interpolation and quadrature problems, another couple of Chebyshev polynomials have been considered. They correspond to different choices of weights:

$$V_n(x) = P_n^{(1/2,-1/2)}(x), \qquad W_n(x) = P_n^{(-1/2,1/2)}(x).$$

These were called by Gautschi [20] the third and fourth kind Chebyshev polynomials, and will be considered in what follows.

3.3. Basic Properties of the Chebyshev Polynomials of Third and Fourth Kind

The third and fourth kind Chebyshev polynomials have been studied and applied by several scholars (see e.g. [8,21]), because they are useful in quadrature rules, when the singularities occur only at one of the end points (+1 or −1) (see [22]). Furthermore, recently they have been applied in Numerical Analysis for solving high odd-order boundary value problems with homogeneous or nonhomogeneous boundary conditions [21].

The third and fourth kind Chebyshev polynomials are defined in $[-1, 1]$ as follows:

$$\begin{aligned}
V_n(x) &= \frac{\cos\left[\left(n+\frac{1}{2}\right)\arccos x\right]}{\cos\left[\frac{(\arccos x)}{2}\right]},\\
W_n(x) &= \frac{\sin\left[\left(n+\frac{1}{2}\right)\arccos x\right]}{\sin\left[\frac{(\arccos x)}{2}\right]}.
\end{aligned} \tag{24}$$

Since $W_n(x) = (-1)^n V_n(-x)$, as it can be see by their graphs (Figures 10 and 11), the third and fourth kind Chebyshev polynomials are essentially the same polynomial set, but interchanging the ends of the interval [−1, 1].

The orthogonality properties hold [8]:

$$\int_{-1}^{1} V_n(x)V_m(x)\sqrt{\frac{1+x}{1-x}}\,dx = \int_{-1}^{1} W_n(x)W_m(x)\sqrt{\frac{1-x}{1+x}}\,dx = \pi\delta_{n,m},$$

(δ is the *Kronecher delta*).

One of the explicit advantages of Chebyshev polynomials of third and fourth kind is to estimate some definite integrals as

$$\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\,f(x)dx \quad \text{and} \quad \int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\,f(x)dx$$

with the precision degree $2n-1$, by using the n interpolatory points $x_k = \cos\frac{(2k-1)\pi}{2n+1}$, ($k = 1, 2, ..., n$), in the interval [−1, 1] [7,21,22].

4. THE GRANDI (RHODONEA) CURVES

A few graphs of Rhodonea curves are shown in Figures 13–14.

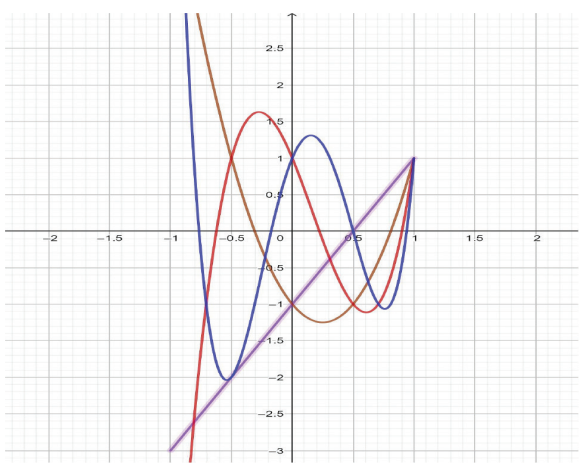


Figure 10 | $V_k(x)$, $k = 1, 2, 3, 4$. 1. violet, 2. brown, 3. red, 4. blue.

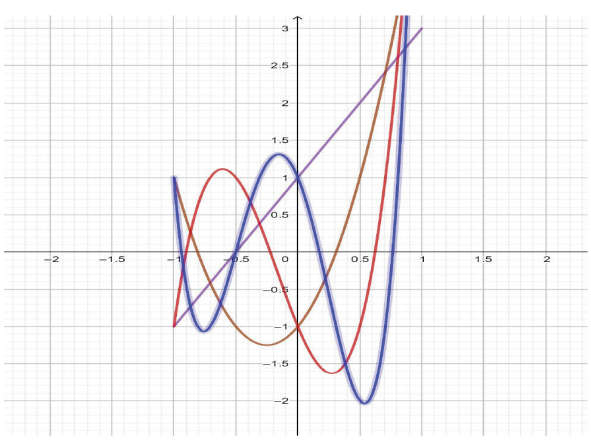


Figure 11 | $W_k(x)$, $k = 1, 2, 3, 4$. 1. violet, 2. brown, 3. red, 4. blue.

Figure 12 | G. G. Grandi and G. W. Leibniz.

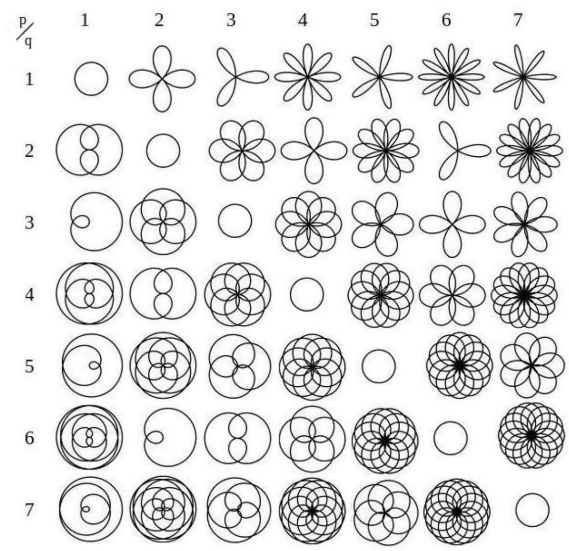


Figure 13 | Rhodonea $\cos\left(\frac{p}{q}\theta\right)$.

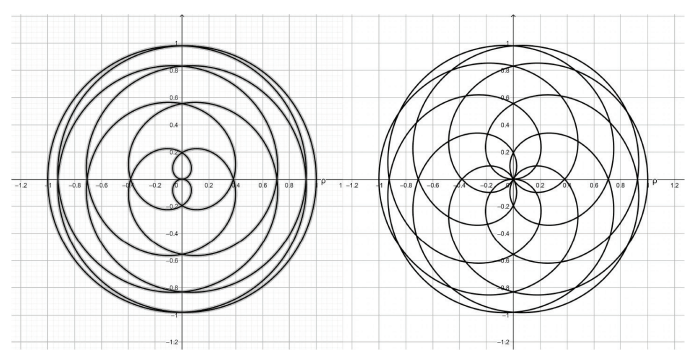


Figure 14 | Rodoneas $\cos\left(\frac{1}{8}\theta\right)$ and $\cos\left(\frac{3}{8}\theta\right)$.

4.1. Cartesian vs Polar Coordinates

The study of curves is of great importance for the modelling of natural objects and has attracted many scholars [9,23]. In particular, the so-called *superformula* by Gielis [2,3,5] has made it possible to construct the most diverse figures by varying a few parameters (see also [4,6,24]).

By exploiting the correspondence, mentioned in Section 2.2, between curves in the cartesian or polar form, we can find further graphs both of symmetric and asymmetric type.

In what follows we show some symmetric graphs, which does not coincide with the shapes of Grandi curves, and several others which are not symmetric. These graphs generalize in a wide way both the Grandi curves and the Bernoulli lemniscate recalled in Section 2.3.

To this aim, we consider first circular functions of the type:

$$y = \sin^h(mx)\cos^k(nx), \quad (h, k, m, n \in \mathbf{N}^+, m \neq n),$$

The corresponding polar curves are:

$$\rho = \sin^h(m\theta)\cos^k(n\theta), \quad (h,k,m, n \in \mathbf{N}^+, m \neq n),$$

and, by choosing m, n, h, k as particular positive integers, we find the few graphs depicted in Figures 15–17.

Of course we have more examples by changing sine by cosine or increasing the number of sine/cosine factors in the above trigonometric products.

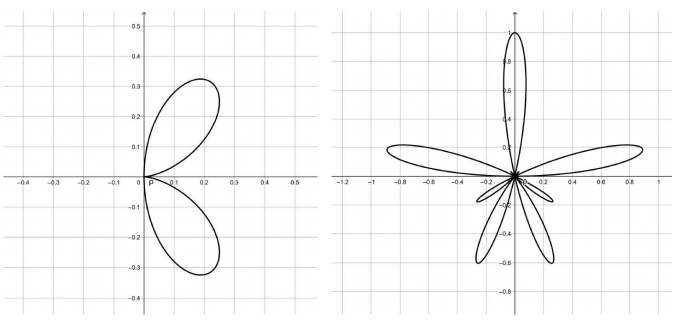


Figure 15 | $\rho = \cos(\theta)\sin^2(\theta)$, $\rho = \sin(7\theta)\cos(2\theta)$.

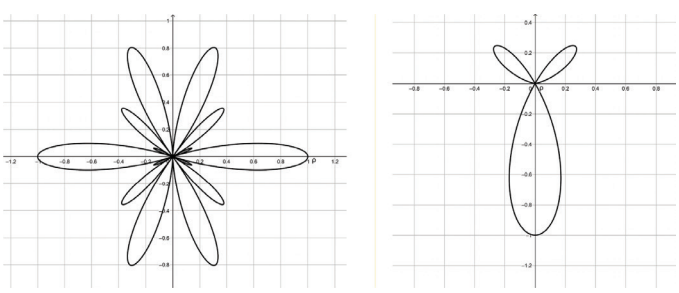


Figure 16 | $\rho = \cos(3\theta)\cos(5\theta)$, $\rho = \sin^2(\theta)\sin(3\theta)$.

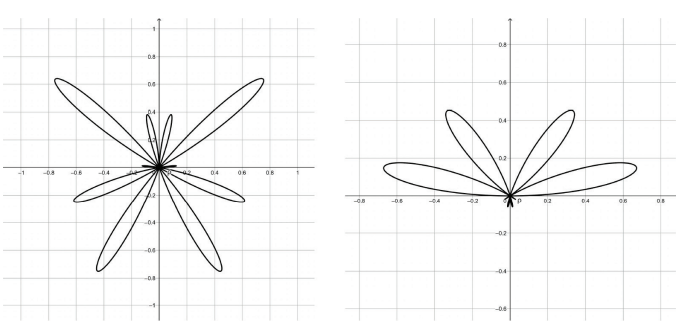


Figure 17 | $\rho = \sin(2\theta)\cos(9\theta)$, $\rho = \cos(\theta)\cos(3\theta)\sin(5\theta)$.

The curves obtained in this way can be interpreted as a generalization of the Grandi roses.

A first example of this kind of figures can be found in Thompson [1] (Figure 504, page 1047), where the polar equation $\rho = \sin(\theta/2)\sin(n\theta)$ can be found.

A infinite number of non-symmetric possibilities (generalizing in particular the Bernoulli lemniscate), are obtained by starting from the above circular functions multiplied by powers of the x variable (of polynomials $P_q(x)$ of degree q), and taking positive rational numbers for the parameters m, n:

$$y = x^q \sin^h(mx)\cos^k(nx), \quad (h,k,m,n \in \mathbf{K}^+),$$
$$y = P_q(x)\sin^h(mx)\cos^k(nx), \quad (h,k,m,n \in \mathbf{K}^+),$$

therefore, finding the polar equations:

$$\rho = \theta^q \sin^h(m\theta)\cos^k(n\theta), \quad (h,k,m,n \in \mathbf{K}^+),$$
$$\rho = P_q(\theta)\sin^h(m\theta)\cos^k(n\theta), \quad (h,k,m,n \in \mathbf{K}^+).$$

The graphic results obtained in these cases recall the inflatable balloons of children's games. Note that in this case, the resulting curve may depend on the considered interval.

A few examples of this type are shown in Figures 18–22.

5. PSEUDO-CHEBYSHEV FUNCTIONS OF THE FIRST, SECOND, THIRD, FOURTH KIND

5.1. Basic properties of the First and Second Kind

We put, by definition:

$$T_{\frac{p}{q}}(x) = \cos\left(\frac{p}{q}\arccos(x)\right), \tag{25}$$

$$\sqrt{1-x^2}\, U_{\frac{p}{q}}(x) = \sin\left(\frac{p}{q}\arccos(x)\right), \tag{26}$$

where p and q are integer numbers, $(q \neq 0)$.

Note that definitions (25) and (26) hold even for negative indexes, that is for $p/q < 0$, according to the parity properties of the trigonometric functions.

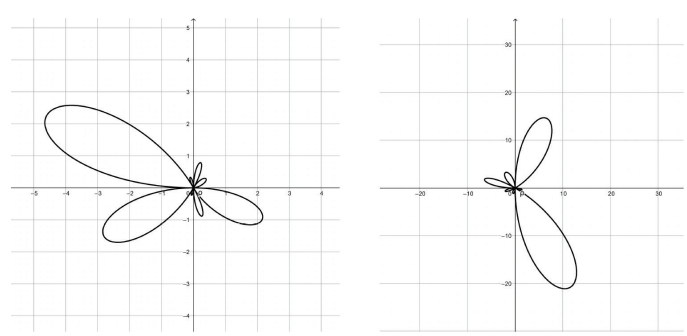


Figure 18 | $\rho = \theta\sin(3\theta)\cos(\theta)$, $\rho = \theta^2\sin(\theta)\cos(3\theta)$.

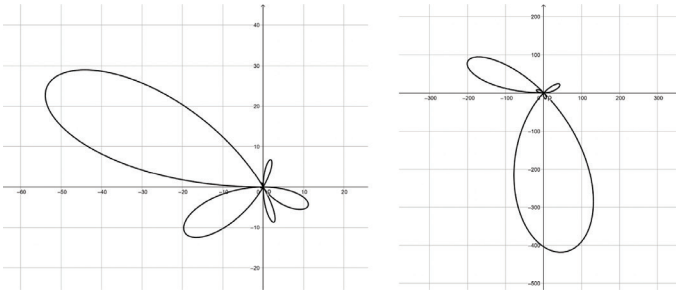


Figure 19 | $\rho = (2\theta^2 - 1)\cos(\theta)\sin(3\theta)$, $\rho = (4\theta^3 - 3\theta)\cos(2\theta)\sin(\theta)$.

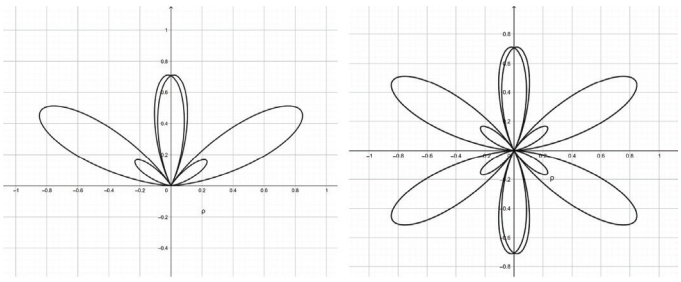


Figure 20 | $\rho = \cos(\theta/2)\sin^2(3\theta)$ in $[0, 2\pi]$ & in $[0, 4\pi]$.

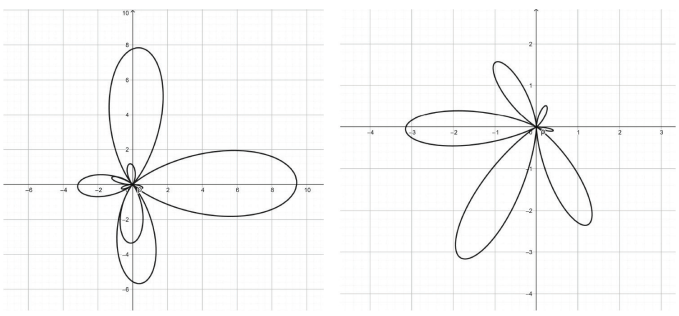


Figure 21 | $\rho = \theta\sin(\theta/2)\cos^2(2\theta)$ & $\rho = \theta\sin(\theta/2)\cos^2(3\theta)$ in $[0, 2\pi]$.

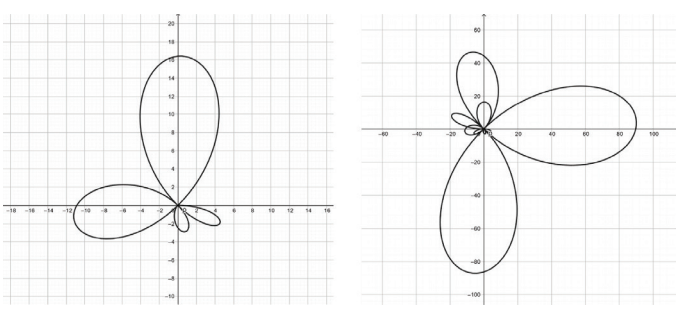


Figure 22 | $\rho = (\theta^2 + 1)\sin(\theta/2)\cos(2\theta)$ in $[0, 2\pi]$ & in $[0, 4\pi]$.

5.2. Pseudo-Chebyshev Functions of the First Kind

The following theorems hold:

Theorem 5.1. *The pseudo-Chebyshev functions $T_{p/q}(x)$ satisfy the recurrence relation*

$$T_{\frac{p}{q}+1}(x) = 2xT_{\frac{p}{q}}(x) - T_{\frac{p}{q}-1}(x). \tag{27}$$

Proof - Write Eq. (27) in the form:

$$T_{\frac{p}{q}+1}(x) + T_{\frac{p}{q}-1}(x) = 2xT_{\frac{p}{q}}(x),$$

then use definition (25) and the trigonometric identity:

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right).$$

Theorem 5.2. *The first kind pseudo-Chebyshev functions $T_{p/q}(x)$ satisfy the differential equation:*

$$(1-x^2)y'' - xy' + \left(\frac{p}{q}\right)^2 y = 0. \tag{28}$$

Proof - Note that

$$D_x T_{\frac{p}{q}}(x) = \left(\frac{p}{q}\right)U_{\frac{p}{q}-1}(x),$$

$$D_x^2 T_{\frac{p}{q}}(x) = -\left(\frac{p}{q}\right)^2(1-x^2)^{-1}T_{\frac{p}{q}}(x) + x(1-x^2)^{-1}\left(\frac{p}{q}\right)U_{\frac{p}{q}-1}(x),$$

$$(1-x^2)D_x^2 T_{\frac{p}{q}}(x) - xD_x T_{\frac{p}{q}}(x) = -\left(\frac{p}{q}\right)^2 T_{\frac{p}{q}}(x),$$

so that Eq. (28) follows.

5.3. Pseudo-Chebyshev Functions of the Second Kind

The following theorems hold:

Theorem 5.3. *The pseudo-Chebyshev functions $U_{\frac{p}{q}}(x)$ satisfy the recurrence relation*

$$U_{\frac{p}{q}+1}(x) = 2xU_{\frac{p}{q}}(x) - U_{\frac{p}{q}-1}(x). \tag{29}$$

Proof - Write Eq. (29) in the form:

$$U_{\frac{p}{q}+1}(x) + U_{\frac{p}{q}-1}(x) = 2xU_{\frac{p}{q}}(x),$$

then use definition (26) and the trigonometric identity:

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right).$$

Theorem 5.4. *The pseudo-Chebyshev functions $y(x) = U_{\frac{p}{q}}(x)$ satisfy the differential equation*

$$(1-x^2)y'' - 3xy' + \left[\left(\frac{p}{q}\right)^2 - 1\right]y = 0. \tag{30}$$

The proof is obtained in a similar way to that of the first kind functions.

5.4. Basic Relations of the Pseudo-Chebyshev Functions of Third and Fourth Kind

According to Eq. (24), we put by definition:

$$V_{\frac{p}{q}}(x)=\frac{\cos\left[\left(\frac{p}{q}+\frac{1}{2}\right)\arccos x\right]}{\cos\left[\frac{(\arccos x)}{2}\right]},\qquad W_{\frac{p}{q}}(x)=\frac{\sin\left[\left(\frac{p}{q}+\frac{1}{2}\right)\arccos x\right]}{\sin\left[\frac{(\arccos x)}{2}\right]}. \tag{31}$$

Theorem 5.5. *The third and fourth kind pseudo-Chebyshev functions are related to the 1st and 2nd kind ones by the equations:*

$$V_{\frac{p}{q}}(x)=T_{\frac{p}{q}}(x)-(1-x)U_{\frac{p}{q}}(x),\qquad W_{\frac{p}{q}}(x)=T_{\frac{p}{q}}(x)+(1+x)U_{\frac{p}{q}}(x). \tag{32}$$

Proof - It is sufficient to use the addition formulas for the cosine and sine functions.

Therefore, we can derive the equations:

$$W_{\frac{p}{q}}(x)-V_{\frac{p}{q}}(x)=2U_{\frac{p}{q}}(x),\qquad W_{\frac{p}{q}}(x)+V_{\frac{p}{q}}(x)=2T_{\frac{p}{q}}(x)+2xU_{\frac{p}{q}}(x). \tag{33}$$

5.5. Some General Formulas

By using cosine addition formulas, putting:

$$\frac{m}{n}=\frac{p}{q}+\frac{r}{s}, \tag{34}$$

we find:

$$T_{\frac{m}{n}}(x)=T_{\frac{p}{q}}(x)T_{\frac{r}{s}}(x)-(1-x^2)\,U_{\left(\frac{p}{q}\right)-1}(x)U_{\left(\frac{r}{s}\right)-1}(x), \tag{35}$$

and by using the sine addition formulas:

$$U_{\frac{m}{n}}(x)=U_{\left(\frac{p}{q}\right)-1}(x)T_{\frac{r}{s}}(x)+U_{\left(\frac{r}{s}\right)-1}(x)T_{\frac{p}{q}}(x). \tag{36}$$

5.5.1. Particular results

$$T_1(x)=T_{\frac{1}{3}}(x)T_{\frac{2}{3}}(x)-(1-x^2)\,U_{-\frac{2}{3}}(x)U_{-\frac{1}{3}}(x). \tag{37}$$

$$T_1(x)=\cos\left[3\cdot\frac{1}{3}\arccos(x)\right]=4T^3_{\frac{1}{3}}(x)-3T_{\frac{1}{3}}(x). \tag{38}$$

$$T_2(x)=\cos\left[3\cdot\frac{2}{3}\arccos(x)\right]=4T^3_{\frac{2}{3}}(x)-3T_{\frac{2}{3}}(x). \tag{39}$$

$$\begin{aligned}T_{\frac{2}{3}}(x)&=\cos\left[2\cdot\frac{1}{3}\arccos(x)\right]=1-2\sin^2\left[\frac{1}{3}\arccos(x)\right]\\&=1-2(1-x^2)U_{-\frac{2}{3}}(x).\end{aligned} \tag{40}$$

$$\begin{aligned}U_{-\frac{1}{3}}(x)&=\frac{\sin\left[2\cdot\frac{1}{3}\arccos(x)\right]}{\sqrt{1-x^2}}\\&=\frac{2}{\sqrt{1-x^2}}\sin\left[\frac{1}{3}\arccos(x)\right]\cos\left[\frac{1}{3}\arccos(x)\right]\\&=2U_{-\frac{2}{3}}(x)T_{\frac{1}{3}}(x).\end{aligned} \tag{41}$$

$$U_{-\frac{2}{3}}(x)=\frac{\sin\left[\frac{1}{3}\arccos(x)\right]}{\sqrt{1-x^2}}=\sqrt{\frac{1-T^2_{\frac{1}{3}}(x)}{1-x^2}}. \tag{42}$$

Combining the above equations, we find:

$$\begin{aligned}T_1(x)&=T_{\frac{1}{3}}(x)T_{\frac{2}{3}}(x)-2T_{\frac{1}{3}}(x)\left(1-T^2_{\frac{1}{3}}(x)\right)\\&=T_{\frac{1}{3}}(x)\left(2T^2_{\frac{1}{3}}(x)+T_{\frac{2}{3}}(x)-2\right).\end{aligned} \tag{43}$$

5.6. Links with the Pseudo-Chebyshev Functions

Actually, the definitions of the third and fourth kind Chebyshev polynomials are as follows:

$$\begin{aligned}V_n(x)&=\frac{\cos\left[\left(n+\frac{1}{2}\right)\arccos x\right]}{\cos\left[\frac{(\arccos x)}{2}\right]}=\sqrt{\frac{2}{1+x}}T_{n+\frac{1}{2}}(x)\\&=T^{-1}_{\frac{1}{2}}(x)\,T_{n+\frac{1}{2}}(x),\end{aligned} \tag{44}$$

$$\begin{aligned}W_n(x)&=\frac{\sin\left[\left(n+\frac{1}{2}\right)\arccos x\right]}{\sin\left[\frac{(\arccos x)}{2}\right]}=2\sqrt{\frac{1+x}{2}}U_{n-\frac{1}{2}}(x)\\&=2T_{\frac{1}{2}}(x)U_{n-\frac{1}{2}}(x).\end{aligned} \tag{45}$$

Therefore, we find the equations:

$$V_{\frac{p}{q}}(x)=\sqrt{\frac{2}{1+x}}T_{\frac{p}{q}+\frac{1}{2}}(x)=T^{-1}_{\frac{1}{2}}(x)\,T_{\frac{p}{q}+\frac{1}{2}}(x), \tag{46}$$

$$W_{\frac{p}{q}}(x)=2\sqrt{\frac{1+x}{2}}U_{\frac{p}{q}-\frac{1}{2}}(x)=2T_{\frac{1}{2}}(x)U_{\frac{p}{q}-\frac{1}{2}}(x). \tag{47}$$

6. THE CASE OF HALF-INTEGER DEGREE

In what follows, we consider the case of the half-integer degree, which seems to be the most interesting one, since the resulting pseudo-Chebyshev functions satisfy the orthogonality properties in the interval [−1, 1] with respect to the same weights of the corresponding Chebyshev polynomials [25].

Definition: Let, for any integer k:

$$\begin{aligned} T_{k+\frac{1}{2}}(x) &= \cos\left(\left(k+\frac{1}{2}\right)\arccos(x)\right), \\ \sqrt{1-x^2}\, U_{k-\frac{1}{2}}(x) &= \sin\left(\left(k+\frac{1}{2}\right)\arccos(x)\right), \\ \sqrt{1-x^2}\, V_{k+\frac{1}{2}}(x) &= \cos\left(\left(k+\frac{1}{2}\right)\arccos(x)\right), \\ W_{k+\frac{1}{2}}(x) &= \sin\left(\left(k+\frac{1}{2}\right)\arccos(x)\right). \end{aligned} \tag{48}$$

Note that the above definition holds even for $k + 1/2 < 0$, taking into account the parity properties of the circular functions.

The pseudo-Chebyshev functions $T_{k+1/2}(x)$, $U_{k-1/2}(x)$, $V_{k+1/2}(x)$ and $W_{k+1/2}(x)$ can be represented, in terms of the third and fourth kind Chebyshev polynomials as follows:

$$\begin{aligned} T_{k+\frac{1}{2}}(x) &= \sqrt{\frac{1+x}{2}}\, V_k(x), \\ \sqrt{1-x^2}\, U_{k-\frac{1}{2}}(x) &= \sqrt{\frac{1}{2(1+x)}}\, W_k(x), \\ \sqrt{1-x^2}\, V_{k+\frac{1}{2}}(x) &= \sqrt{\frac{1}{2(1-x)}}\, V_k(x), \\ W_{k+\frac{1}{2}}(x) &= \sqrt{\frac{1-x}{2}}\, W_k(x). \end{aligned} \tag{49}$$

We will show that, in the case of half-integer degree, the pseudo-Chebyshev functions satisfy not only the recurrence relations and differential equations analogues to the classical ones, but even the orthogonality properties.

6.1. Orthogonality of the $T_{k+1/2}(x)$ and $U_{k+1/2}(x)$ Functions

A few graphs of the $T_{k+\frac{1}{2}}$ functions are shown in Figure 23.

Theorem 6.1. *The pseudo-Chebyshev functions $T_{k+1/2}(x)$ satisfy the orthogonality property:*

$$\int_{-1}^{1} T_{h+\frac{1}{2}}(x) T_{k+\frac{1}{2}}(x) \frac{1}{\sqrt{1-x^2}}\, dx = 0, \qquad (h \neq k), \tag{50}$$

where h, k are integer numbers,

$$\int_{-1}^{1} T^2_{k+\frac{1}{2}}(x) \frac{1}{\sqrt{1-x^2}}\, dx = \frac{\pi}{2}. \tag{51}$$

A few graphs of the $U_{k+1/2}$ functions are shown in Figure 24.

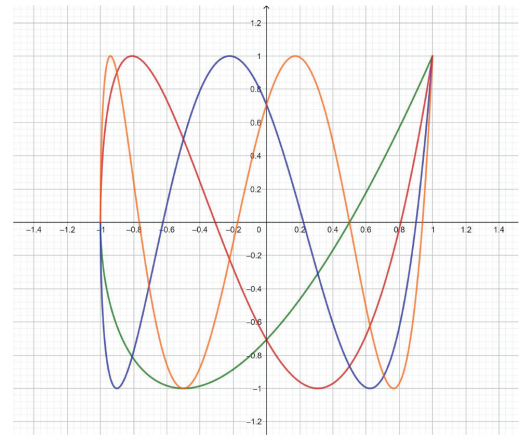


Figure 23 | $T_{k+1/2}(x)$, $k = 1, 2, 3, 4$. 1. green, 2. red, 3. blue, 4. orange.

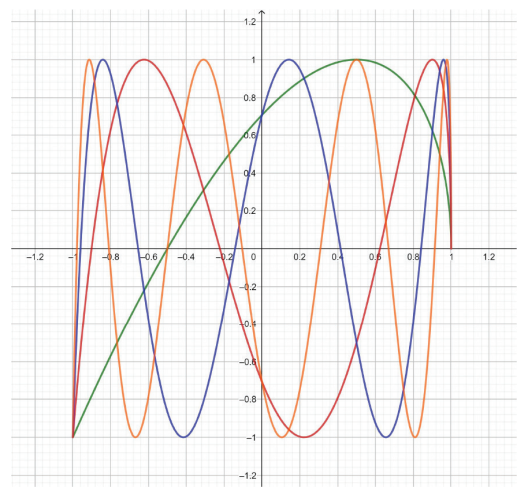


Figure 24 | $U_{k+1/2}(x)$, $k = 1, 2, 3, 4$. 1. green; 2. red; 3. blue; 4. orange.

Theorem 6.2. *The pseudo-Chebyshev functions $U_{k+\frac{1}{2}}(x)$ satisfy the orthogonality property:*

$$\int_{-1}^{1} U_{h+\frac{1}{2}}(x) U_{k+\frac{1}{2}}(x) \sqrt{1-x^2}\, dx = 0, \qquad (m \neq n), \tag{52}$$

where h, k are integer numbers,

$$\int_{-1}^{1} U^2_{k+\frac{1}{2}}(x) \sqrt{1-x^2}\, dx = \frac{\pi}{2}. \tag{53}$$

Proof - We prove only Theorem 6.1, since the proof of Theorem 6.2 is similar.

From the Werner formulas, we have:

$$\begin{aligned} &\int_{-1}^{1} \cos\left[\left(h+\frac{1}{2}\right)\arccos(x)\right]\cos\left[\left(k+\frac{1}{2}\right)\arccos(x)\right]\frac{1}{\sqrt{1-x^2}}\, dx \\ &= 2\int_0^{\pi/2} \cos[(2h+1)t]\cos[(2k+1)t]\, dt = 0, \end{aligned}$$

and

$$\int_{-1}^{1} \cos^2\left[\left(k+\frac{1}{2}\right)\arccos(x)\right]\frac{1}{\sqrt{1-x^2}}\, dx = 2\int_0^{\pi/2} \cos^2((2k+1)t)\, dt = \frac{\pi}{2}.$$

6.2. The Third and Fourth Kind Pseudo-Chebyshev Functions

The results of this section are based on the excellent survey by Aghigh et al. [8]. By using that article, it is possible to derive, in an almost trivial way, the links among the pseudo-Chebyshev functions and the third and fourth kind Chebyshev polynomials.

We recall here only the principal properties, without proofs. Proofs and other properties are reported in Cesarano et al. [27]. In Figures 25 and 26, we show the graphs of the first few third and fourth kind pseudo-Chebyshev functions.

6.3. The Third Kind Pseudo-Chebyshev $V_{k+1/2}$

The third kind pseudo-Chebyshev functions satisfy the recurrence relation:

$$\begin{cases} V_{k+\frac{1}{2}}(x) = 2xV_{k-\frac{1}{2}}(x) - V_{k-\frac{3}{2}}(x), \\ V_{\pm\frac{1}{2}}(x) = \dfrac{1}{\sqrt{2(1-x)}}. \end{cases} \tag{54}$$

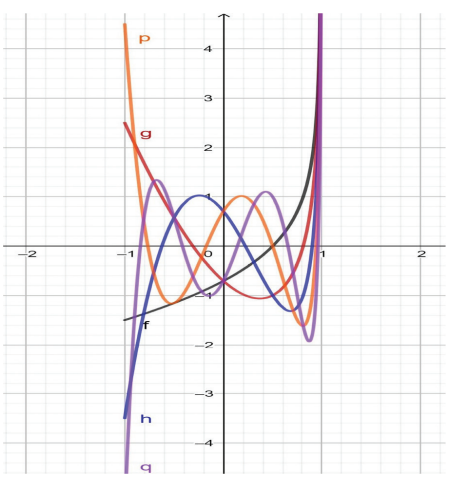


Figure 25 | $V_{k+1/2}(x)$, $k = 1, 2, 3, 4, 5$. 1. grey, 2. red, 3. blue, 4. orange, 5. violet.

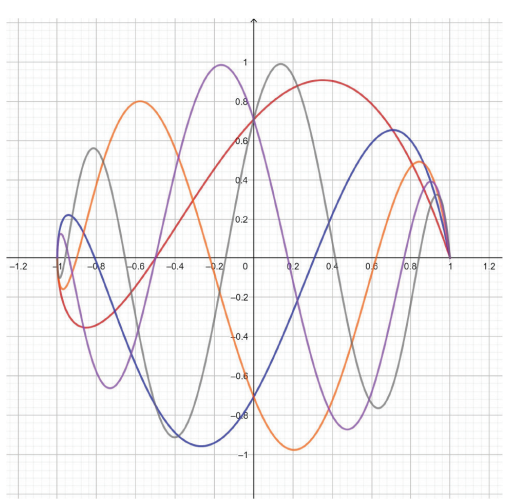


Figure 26 | $W_{k+1/2}(x)$, $k = 1, 2, 3, 4, 5$. 1. red, 2. blue, 3. orange, 4. violet, 5. grey.

Theorem 6.3. *The pseudo-Chebyshev functions $V_{k+1/2}(x)$ verify the differential equation:*

$$(1-x^2)y_k'' - 3xy_k' + \left[\left(k+\frac{1}{2}\right)^2 - 1\right]y_k = 0. \tag{55}$$

The orthogonality property holds:

$$\int_{-1}^{1} V_{h+\frac{1}{2}}(x)V_{k+\frac{1}{2}}(x)\sqrt{1-x^2}\,dx = 0, \quad (h \neq k), \tag{56}$$

$$\int_{-1}^{1} V^2_{k+\frac{1}{2}}(x)\sqrt{1-x^2}\,dx = \frac{\pi}{2}. \tag{57}$$

6.4. The Fourth Kind Pseudo-Chebyshev $W_{k+1/2}$

The fourth kind pseudo-Chebyshev functions satisfy the recurrence relation:

$$\begin{cases} W_{k+\frac{1}{2}}(x) = 2xW_{k-\frac{1}{2}}(x) - W_{k-\frac{3}{2}}(x), \\ W_{\pm\frac{1}{2}}(x) = \pm\sqrt{\dfrac{1-x}{2}}. \end{cases} \tag{58}$$

Theorem 6.4. *The pseudo-Chebyshev functions $W_{k+1/2}(x)$ verify the differential equation:*

$$(1-x^2)y_k'' - xy_k' + \left(k+\frac{1}{2}\right)^2 y_k = 0, \tag{59}$$

The orthogonality property holds:

$$\int_{-1}^{1} W_{h+\frac{1}{2}}(x)W_{k+\frac{1}{2}}(x)\frac{1}{\sqrt{1-x^2}}dx = 0, \quad (h \neq k), \tag{60}$$

$$\int_{-1}^{1} W^2_{k+\frac{1}{2}}(x)\frac{1}{\sqrt{1-x^2}}dx = \frac{\pi}{2}. \tag{61}$$

6.5. Explicit Forms

Theorem 6.5. *It is possible to represent explicitly the pseudo-Chebyshev functions as follows:*

$$\begin{aligned} T_{k+\frac{1}{2}}(x) &= \sqrt{\frac{1+x}{2}}\sum_{h=0}^{k}(-1)^h\left(\frac{2k+1}{2h}\right)\left(\frac{1-x}{2}\right)^h\left(\frac{1+x}{2}\right)^{k-h}, \\ U_{k-\frac{1}{2}}(x) &= \sqrt{\frac{1}{2(1+x)}}\sum_{h=0}^{k}(-1)^h\left(\frac{2k+1}{2h+1}\right)\left(\frac{1-x}{2}\right)^h\left(\frac{1+x}{2}\right)^{k-h}, \\ V_{k+\frac{1}{2}}(x) &= \sqrt{\frac{1}{2(1-x)}}\sum_{h=0}^{k}(-1)^h\left(\frac{2k+1}{2h}\right)\left(\frac{1-x}{2}\right)^h\left(\frac{1+x}{2}\right)^{k-h}, \\ W_{k+\frac{1}{2}}(x) &= \sqrt{\frac{1-x}{2}}\sum_{h=0}^{k}(-1)^h\left(\frac{2k+1}{2h+1}\right)\left(\frac{1-x}{2}\right)^h\left(\frac{1+x}{2}\right)^{k-h}. \end{aligned} \tag{62}$$

6.6. Location of Zeros

By Eq. (48), the zeros of the pseudo-Chebyshev functions $T_{k+1/2}(x)$ and $V_{k+1/2}(x)$ are given by

$$x_{k,h} = \cos\left(\frac{(2h-1)\pi}{2k+1}\right), \quad (h = 1,2,\ldots,k), \tag{63}$$

and the zeros of the pseudo-Chebyshev functions $U_{k+1/2}(x)$ and $W_{k+1/2}(x)$ are given by

$$x_{k,h} = \cos\left(\frac{2h\pi}{2k+1}\right), \quad (h = 1,2,\ldots,k), \tag{64}$$

furthermore, the $W_{k+1/2}(x)$ functions always vanish at the end of the interval [−1, 1].

Remark 6.6. More technical properties as the Hypergeometric representations and the Rodrigues-type formulas are reported in Cesarano et al. [26].

6.7. Links with First and Second Kind Chebyshev Polynomials

Theorem 6.7. *The pseudo-Chebyshev functions are connected with the first and second kind Chebyshev polynomials by means of the equations:*

$$\begin{aligned}
T_{k+\frac{1}{2}}(x) &= T_{2k+1}\left(\sqrt{\frac{1+x}{2}}\right) = T_{2k+1}\left(T_{1/2}(x)\right),\\
U_{k-\frac{1}{2}}(x) &= \frac{1}{1+x}\sqrt{\frac{1}{2(1-x)}}\,U_{2k}\left(\sqrt{\frac{1+x}{2}}\right),\\
V_{k+\frac{1}{2}}(x) &= \frac{1}{1-x^2}T_{2k+1}\left(\sqrt{\frac{1+x}{2}}\right) = \frac{1}{1-x^2}T_{k+\frac{1}{2}}(x),\\
W_{k+\frac{1}{2}}(x) &= \sqrt{\frac{1-x}{2}}\,U_{2k}\left(\sqrt{\frac{1+x}{2}}\right) = (1-x^2)U_{k-\frac{1}{2}}(x).
\end{aligned} \tag{65}$$

Proof - The results follow from the equations:

$$\begin{aligned}
V_k(x) &= \sqrt{\frac{2}{1+x}}\,T_{2k+1}\left(\sqrt{\frac{1+x}{2}}\right),\\
W_k(x) &= U_{2k}\left(\sqrt{\frac{1+x}{2}}\right),
\end{aligned} \tag{66}$$

(see [8]), by using definitions (49).

Remark 6.8. Note that the first equation in (65), extends the known nesting property verified by the first kind Chebyshev polynomials:

$$T_m\left(T_n(x)\right) = T_{mn}(x). \tag{67}$$

This property, already considered in Brandi and Ricci [27] for the first kind pseudo-Chebyshev functions, actually holds in general, as a consequence of the definition $T_k(x) = \cos(k \arccos(x))$. Note that this composition identity even holds for the first kind Chebyshev polynomials in several variables, as it was proven in Ricci [28].

7. CONCLUSION

The growth of living organisms is often described by the Bernoulli's logarithmic spiral, which is one of the first examples of fractals. The study of natural forms (see e.g. Bini et al. [29]) is commonly associated with mathematical entities like extensions of Lamé's curves, Grandi's roses, Bernoulli's lemniscate, etc.

In this article, it has been shown that innumerable plane forms can be described by means of polar equations that extend some of the above mentioned geometrical entities. Moreover, the consideration of Grandi's roses in the case of fractional indexes gives rise, in a natural way, to mathematical functions that generalize to the case of fractional degree the classical Chebyshev polynomials. The resulting functions, called of pseudo-Chebyshev type, verify many properties of the corresponding polynomials and, in the case of half-integer degree, also the orthogonality properties.

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