

Algorithm for Generating Compound Poisson Process Which has Nonhomogeneous Poisson process and Exponential Distribution Components

1st Syarif Abdullah

*Department of Mechanical
Engineering, University of Sultan
Ageng Tirtayasa*
 Jl. Jenderal Sudirman Km 03 Cilegon,
 Banten 42435, Indonesia
 abdullahsyarifayis@gmail.com

2nd Sidik Susilo

*Department of Mechanical
Engineering, University of Sultan
Ageng Tirtayasa*
 Jl. Jenderal Sudirman Km 03 Cilegon,
 Banten 42435, Indonesia
 sidik@untirta.ac.id

3rd I Wayan Mangku

*Department of Mathematics, Bogor
Agricultural University*
 Jl. Meranti, Kampus IPB Dramaga,
 Bogor 16680, Indonesia
 wayan.mangku@gmail.com

4th Fajri Ikhsan

*Department of Metallurgy Engineering,
University of Sultan Ageng Tirtayasa*
 Jl. Jenderal Sudirman Km 03 Cilegon,
 Banten 42435, Indonesia
 fajri.ikhsan@untirta.ac.id

5th Shofiatul Ula

*Department of Mechanical
Engineering, University of Sultan
Ageng Tirtayasa*
 Jl. Jenderal Sudirman Km 03 Cilegon,
 Banten 42435, Indonesia
 shofiatulula@yahoo.com

6th Yazid Rukmayadi

*Department of Mechanical
Engineering, University of Sultan
Ageng Tirtayasa*
 Jl. Jenderal Sudirman Km 03 Cilegon,
 Banten 42435, Indonesia
 yazid.rukmayadi86@gmail.com

Abstract—Compound Poisson process (CPP) is one of the models of stochastic processes in which this model can model a real phenomenon that has an element of uncertainty in the process. CPP has two main components, those is a Poisson process on the component of the poses of an event that occurs and a sequence component of the magnitude as a result of the process of events that occur. This research aims to develop an algorithm to generate random numbers from CPP with a component in the Poisson process in the form of a nonhomogeneous Poisson process (NHPP) and a component in the magnitude of the effect in the form of an exponential distribution (CPP-NHPP-ED). The method used is using the acceptance and rejection method in the form of Thinning process techniques. The results of the study obtained several algorithms, namely the algorithm for CPP-HPP-ED, CPP-NHPP-ED type 1, and CPP-NHPP-ED type 2. These algorithms can be used for computer simulation analysis that can be applied to various fields of science and engineering.

Keywords: *algorithm, Compound Poisson Process (CPP), exponential distribution, Nonhomogeneous Poisson Process (NHPP), Thinning process*

I. INTRODUCTION

In real phenomena, there are a lot of events that contain random events. Not necessarily events that have occurred in the past will recur in the present and future. But it could be that the event will recur, such as cyclic or periodic events. So that from these real phenomena a model is created that can resemble the actual model. The appropriate model is expected to be able to predict future events in terms of both profits and losses. In probability statistics, if a real phenomenon contains an element of uncertainty and contains rules of opportunity, then the event is classified as a stochastic process event.

In terms of time, stochastic processes can be divided into discrete stochastic processes and continuous stochastic processes. One continuous stochastic process that is often used to count the number of events that occur, both success or failure at a certain time or a certain time interval, the Poisson process. According to its intensity function, this process is divided into a homogeneous Poisson process (HPP) and a nonhomogeneous Poisson process (NHPP). HPP has an intensity function that is assumed to have a constant time-intensity function (not time-dependent), whereas NHPP has an intensity function that is not constant (time-dependent). One example of HPP and NHPP in engineering can be seen in [1], [2], and [3].

Stochastic process modeling that not only counts the number of events that occur at certain time intervals but also calculates the effects or effects of events that occur is called a compound process. A compound process consists of at least 2 components, namely the number of events (frequency) and the magnitude component (severity). In this case, the frequency distribution component in question is the distribution of the number of events in a given period and the distribution of severity is a large distribution of consequences caused when the event occurs, can be in the form of money or other amounts. If it is assumed that the frequency component is a Poisson process, then the compound process is called the compound Poisson process (CPP). Whereas the severity component can be assumed to be in the form of the exponential distribution, gamma, Pareto, and others which correspond to the magnitude of the consequences of the event.

The CPP model with the assumption on the frequency component has an HPP component, where the HPP is

assumed to have a constant intensity function, has many applications in various fields, for example in the fields of biology, seismography, demographics, insurance and finance (see reference in [4]), and engineering ([5] and [6]). However, if in an event that happens to have an event that increase or decrease at a certain period, which has an intensity function that is no longer constant (time-dependent), then the model is not appropriate. So it is necessary to develop a new model where the frequency component in CPP is assumed to have the form of NHPP. Furthermore, if in an event that has an event that tends to be repeated so that it has cyclic or periodic properties, then the model must be changed to the form of NHPP with an intensity function in the form of periodic functions ([7], [8] and [9]). Because this intensity function has a repetitive function in the next period, this model is good for describing phenomena that occur periodically. Furthermore, if the model has properties that not only have periodic properties but also have periodic properties that have linear trends, then the model must be changed to the form of NHPP with intensity functions in the form of periodic functions with linear trends ([10] and [4]).

In formulating a stochastic model to describe real phenomena, people used to agree to choose between a model that is a realistic replica of an actual situation and a model for which mathematical analysis is easy to do. However, the relatively new progress of rapid computing opens up a new approach, namely by trying to make phenomena as precisely as possible and then relying on simulation studies to analyze them. In particular, the computer can be used to produce random numbers (pseudo-random) and then these random numbers can be used to produce random variable values from arbitrary distributions. So from the right random variable can produce the behavior of stochastic models at the right time.

The algorithm for conducting simulations on CPP models with frequency components in the form of NHPP and assumptions on the severity component in the form of exponential distribution has not been made. So in this study aims to create algorithms to generate a random number in the compound Poisson process (CPP) model which has a non-homogeneous Poisson process (NHPP) and exponential distribution components. The results of this algorithm can be used in the analysis of models in computational simulations, for example, for the analysis of reliability, total loss (aggregate loss) and risk size (VaR) and expected shortfall (ES) in risk theory. So the company can provide protection or protection against the possibility of future events that can cause financial losses.

II. COMPONENTS OF COMPOUND POISSON PROCESS (CPP)

This section discusses the theory of compound Poisson processes (CPP) that have homogeneous Poisson processes (HPP) and exponential distribution components (CPP-HPP-ED), and compound Poisson processes (CPP) that have nonhomogeneous Poisson processes (NHPP) and exponential distribution components (CPP-NHPP-ED).

A. Compound Poisson process having homogeneous Poisson process (HPP) and exponential distribution components (CPP-HPP-ED).

The counting process $\{N(t), t \geq 0\}$ represents the number of events that occurred at time T . This process is said to be a homogeneous Poisson process (HPP) if it has a Poisson distribution and has an intensity function $\lambda > 0$. Because it has a constant intensity function, it is said that this process is not time-dependent. Formal definitions and theorems of HPP can be seen in Ross [11]. The expectation and variance of HPP respectively as follows.

$$E[N(t)] = \lambda t \text{ and } \text{var}[N(t)] = \lambda t. \tag{1}$$

Poisson compound process has two basic components, namely the component number of events (frequency) which is assumed to have the Poisson process and the component of the magnitude of the result (severity). The definition of a Poisson compound process is as follows.

Definition 2.1. A stochastic process $\{Y(t), t \geq 0\}$ is said to be a CPP if it can be represented as

$$Y(t) = \sum_{i=1}^{N(t)} X_i, t \geq 0, \tag{2}$$

where $\{N(t), t \geq 0\}$ is a Poisson process, and $\{X_i, i \geq 1\}$ is a family of independent and identically distributed (i.i.d) random variables that is also independent of $\{N(t), t \geq 0\}$ [11].

The compound Poisson process with its frequency component in the form of the Poisson process with a constant intensity function has an expectation value (denoted by $\psi_1(t)$) and variants (denoted by $V_1(t)$) respectively as follows Ross [11]:

$$\psi_1(t) = E[Y(t)] = E[N(t)]E[X_1] \tag{3}$$

$$V_1(t) = \text{var}[Y(t)] = E[N(t)]E[X_1^2]. \tag{4}$$

Assumed that X_i has an exponent distribution with the parameter θ . Then the expectation, variance and second-order of X_i are as follows.

$$E[X_1] = \frac{1}{\theta}, \text{var}[X_1] = \frac{1}{\theta^2} \text{ and } E[X_1^2] = \frac{1}{\theta^2}. \tag{5}$$

By substituting equations (1), and (5) to equations (3) and (4), we get, respectively,

$$\psi_1(t) = \frac{\lambda t}{\theta}, \text{ and } \tag{6}$$

$$V_1(t) = \frac{\lambda t}{\theta^2} \tag{7}$$

B. Compound Poisson process having nonhomogeneous Poisson process (NHPP) and exponential distribution components (CPP-NHPP-ED).

The counting process $\{N^*(t), t \geq 0\}$ is said to be a nonhomogeneous Poisson process (NHPP) if it has a Poisson distribution and has a non-constant intensity function $\lambda(t) > 0$. Because it has a non-constant intensity function, it is said to be a time-dependent process. Formal definitions and theorems of NHPP can be seen in Ross [11]. The expectation and variance of NHPP respectively as follows.

$$E[N^*(t)] = \Lambda(t) = \text{var}[N^*(t)] \tag{8}$$

with $\Lambda(t) = \int_0^t \lambda(s) ds$.

Compound Poisson process with its frequency component in the form of Poisson process with the function of the non-constant intensity (denoted by $Y^*(t)$) has an expected value (denoted by $\psi_2(t)$) and variants (denoted by $V_2(t)$) respectively as follows ([12] and [13]):

$$\psi_2(t) = E[Y^*(t)] = E[N^*(t)]E[X_1] \tag{9}$$

$$V_2(t) = \text{var}[Y^*(t)] = E[N^*(t)]E[X_1^2] \tag{10}$$

By substituting equations (8) and (5) to equations (9) and (10), we get, respectively,

$$\psi_2(t) = \frac{\Lambda(t)}{\theta}, \text{ and} \tag{11}$$

$$V_2(t) = \frac{\Lambda(t)}{\theta^2}. \tag{12}$$

It should be noted because this process depends on time, the mean and its variance are functions of time as well.

The algorithm for generating random numbers contained in [14] and [15] is only limited to HPP and NHPP. The algorithm for generating random numbers using the thinning process method for CPP-HPP and CPP-NHPP has been discussed in Abdullah [16], but it has not been discussed if X_i is an exponential distribution. Furthermore, in this paper, the algorithm and its generalizations will be modified and developed for generating CPP-HPP-ED and CPP-NHPP-ED.

III. RESULTS AND DISCUSSION

A. CPP-HPP-ED Algorithm

The algorithm generates a compound Poisson process with HPP components and exponential distribution (CPP-HPP-ED) has 3 main steps as follows:

- Generating an HPP process (denoted Z_i).
 - Generating a random number X_i that has an exponential distribution (ED).
- Calculate CPP (denoted $Y(t)$).

Per the definition of HPP and Definition 1 (equation (2)), suppose the process $\{N(t), t \geq 0\}$ is defined in space Ω , so that for each $\omega \in \Omega$ is defined in the probability space (Ω, F, P) . This process has an intensity function λ and is observed. The function $N(\omega)$ is a single realization of the number of events that occur at intervals $(0, t]$ with $N(0) = 0$. Suppose that for each data point observed in correspondence with a family of random variables i.i.d with an exponential distribution with the parameter θ ($X \sim \exp(\theta)$). Each Z_i is observed, then Z_i produces a single realization to start generating X_i and is given a value with an exponential distribution. Following is the modification of the algorithm to generate CPP-HPP-ED which is illustrated in the flowchart of Figure 1.

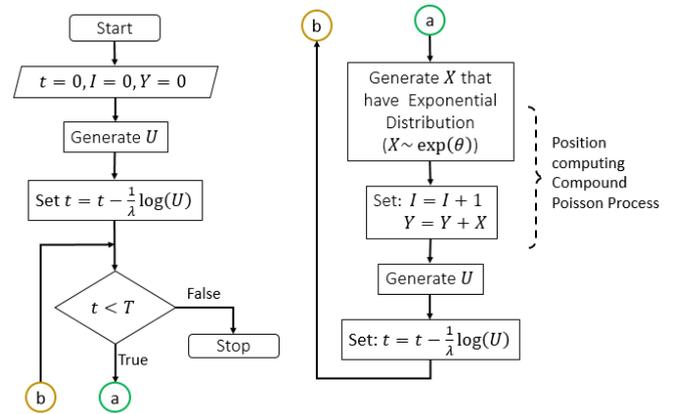


Fig 1. Flowchart of CPP-HPP-ED algorithm

The following are the algorithm modification steps for CPP-HPP-ED.

Step	Instructions	Step	Instructions
1	$t = 0, I = 0, Y = 0$.	5	Generate X ($X \sim \exp(\theta)$).
2	Generate U .	6	Set $I = I + 1$, $Y = Y + X$.
	Set	7	Go to step 2.
3	$t = t - \frac{1}{\lambda} \log(U)$.		
4	If $t > T$, stop.		

The meanings of the symbols in the flowchart and the algorithm above are respectively: t represents time, I represents the number of events that occur at time t , Y represents CPP-HPP-ED, U represents random numbers with uniform distribution, λ represents constant intensity

function, T represents the first time unit and X represents an exponential random number with parameter θ .

B. CPP-NHPP-ED Algorithm type 1

The algorithm generates a compound Poisson process with NHPP components and exponential distribution (CPP-NHPP-ED) has 3 main steps as follows:

- Generating an NHPP process (denoted Z_i).
- Generating a random number X_i that has an exponential distribution (ED).
- Calculate CPP (denoted $Y(t)$).

Per the definition of NHPP and Definition 1 (equation (2)), suppose the process $\{N(t), t \geq 0\}$ is defined in space Ω , so that for each $\omega \in \Omega$ is defined in the probability space (Ω, F, P) . This process has an intensity function λ and is observed. The function $N(\omega)$ is a single realization of the number of events that occur at intervals $(0, t]$ with $N(0) = 0$. Suppose that for each data point observed in correspondence with a family of random variables i.i.d with an exponential distribution with the parameter θ ($X \sim \exp(\theta)$). Remember that to generate NHPP, the value of λ must be chosen so that $\lambda(t) \leq \lambda$ for all $t \leq T$ and with the probability value of $\lambda(t)/\lambda$. Each Z_i is observed, then Z_i produces a single realization to start generating X_i and is given a value with an exponential distribution. The following proposition is the basis of the method for generating CPP-NHPP-ED with a thinning process.

Proposition 1. Consider form of CCP-NHPP in equation (2). Suppose $Y(T)$ is representing CCP, $X_1^*, X_2^*, \dots, X_n^*$ are representing a family of i.i.d random variables that is independent of $\{N(t), t \geq 0\}$, and $T_1^*, T_2^*, \dots, T_n^*$ are random variables representing event times from the NHPP with intensity function, $\lambda^*(t) \geq 0$, and lying in the fixed interval $(0, t_0]$. Let $\lambda(t)$ be an intensity functions such that $0 \leq \lambda(t) \leq \lambda^*(t)$ for all $t \in [0, t_0]$. If the i th event time T_i^* is independently deleted with probability $1 - \lambda(t)/\lambda^*(t)$ for $i = 1, 2, \dots, n$, then the remaining event time of NHPP with the intensity function $\lambda(t)$ in the interval $(0, t_0]$, bring in an observed realization of $X_{j \neq i}^*$ for $j = 1, 2, \dots, n$ and $Y(T)$ can also be calculated at that interval [16].

Following is the modification of the algorithm to generate CPP-NHPP-ED which is illustrated in the flowchart of Figure 2.

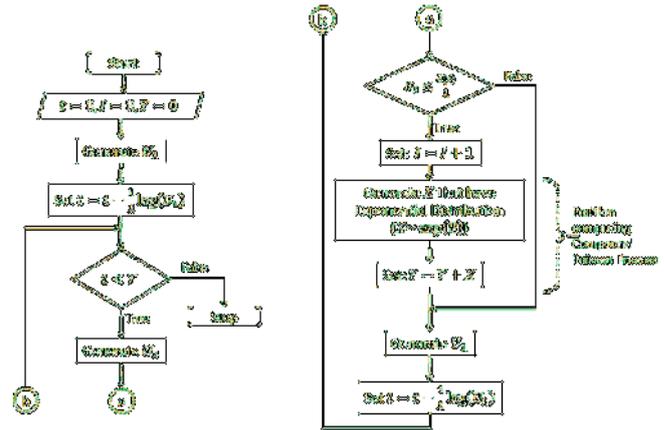


Fig 2. Flowchart of CPP-NHPP-ED algorithm type 1

The steps of the algorithm to generate the CPP-NHPP-ED type 1 as follows

Ste P	Instructions	Ste P	Instructions
1	$t = 0, I = 0, Y^* = 0$	6	If $U_2 \leq \lambda(t)/\lambda$
2	Generate U_1 . Set	7	Set $I = I + 1$.
3	$t = t - \frac{1}{\lambda} \log(U_1)$.	8	Generate X ($X \sim \exp(\theta)$).
4	If $t > T$, stop.	9	Set $Y^* = Y^* + X$
5	Generate U_2 .	10	Go to step 2.

The meaning of the symbols in the flowchart and the algorithm are: t represented as time, I represented a number of events that have occurred by time t , Y^* represented as CPP-NHPP, U_1 and U_2 represented as random number, $\lambda(t)$ represented as intensity function of NHPP that is non-constant function with $\lambda(t) \leq \lambda$, T represented as first T time units and X represented as random variables that exponential distributed which has i.i.d properties and θ represented as parameter of exponential distribution.

C. CPP-NHPP-ED algorithm type 2

The algorithm in type 1 is modified and developed by dividing the interval into subintervals. After being divided, procedures are used such as type 1. The algorithm generates a compound Poisson process with NHPP components and exponential distribution (CPP-NHPP-ED) type 2 has 4 main steps as follows:

- Break the interval of time t into subinterval.
- Generating an NHPP process (denoted Z_i).
- Generating a random number X_i that has an exponential distribution (ED).
- Calculate CPP (denoted $Y(t)$).

Per the definition of NHPP and Definition 1 (equation (2)), suppose the process $\{N(t), t \geq 0\}$ is defined in space Ω , so that for each $\omega \in \Omega$ is defined in the probability space (Ω, F, P) . This process has an intensity function λ and is observed. The function $N(\omega)$ is a single realization of the number of events that occur at intervals $(0, t]$ with $N(0) = 0$. Suppose that for each data point observed in correspondence with a family of random variables i.i.d with an exponential distribution with the parameter θ ($X \sim \exp(\theta)$). Remember that to generate NHPP, the value of λ_i must be chosen so that $\lambda(s) \leq \lambda_i$ for all $s \in (t_{i-1}, t_i)$ and with the probability value of $\lambda(t) / \lambda_i$. Because exponential has no memory, use $\lambda_i [Z - (t_i - t)] / \lambda_{i+1}$ for the next exponential form with the rate λ_{i+1} . Each Z_i is observed, then Z_i produces a single realization to start generating X_i and is given a value with an exponential distribution. The steps to generate the CPP-NHPP-ED type 2 describe in the flowchart in figure 3 as follows.

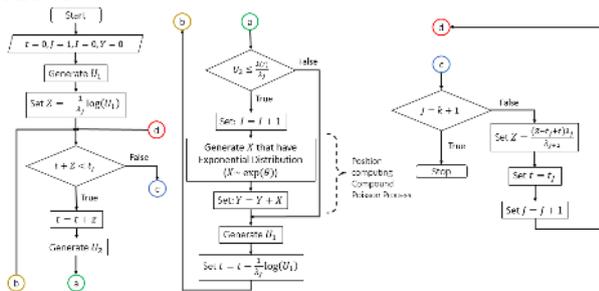


Fig 3. Flowchart of CPP-NHPP-ED algorithm type 2

The steps of the algorithm to generate the CPP-NHPP-ED type 2 as follows

Step	Instructions	Step	Instructions
1	$t = 0, J = 1, I = 0, Y^* =$	9	Generate X ($X \sim \exp(\theta)$).
2	Generate U_1 .	10	Set $Y^* = Y^* + X$.
3	Set $Z = -\frac{1}{\lambda_J} \log(U_1)$.	11	Go to Step 2.
4	If $t + Z > t_J$, go to Step 12.	12	If $J = k + 1$, stop.
5	Set $t = t + Z$.	13	Set $Z = \frac{(Z - t_J + t)}{\lambda_{J+1}}$.
6	Generate U_2 .	14	Set $t = t_J$.
7	If $U_2 \leq \lambda(t) / \lambda_J$.	15	Set $J = J + 1$.
8	Set $I = I + 1$.	16	Go to Step 4.

The meaning of the symbols in the flowchart and the algorithm are: t represented as present time, J represented as present interval, I represented a number of events that have occurred by time t , Y^* represented as CPP-NHPP, U_1 and U_2 represented as uniform random number, $\lambda(t)$ represented as intensity function of NHPP that is non-constant function with $\lambda(t) \leq \lambda$, X represented as random variables that has exponential distributed which has i.i.d properties, and θ represented as parameter of exponential distribution.

In the last algorithm, it can be seen that time t is broken down into several subintervals. This results in the intensity function $\lambda(t)$ being corrected and also broken down, resulting in fairly complex processes and consuming a lot of memory, but produces more accurate results and the model approaches the desired real phenomenon.

From the results of the random number generation algorithms discussed above, then the expected value and variance can be calculated. Furthermore, these results can be used for model analysis in computational simulations, for example, for reliability analysis, total loss (aggregate loss) and risk measures, for example, the value of risk (VaR) and expected shortfall (ES), in risk theory in risk theory, in various fields of science and engineering.

IV. CONCLUSIONS

The results of the study obtained several algorithms, namely CPP-HPP-ED algorithm, CPP-NHPP-ED algorithm type 1, and CPP-NHPP-ED algorithm type 2. The first algorithm is generated with components in the Poisson process in the form of constant functions and distributed components exponential. The second algorithm is generated with components in the Poisson process in the form of a non-constant function and an exponential distribution component. The third algorithm is produced by broken down the time interval in the second algorithm and producing more accurate results and the model approaches the desired real phenomenon. These algorithms can be used for computer simulation analysis that can be applied to various fields of science and engineering.

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Brief biography of first author: Syarif Abdullah. Lecturer from Department of Mechanical Engineering, University of Sultan Ageng Tirtayasa, Indonesia. Specifications of the research is in mathematics, stochastic process, Poisson process, computer analysis, mechanical engineering and computer analysis for engineering.

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