Journal of Statistical Theory and Applications
Vol. 19(1), March 2020, pp. 109–117
DOI: https://doi.org/10.2991/jsta.d.200303.001; ISSN 1538-7887
https://www.atlantis-press.com/journals/jsta

Research Article

Sample Design and Estimation of Parameters of Half Logistic Distribution Using Generalized Ranked-Set Sampling

A. Adatia[1,*], A.K.MD. Ehsanes Saleh[2]

[1] Department of Computer Science and Mathematics, University of Lethbridge, Lethbridge, Alberta, Canada
[2] Professor Emeritus and Distinguished Research Professor, School of Mathematics and Statistics, Carleton University, Ottawa, Ontario, Canada

ARTICLE INFO

Article History

Received 01 Sep 2017
Accepted 27 May 2019

Keywords

Half logistic distribution
Order statistics
Linear estimation
Generalized ranked-set sampling
Sampling efficiency

2000 Mathematics Subject Classification: 62D05; 62G08

ABSTRACT

The ranked-set sampling technique has been generalized so that a more efficient estimator may be obtained. This technique allows more than one unit from each set to be quantified. Consequently, the number of units to be sampled may be reduced significantly and as a result, the corresponding cost would also be reduced. The generalized ranked-set sampling technique is applied in the estimation of parameters of the half logistic distribution. New estimators are proposed which include linear minimum variance unbiased estimators and ranked-set sample estimators. The coefficients, variances and relative efficiencies are tabulated. The estimators are compared to the best linear unbiased estimator of the parameters. Sample design strategy is also considered.

1. INTRODUCTION

McIntyre [1] advocated the use of ranked-set sampling when experimenters encountered situations where the actual measurements of the sample observations were difficult to make due to constraints like cost, time and other factors. However, ranking of the potential sample data is relatively easy. Since then, this technique has been studied and applied to several areas of applied research. Some of the applications are in forestry [2], medicine [3], environmental monitoring [4,5], population genetics [6], clinical trials [6], agriculture and entomology [7]. A comprehensive review of the subject has been provided by Wolfe [8].

In this paper, the generalized ranked-set sampling (RSS) procedure proposed in Adatia [9] is extended to allow more than one unit from each set to be quantified. Unbiased estimators of the (location, scale) parameters of any location-scale family of distributions can be obtained. The performance of these estimators can be investigated by comparing their efficiency against that of best linear unbiased estimators (BLUEs). Also, the number of units to be sampled is less than that required for balanced RSS.

The generalized ranked-set sampling procedure proposed in this paper also enables us to provide a sampling design strategy to explore the relationship between the efficiency of the estimators and the number of units sampled. Suppose we can afford to take up to N = 15 ranked-set observations. We can select one ranked-set sample of size N = 15, sample design 15×1 (with 225 units selected) or in replicated smaller samples (e.g. five ranked-set samples of size N = 3, sample design 3×5 (with 45 units selected) or three ranked-set samples of size N = 5, sample design 5×3 (with 75 units selected)). For each sample design and number of sampling units selected, the efficiency is computed and can be compared. This is relevant in applications where RSS might be beneficial, but the cost involved in sampling and ranking cannot be completely ignored. Several authors have addressed this concern by introducing cost models [10–12]. Also, since ranking large sets is difficult and introduces room for error, an appropriate sample design can be selected with the advantage of known efficiency.

2. APPLICATION

The generalized ranked-set sampling technique is applied in the estimation of parameters of the half logistic distribution. The development of the generalized minimum variance estimator with 2 or 3 observations kept in each set to be quantified, is a continuation of the method presented by Adatia [9]. This paper provides sufficient details for practitioners to construct this type of estimators for their applications. New estimators proposed are generalized ranked-set minimum variance unbiased estimators (GR-MVUEs) and generalized ranked-set

*Corresponding author. Email: aminmohamed.adatia@uleth.ca

sample estimators (GR-RSSs). Coefficients, variances and relative efficiencies are derived. The estimators are compared to the BLUEs of the parameters. In the case of half logistic distribution, GR-MVUE have advantage compared to maximum likelihood estimators (MLEs) as it can be directly calculated from a sample even if both (location, scale) parameters are unknown. Giles [13] gives a good account on the complexity of MLE's for the half logistic distribution.

In generalized ranked-set sampling, first a set of N elements is randomly selected from a given population. The sample is ordered without making actual measurements. The unit identified with the $N_1^{(1)}$ rank is accurately measured. Next, a second set of N elements is randomly selected from the population. Again the units are ordered and the unit with the $N_1^{(2)}$ rank is accurately measured. The process is continued until N set of N elements is selected. The units are again ordered and the unit with $N_1^{(N)}$ rank is accurately measured. The ordered sample of the N sets can be represented as follows:

$$\begin{array}{lcccc} \text{Set 1} & X_{(11)} & X_{(12)} & \cdots & X_{(1N)} \\ \text{Set 2} & X_{(21)} & X_{(22)} & \cdots & X_{(2N)} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ \text{Set N} & X_{(N1)} & X_{(N2)} & \cdots & X_{(NN)} \end{array}$$

In generalized ranked-set sampling where two units are selected from each set, first a set of N elements is randomly selected from a given population. The sample is ordered without making actual measurements. The units identified with the $N_1^{(1)}$ and $N_2^{(1)}$ ranks are accurately measured. Next, a second set of N elements is randomly selected from the population. Again the units are ordered and the units with the $N_1^{(2)}$ and $N_2^{(2)}$ ranks are accurately measured. The process is continued until M = N/2 set of N elements is selected. The units are again ordered and the units with $N_1^{(M)}$ and $N_2^{(M)}$ ranks are accurately measured. The ordered sample of the M = N/2 sets can be represented as follows: $\left(X_{\left(N_1^{(1)}\right)}, X_{\left(N_2^{(1)}\right)}, \dots, X_{\left(N_1^{(M)}\right)}, X_{\left(N_2^{(M)}\right)}\right)$ where $1 \le N_1^{(i)} < N_2^{(i)} \le N$ and $1 \le i \le M$.

Similarly, the generalized ranked-set sample of size N where three units are selected from each set consists of units which are accurately measured, i.e. $\left(X_{\left(N_1^{(1)}\right)}, X_{\left(N_2^{(1)}\right)}, X_{\left(N_3^{(1)}\right)}, \dots, X_{\left(N_1^{(M)}\right)}, X_{\left(N_2^{(M)}\right)}, X_{\left(N_3^{(M)}\right)}\right)$ where $1 \le N_1^{(i)} < N_2^{(i)} < N_3^{(i)} \le N$, $1 \le i \le M$ and $M = N/3$.

We note that from the generalized ranked-set sampling procedure, the balanced ranked-set sample is obtained by selecting the ordered unit $N_1^{(i)} = i^{(i)}$ from set i $(1 \le i \le N)$.

2.1. Estimators Based on Generalized Ranked-Set Sampling

Let the random variable X have a half logistic distribution with probability density function ($f(x)$)

$$f(x) = \frac{2\exp[-(x-\mu)/\sigma]}{\sigma[1+\exp[-(x-\mu)/\sigma]]^2}, \quad x \ge \mu,\ \sigma > 0$$

where μ and σ are the location and the scale parameters respectively.

Let

$$Z_{\left(N_j^{(i)}\right)} = \left(X_{\left(N_j^{(i)}\right)} - \mu\right) \Big/ \sigma$$

$$\alpha_{N_j^{(i)}} = E\left(Z_{\left(N_j^{(i)}\right)}\right)$$

$$\omega_{N_j^{(i)} N_k^{(i)}} = Cov\left(Z_{\left(N_j^{(i)}\right)}, Z_{\left(N_k^{(i)}\right)}\right)$$

$\underline{X}^{(i)} = \left(X_{\left(N_1^{(i)}\right)}, X_{\left(N_2^{(i)}\right)}\right)^T$ be the two order statistics of ranks $N_1^{(i)}$ and $N_2^{(i)}$ in the i$^{\text{th}}$ set.

Then the generalized ranked-set sample is given by $\underline{X}_{S_2} = \left(\underline{X}^{(1)}, \underline{X}^{(2)}, \cdots, \underline{X}^{(M)}\right)$, the expected value of the standardized $\underline{X}_{S_2}$ is given by $\underline{\alpha}_{S_2}^T = \left(\underline{\alpha}_1^T, \cdots, \underline{\alpha}_M^T\right)$ and the variance and covariance matrix is given by $\Omega_{S_2} = Diagonal\left(\underline{\Omega}_1, \cdots, \underline{\Omega}_M\right)$

where

$$S_2 = \left(N_1^{(1)}, N_2^{(1)}, \cdots, N_1^{(M)}, N_2^{(M)}\right)$$

$$\underline{\Omega}_i = \begin{pmatrix} \omega_{N_1^{(i)}N_1^{(i)}} & \omega_{N_1^{(i)}N_2^{(i)}} \\ \omega_{N_1^{(i)}N_2^{(i)}} & \omega_{N_1^{(i)}N_1^{(i)}} \end{pmatrix}, N_1^{(i)} < N_2^{(i)} \; and \, i = 1, \dots, M$$

$$\underline{\alpha}_i^T = \left(\alpha_{N_1^{(i)}}, \alpha_{N_2^{(i)}}\right)^T$$

Least squares estimator of the parameter based on generalized ranked-set sample when two order statistics are selected from each sample:

$$\mathrm{S}_2 = \left(\mathrm{N}_1^{(1)}, \mathrm{N}_2^{(1)}, \dots, \mathrm{N}_1^{(\mathrm{M})}, \mathrm{N}_2^{(\mathrm{M})}\right)$$

$$\hat{\mu}_{S_2} = \sum_{i=1}^{M} a_{N_1^{(i)}}^{(i)} X_{N_1^{(i)}} + a_{N_2^{(i)}}^{(i)} X_{N_2^{(i)}}$$
$$\hat{\sigma}_{S_2} = \sum_{i=1}^{M} b_{N_1^{(i)}}^{(i)} X_{N_1^{(i)}} + b_{N_2^{(i)}}^{(i)} X_{N_2^{(i)}}$$

where

$$a_{N_1^{(i)}}^{(i)} = \frac{T_1}{\left(T_1T_2 - T_3^2\right)} \frac{\left(\omega_{N_2^{(i)}N_2^{(i)}} - \omega_{N_1^{(i)}N_2^{(i)}}\right)}{\left(\omega_{N_1^{(i)}N_1^{(i)}}\omega_{N_2^{(i)}N_2^{(i)}} - \omega_{N_1^{(i)}N_2^{(i)}}{}^2\right)} - \frac{T_3}{\left(T_1T_2 - T_3^2\right)} \frac{\left(\omega_{N_2^{(i)}N_2^{(i)}}\alpha_{N_1^{(i)}} - \omega_{N_1^{(i)}N_2^{(i)}}\alpha_{N_2^{(i)}}\right)}{\left(\omega_{N_1^{(i)}N_1^{(i)}}\omega_{N_2^{(i)}N_2^{(i)}} - \omega_{N_1^{(i)}N_2^{(i)}}{}^2\right)}$$

$$a_{N_2^{(i)}}^{(i)} = \frac{T_1}{\left(T_1T_2 - T_3^2\right)} \frac{\left(\omega_{N_1^{(i)}N_1^{(i)}} - \omega_{N_1^{(i)}N_2^{(i)}}\right)}{\left(\omega_{N_1^{(i)}N_1^{(i)}}\omega_{N_2^{(i)}N_2^{(i)}} - \omega_{N_1^{(i)}N_2^{(i)}}{}^2\right)} - \frac{T_3}{\left(T_1T_2 - T_3^2\right)} \frac{\left(\omega_{N_1^{(i)}N_1^{(i)}}\alpha_{N_2^{(i)}} - \omega_{N_1^{(i)}N_2^{(i)}}\alpha_{N_1^{(i)}}\right)}{\left(\omega_{N_1^{(i)}N_1^{(i)}}\omega_{N_2^{(i)}N_2^{(i)}} - \omega_{N_1^{(i)}N_2^{(i)}}{}^2\right)}$$

$$b_{N_1^{(i)}}^{(i)} = \frac{T_2}{\left(T_1T_2 - T_3^2\right)} \left(\frac{\left(\omega_{N_2^{(i)}N_2^{(i)}}\alpha_{N_1^{(i)}} - \omega_{N_1^{(i)}N_2^{(i)}}\alpha_{N_2^{(i)}}\right)}{\left(\omega_{N_1^{(i)}N_1^{(i)}}\omega_{N_2^{(i)}N_2^{(i)}} - \omega_{N_1^{(i)}N_2^{(i)}}{}^2\right)}\right) - \frac{T_3}{\left(T_1T_2 - T_3^2\right)} \left(\frac{\left(\omega_{N_2^{(i)}N_2^{(i)}} - \omega_{N_1^{(i)}N_2^{(i)}}\right)}{\left(\omega_{N_1^{(i)}N_1^{(i)}}\omega_{N_2^{(i)}N_2^{(i)}} - \omega_{N_1^{(i)}N_2^{(i)}}{}^2\right)}\right)$$

$$b_{N_2^{(i)}}^{(i)} = \frac{T_2}{\left(T_1T_2 - T_3^2\right)} \left(\frac{\left(\omega_{N_1^{(i)}N_1^{(i)}}\alpha_{N_2^{(i)}} - \omega_{N_1^{(i)}N_2^{(i)}}\alpha_{N_1^{(i)}}\right)}{\left(\omega_{N_1^{(i)}N_1^{(i)}}\omega_{N_2^{(i)}N_2^{(i)}} - \omega_{N_1^{(i)}N_2^{(i)}}{}^2\right)}\right) - \frac{T_3}{\left(T_1T_2 - T_3^2\right)} \left(\frac{\left(\omega_{N_1^{(i)}N_1^{(i)}} - \omega_{N_1^{(i)}N_2^{(i)}}\right)}{\left(\omega_{N_1^{(i)}N_1^{(i)}}\omega_{N_2^{(i)}N_2^{(i)}} - \omega_{N_1^{(i)}N_2^{(i)}}{}^2\right)}\right)$$

$$T_{1S_2} = \alpha_{S_2}^T \Omega_{S_2}^{-1} \alpha_{S_2}$$

$$T_{2S_2} = 1^T \Omega_{S_2}^{-1} 1$$

$$T_{3S_2} = 1^T \Omega_{S_2}^{-1} \alpha_{S_2}$$

The variances and covariances of these estimators are given by

$$\mathrm{Var}\left(\hat{\mu}_{S_2}\right) = T_{1S_2}\sigma^2 / \left(T_{1S_2}T_{2S_2} - T_{3S_2}^2\right)$$
$$\mathrm{Var}\left(\hat{\sigma}_{S_2}\right) = T_{2S_2}\sigma^2 / \left(T_{1S_2}T_{2S_2} - T_{3S_2}^2\right)$$
$$\mathrm{Cov}\left(\hat{\mu}_{S_2}, \hat{\sigma}_{S_2}\right) = -T_{3S_2}\sigma^2 / \left(T_{1S_2}T_{2S_2} - T_{3S_2}^2\right)$$

The generalized variance is

$$\text{Gvar}\left(\hat{\mu}_{S_2}, \hat{\sigma}_{S_2}\right) = V\left(\hat{\mu}_{S_2}\right) V\left(\hat{\sigma}_{S_2}\right) - Cov\left(\hat{\mu}_{S_2}, \hat{\sigma}_{S_2}\right)^2$$

and when three order statistics are selected from each set, the generalized ranked-set sample is $\mathrm{S}_3 = \left(\mathrm{N}_1^{(1)}, \mathrm{N}_2^{(1)}, \mathrm{N}_3^{(1)}, \ldots, \mathrm{N}_1^{(\mathrm{M})}, \mathrm{N}_2^{(\mathrm{M})}, \mathrm{N}_3^{(\mathrm{M})}\right)$. The variance and covariance matrix is given by

$$\Omega_{S_3} = Diagonal\left(\underline{\Omega}_1, \cdots, \underline{\Omega}_M\right)$$

where

$$\underline{\Omega}_i = \begin{pmatrix} \omega_{N_1^{(i)}N_1^{(i)}} & \omega_{N_1^{(i)}N_2^{(i)}} & \omega_{N_1^{(i)}N_3^{(i)}} \\ \omega_{N_2^{(i)}N_1^{(i)}} & \omega_{N_2^{(i)}N_2^{(i)}} & \omega_{N_2^{(i)}N_3^{(i)}} \\ \omega_{N_3^{(i)}N_1^{(i)}} & \omega_{N_3^{(i)}N_2^{(i)}} & \omega_{N_3^{(i)}N_3^{(i)}} \end{pmatrix}, N_1^{(i)} < N_2^{(i)} < N_3^{(i)} \ and \ i = 1, \ldots, M$$

$$\hat{\mu}_{S_3} = \sum_{i=1}^{M} a^{(i)}_{N_1^{(i)}} X_{N_1^{(i)}} + a^{(i)}_{N_2^{(i)}} X_{N_2^{(i)}} + a^{(i)}_{N_3^{(i)}} X_{N_3^{(i)}}$$

$$\hat{\sigma}_{S_3} = \sum_{i=1}^{M} b^{(i)}_{N_1^{(i)}} X_{N_1^{(i)}} + b^{(i)}_{N_2^{(i)}} X_{N_2^{(i)}} + b^{(i)}_{N_3^{(i)}} X_{N_3^{(i)}}$$

$$\begin{aligned} a^{(i)}_{N_1^{(i)}} = & \frac{T_2}{\left(T_1T_2 - T_3^2\right)|\Omega_{S_3}|}\left(\omega_{N_2^{(i)}N_3^{(i)}}\left(\omega_{N_1^{(i)}N_2^{(i)}} - \omega_{N_2^{(i)}N_3^{(i)}}\right) + \omega_{N_1^{(i)}N_3^{(i)}}\left(\omega_{N_2^{(i)}N_3^{(i)}} - \omega_{N_2^{(i)}N_2^{(i)}}\right) + \omega_{N_3^{(i)}N_3^{(i)}}\left(\omega_{N_2^{(i)}N_2^{(i)}} - \omega_{N_1^{(i)}N_2^{(i)}}\right)\right) \\ & - \frac{T_3}{\left(T_1T_2 - T_3^2\right)|\Omega_{S_3}|}\left(\alpha_{N_1^{(i)}}\left(\omega_{N_2^{(i)}N_2^{(i)}}\omega_{N_3^{(i)}N_3^{(i)}} - \omega_{N_2^{(i)}N_3^{(i)}}^{\ 2}\right) + \alpha_{N_2^{(i)}}\left(\omega_{N_1^{(i)}N_3^{(i)}}\omega_{N_2^{(i)}N_3^{(i)}} - \omega_{N_1^{(i)}N_2^{(i)}}\omega_{N_3^{(i)}N_3^{(i)}}\right)\right. \\ & \left. + \alpha_{N_3^{(i)}}\left(\omega_{N_1^{(i)}N_2^{(i)}}\omega_{N_2^{(i)}N_3^{(i)}} - \omega_{N_1^{(i)}N_3^{(i)}}\omega_{N_2^{(i)}N_2^{(i)}}\right)\right) \end{aligned}$$

$$\begin{aligned} a^{(i)}_{N_2^{(i)}} = & \frac{T_2}{\left(T_1T_2 - T_3^2\right)|\Omega_{S_3}|}\left(\omega_{N_1^{(i)}N_3^{(i)}}\left(\omega_{N_1^{(i)}N_2^{(i)}} - \omega_{N_1^{(i)}N_3^{(i)}}\right) + \omega_{N_2^{(i)}N_3^{(i)}}\left(\omega_{N_1^{(i)}N_3^{(i)}} - \omega_{N_2^{(i)}N_3^{(i)}}\right) + \omega_{N_3^{(i)}N_3^{(i)}}\left(\omega_{N_1^{(i)}N_1^{(i)}} - \omega_{N_1^{(i)}N_2^{(i)}}\right)\right) \\ & - \frac{T_3}{\left(T_1T_2 - T_3^2\right)|\Omega_{S_3}|}\left(\alpha_{N_1^{(i)}}\left(\omega_{N_1^{(i)}N_3^{(i)}}\omega_{N_2^{(i)}N_3^{(i)}} - \omega_{N_1^{(i)}N_2^{(i)}}\omega_{N_3^{(i)}N_3^{(i)}}\right) + \alpha_{N_2^{(i)}}\left(\omega_{N_1^{(i)}N_1^{(i)}}\omega_{N_3^{(i)}N_3^{(i)}} - \omega_{N_1^{(i)}N_3^{(i)}}^{\ 2}\right)\right. \\ & \left. + \alpha_{N_3^{(i)}}\left(\omega_{N_1^{(i)}N_2^{(i)}}\omega_{N_1^{(i)}N_3^{(i)}} - \omega_{N_1^{(i)}N_1^{(i)}}\omega_{N_2^{(i)}N_3^{(i)}}\right)\right) \end{aligned}$$

$$\begin{aligned} a^{(i)}_{N_3^{(i)}} = & \frac{T_2}{\left(T_1T_2 - T_3^2\right)|\Omega_{S_3}|}\left(\omega_{N_2^{(i)}N_2^{(i)}}\left(\omega_{N_1^{(i)}N_1^{(i)}} - \omega_{N_1^{(i)}N_3^{(i)}}\right) + \omega_{N_2^{(i)}N_2^{(i)}}\left(\omega_{N_1^{(i)}N_3^{(i)}} - \omega_{N_1^{(i)}N_2^{(i)}}\right) + \omega_{N_2^{(i)}N_3^{(i)}}\left(\omega_{N_1^{(i)}N_2^{(i)}} - \omega_{N_1^{(i)}N_1^{(i)}}\right)\right) \\ & - \frac{T_3}{\left(T_1T_2 - T_3^2\right)|\Omega_{S_3}|}\left(\alpha_{N_1^{(i)}}\left(\omega_{N_1^{(i)}N_2^{(i)}}\omega_{N_2^{(i)}N_3^{(i)}} - \omega_{N_1^{(i)}N_3^{(i)}}\omega_{N_2^{(i)}N_2^{(i)}}\right) + \alpha_{N_2^{(i)}}\left(\omega_{N_1^{(i)}N_2^{(i)}}\omega_{N_1^{(i)}N_3^{(i)}} - \omega_{N_1^{(i)}N_1^{(i)}}\omega_{N_2^{(i)}N_3^{(i)}}\right)\right. \\ & \left. + \alpha_{N_3^{(i)}}\left(\omega_{N_1^{(i)}N_1^{(i)}}\omega_{N_2^{(i)}N_2^{(i)}} - \omega^2_{N_1^{(i)}N_2^{(i)}}\right)\right) \end{aligned}$$

$$\begin{aligned} b^{(i)}_{N_1^{(i)}} = & \frac{T_2}{\left(T_1T_2 - T_3^2\right)|\Omega_{S_3}|}\left(\alpha_{N_1^{(i)}}\left(\omega_{N_2^{(i)}N_2^{(i)}}\omega_{N_3^{(i)}N_3^{(i)}} - \omega_{N_2^{(i)}N_3^{(i)}}^{\ 2}\right) + \alpha_{N_2^{(i)}}\left(\omega_{N_1^{(i)}N_3^{(i)}}\omega_{N_2^{(i)}N_3^{(i)}} - \omega_{N_1^{(i)}N_2^{(i)}}\omega_{N_3^{(i)}N_3^{(i)}}\right)\right. \\ & \left. + \alpha_{N_3^{(i)}}\left(\omega_{N_1^{(i)}N_2^{(i)}}\omega_{N_2^{(i)}N_3^{(i)}} - \omega_{N_1^{(i)}N_3^{(i)}}\omega_{N_2^{(i)}N_2^{(i)}}\right)\right) \\ & - \frac{T_3}{\left(T_1T_2 - T_3^2\right)|\Omega_{S_3}|}\left(\omega_{N_2^{(i)}N_3^{(i)}}\left(\omega_{N_1^{(i)}N_2^{(i)}} - \omega_{N_2^{(i)}N_3^{(i)}}\right) + \omega_{N_1^{(i)}N_3^{(i)}}\left(\omega_{N_2^{(i)}N_3^{(i)}} - \omega_{N_2^{(i)}N_2^{(i)}}\right) + \omega_{N_3^{(i)}N_3^{(i)}}\left(\omega_{N_2^{(i)}N_2^{(i)}} - \omega_{N_1^{(i)}N_2^{(i)}}\right)\right) \end{aligned}$$

$$\begin{aligned}b^{(i)}_{N_2^{(i)}} = \frac{T_2}{\left(T_1T_2-T_3^2\right)|\Omega_{S_3}|}&\Bigg(\alpha_{N_1^{(i)}}\left(\omega_{N_1^{(i)}N_3^{(i)}}\omega_{N_2^{(i)}N_3^{(i)}}-\omega_{N_1^{(i)}N_2^{(i)}}\omega_{N_3^{(i)}N_3^{(i)}}\right)+\alpha_{N_2^{(i)}}\left(\omega_{N_1^{(i)}N_1^{(i)}}\omega_{N_3^{(i)}N_3^{(i)}}-\omega_{N_1^{(i)}N_3^{(i)}}^{2}\right)\\&+\alpha_{N_3^{(i)}}\left(\omega_{N_1^{(i)}N_2^{(i)}}\omega_{N_1^{(i)}N_3^{(i)}}-\omega_{N_1^{(i)}N_1^{(i)}}\omega_{N_2^{(i)}N_3^{(i)}}\right)\Bigg)\\&-\frac{T_3}{\left(T_1T_2-T_3^2\right)|\Omega_{S_3}|}\Bigg(\omega_{N_1^{(i)}N_3^{(i)}}\left(\omega_{N_1^{(i)}N_2^{(i)}}-\omega_{N_1^{(i)}N_3^{(i)}}\right)+\omega_{N_2^{(i)}N_3^{(i)}}\left(\omega_{N_1^{(i)}N_3^{(i)}}-\omega_{N_2^{(i)}N_3^{(i)}}\right)+\omega_{N_3^{(i)}N_3^{(i)}}\left(\omega_{N_1^{(i)}N_1^{(i)}}-\omega_{N_1^{(i)}N_2^{(i)}}\right)\Bigg)\end{aligned}$$

$$\begin{aligned}b^{(i)}_{N_3^{(i)}} = \frac{T_2}{\left(T_1T_2-T_3^2\right)|\Omega_{S_3}|}&\Bigg(\alpha_{N_1^{(i)}}\left(\omega_{N_1^{(i)}N_2^{(i)}}\omega_{N_2^{(i)}N_3^{(i)}}-\omega_{N_1^{(i)}N_3^{(i)}}\omega_{N_2^{(i)}N_2^{(i)}}\right)+\alpha_{N_2^{(i)}}\left(\omega_{N_1^{(i)}N_2^{(i)}}\omega_{N_1^{(i)}N_3^{(i)}}-\omega_{N_1^{(i)}N_1^{(i)}}\omega_{N_2^{(i)}N_3^{(i)}}\right)\\&+\alpha_{N_3^{(i)}}\left(\omega_{N_1^{(i)}N_1^{(i)}}\omega_{N_2^{(i)}N_2^{(i)}}-\omega_{N_1^{(i)}N_2^{(i)}}^{2}\right)\Bigg)\\&-\frac{T_3}{\left(T_1T_2-T_3^2\right)|\Omega_{S_3}|}\Bigg(\omega_{N_2^{(i)}N_2^{(i)}}\left(\omega_{N_1^{(i)}N_1^{(i)}}-\omega_{N_1^{(i)}N_3^{(i)}}\right)+\omega_{N_2^{(i)}N_2^{(i)}}\left(\omega_{N_1^{(i)}N_3^{(i)}}-\omega_{N_1^{(i)}N_2^{(i)}}\right)+\omega_{N_2^{(i)}N_3^{(i)}}\left(\omega_{N_1^{(i)}N_2^{(i)}}-\omega_{N_1^{(i)}N_1^{(i)}}\right)\Bigg)\end{aligned}$$

where

$$T_{1S_3} = \alpha_{S_3}^T\Omega_{S_3}^{-1}\alpha_{S_3}$$

$$T_{2S_3} = 1^T\Omega_{S_3}^{-1}1$$

$$T_{3S_3} = 1^T\Omega_{S_3}^{-1}\alpha_{S_3}$$

The variances and covariance of these estimators are given by

$$\mathrm{Var}\left(\hat{\mu}_{S_3}\right) = T_{1S_3}\sigma^2/\left(T_{1S_3}T_{2S_3}-T_{3S_3}^2\right)$$

$$\mathrm{Var}\left(\hat{\sigma}_{S_3}\right) = T_{2S_3}\sigma^2/\left(T_{1S_3}T_{2S_3}-T_{3S_3}^2\right)$$

$$\mathrm{Cov}\left(\hat{\mu}_{S_3},\hat{\sigma}_{S_3}\right) = -T_{3S_3}\sigma^2/\left(T_{1S_3}T_{2S_3}-T_{3S_3}^2\right)$$

The generalized variance is

$$\mathrm{Gvar}\left(\hat{\mu}_{S_3},\hat{\sigma}_{S_3}\right) = V\left(\hat{\mu}_{S_3}\right)V\left(\hat{\sigma}_{S_3}\right) - Cov\left(\hat{\mu}_{S_3},\hat{\sigma}_{S_3}\right)^2$$

2.2. Generalized Ranked-Set Minimum Variance Unbiased Estimator

GR-MVUEs are obtained from the generalized ranked-set estimators (RSSs) when all possible choices of S are considered. The best choice of S is the one which gives the minimum generalized variance of the estimators. This S is denoted by $S_{\text{GR-MVUE}}$,The estimators are denoted by $\hat{\mu}_{GR-MVUE}$ and $\hat{\sigma}_{GR-MVUE}$. Table 1 provides ranks $S_{\text{GR-MVUE}}$, variances, covariances and generalized variances of the estimators for N = 2(2)10 when two order statistics are selected from each sample. Table 2 provides ranks $S_{\text{GR-MVUE}}$, variances, covariances and generalized variances of the estimators for N = 3(3)15 when three order statistics are selected from each sample.

2.3. Generalized Ranked-Set Sample Estimator

The RSSs for μ and σ are obtained from the generalized RSSs when S = {1, 2,.., N}. These estimators are denoted by $\hat{\mu}_{GR-RSS}$ and $\hat{\sigma}_{GR-RSS}$ respectively. Table 3 provides coefficients for estimating the parameters and Table 4 provides variances, covariances and generalized variances of the estimators for N = 2(2)10 when two order statistics are selected from each sample. Table 5 provides coefficients for estimating the parameters and Table 6 provides variances, covariances and generalized variances of the estimators for N = 3(3)15 when three order statistics are selected from each sample.

As the estimates of σ can be negative a nonnegative unbiased estimator $\hat{\sigma}_{GR-RSS,NN}$ is obtained by taking the absolute value of $\hat{\sigma}_{GR-RSS}$ multiplied by an unbiasing constant, i.e.

let $\hat{\sigma}_{GR-RSS},_{NN} = \mathrm{K}|\hat{\sigma}_{GR-RSS}|$ where K is chosen such that $E\left(\hat{\sigma}_{GR-RSS},_{NN}\right) = \sigma$.

Then $\mathrm{Var}\left(\hat{\sigma}_{GR-RSS},_{NN}\right) = \left(\mathrm{K}^2\mathrm{Var}\left(\hat{\sigma}_{GR-RSS}\right) + \mathrm{K}^2 - 1\right)\sigma^2$

Tables 4 and 6 also includes the values of K and $\mathrm{Var}\left(\hat{\sigma}_{GR-RSS},_{NN}\right)$. The values of K were obtained by simulation. In each case 10^4 generalized ranked-set samples were generated.

Table 1 Variances, covariances, generalized variances and coefficients for $\hat{\mu}_{GR-MVUE}$ and $\hat{\sigma}_{GR-MVUE}$ when two order statistics are selected from each set.

N	$S_{GR\text{-}MVUE}$	$\frac{Var(\hat{\mu}_{GR-MVUE})}{\sigma^2}$	$\frac{Var(\hat{\sigma}_{GR-MVUE})}{\sigma^2}$	$\frac{Cov\left(\hat{\mu}_{GR-MVUE},\hat{\sigma}_{GR-MVUE}\right)}{\sigma^2}$	$\frac{GVar\left(\hat{\mu}_{GR-MVUE},\hat{\sigma}_{GR-MVUE}\right)}{\sigma^4}$	$a_{N_1^{(i)}}^{(i)}$	$a_{N_2^{(i)}}^{(i)}$	$b_{N_1^{(i)}}^{(i)}$	$b_{N_2^{(i)}}^{(i)}$
2	{1,2}	1.00166	0.81616	−0.68028	0.35473	1.62945	−0.62945	−0.81472	0.81472
4	{1,4}	0.10577	0.15425	−0.07402	0.01084	0.59445	−0.09445	−0.22292	−0.22292
6	{1,5}	0.03231	0.06900	−0.02440	0.00163	0.39148	−0.05815	−0.19728	0.19728
8	{1,7}	0.01361	0.03625	−0.00999	0.00040	0.27745	−0.02745	−0.12114	0.12114
10	{1,9}	0.00704	0.02296	−0.00516	0.00014	0.21575	−0.01575	−0.08545	0.08545

Table 2 Variances, covariances, generalized variances and coefficients for $\hat{\mu}_{GR-MVUE}$ and $\hat{\sigma}_{GR-MVUE}$ when three order statistics are selected from each set.

N	$S_{GR\text{-}MVUE}$	$\frac{Var(\hat{\mu}_{GR-MVUE})}{\sigma^2}$	$\frac{Var(\hat{\sigma}_{GR-MVUE})}{\sigma^2}$	$\frac{Cov\left(\hat{\mu}_{GR-MVUE},\hat{\sigma}_{GR-MVUE}\right)}{\sigma^2}$	$\frac{GVar\left(\hat{\mu}_{GR-MVUE},\hat{\sigma}_{GR-MVUE}\right)}{\sigma^4}$	$a_{N_1^{(i)}}^{(i)}$	$a_{N_2^{(i)}}^{(i)}$	$a_{N_3^{(i)}}^{(i)}$	$b_{N_1^{(i)}}^{(i)}$	$b_{N_2^{(i)}}^{(i)}$	$b_{N_3^{(i)}}^{(i)}$
3	{1,2,3}	0.37765	0.39677	−0.24682	0.08892	1.39350	−0.15472	−0.23878	−0.75455	0.33586	0.41869
6	{1,4,6}	0.04616	0.08090	−0.02948	0.00287	0.57992	0.04437	0.03555	−0.29143	0.18421	0.10722
9	{1,7,9}	0.01404	0.03410	−0.00882	0.00040	0.36374	−0.01957	−0.01084	−0.15754	0.11233	0.04521
12	{1,9,12}	0.00608	0.01881	−0.00382	0.00010	0.26736	0.01206	0.00530	−0.11888	0.09072	0.02817
15	{1,10,14}	0.00318	0.01198	−0.00204	0.00003	0.21227	−0.00661	−0.00566	−0.10531	0.06791	0.03740

Table 3 Coefficients $a_{N_j^{(i)}}^{(i)}$ and $b_{N_j^{(i)}}^{(i)}$ for computing $\hat{\mu}_{GR-RSS}$ and $\hat{\sigma}_{GR-RSS}$ when two order statistics are selected from each set.

			N = 2		N = 4		N = 6		N = 8		N = 10	
i	j	$N_j^{(i)}$	$a_{N_j^{(i)}}^{(i)}$	$b_{N_j^{(i)}}^{(i)}$	$a_{N_j^{(i)}}^{(i)}$	$b_{N_j^{(i)}}^{(i)}$	$a_{N_j^{(i)}}^{(i)}$	$b_{N_j^{(i)}}^{(i)}$	$a_{N_j^{(i)}}^{(i)}$	$b_{N_j^{(i)}}^{(i)}$	$a_{N_j^{(i)}}^{(i)}$	$b_{N_j^{(i)}}^{(i)}$
1	1	1	1.62945	−0.81472	1.39598	−0.95936	1.18171	−0.87926	1.04868	−0.81221	0.95884	−0.76147
	2	2	−0.62945	0.81472	−0.33002	0.57960	−0.19158	0.40237	−0.12737	0.30237	−0.09238	0.24027
2	1	3			0.10286	0.11900	0.27059	−0.12321	0.29356	−0.18825	0.28928	−0.20735
	2	4			−0.16883	0.26076	−0.17828	0.33185	−0.13637	0.28852	−0.10409	0.24253
3	1	5					−0.00826	0.14160	0.09360	0.00737	0.12928	−0.05456
	2	6					−0.07418	0.12665	−0.10732	0.20950	−0.09736	0.20999
4	1	7							−0.02403	0.11826	0.03642	0.04345
	2	8							−0.04073	0.07444	−0.07029	0.14261
5	1	9									−0.02430	0.09570
	2	10									−0.02540	0.04884

Table 4 Variances, covariances and generalized variances for ranked-set sampling estimators when two order statistics are selected from each set.

N	$\frac{Var(\hat{\mu}_{GR-RSS})}{\sigma^2}$	$\frac{Var(\hat{\sigma}_{GR-RSS})}{\sigma^2}$	$\frac{Cov\left(\hat{\mu}_{GR-RSS},\hat{\sigma}_{GR-RSS}\right)}{\sigma^2}$	$\frac{GVar\left(\hat{\mu}_{GR-RSS},\hat{\sigma}_{GR-RSS}\right)}{\sigma^4}$	K	$\frac{Var(\hat{\sigma}_{GR-RSS,NN})}{\sigma^2}$
2	1.00166	0.81616	−0.68028	0.35473	1.00000	0.81616
4	0.22636	0.24452	−0.16630	0.02770	0.99312	0.22745
6	0.09334	0.11768	−0.07148	0.00588	0.99979	0.11721
8	0.04959	0.06927	−0.03901	0.00191	0.99988	0.06902
10	0.03031	0.04562	−0.02430	0.00079	1.00015	0.04593

2.4. Best Linear Unbiased Estimators

BLUEs of location and scale parameters of the half logistic distribution have been obtained by Balakrishnan and Puthenpura [14]. Table 7 provides variances, covariances and generalized variances of the estimators for N = 2(2)12 and 3(3)15.

3. SAMPLE DESIGN

Table 8 tabulates for each N (N = 3(3)15 and N = 4(2)12), the number of units selected and the relative efficiency of the generalized ranked-set sampling for each sample design. This will enable researchers to explore the relationship between efficiency of the estimators and the number of units sampled and thus the corresponding cost. It would also enable researchers to decide which sample design to select at what cost.

Table 5 Coefficients $a^{(i)}_{N^{(i)}_j}$ and $b^{(i)}_{N^{(i)}_j}$ for computing $\hat{\mu}_{GR-RSS}$ and $\hat{\sigma}_{GR-RSS}$ when three order statistics are selected from each set.

			N = 3		N = 6		N = 9		N = 12		N = 15	
i	j	$N^{(i)}_j$	$a^{(i)}_{N^{(i)}_j}$	$b^{(i)}_{N^{(i)}_j}$	$a^{(i)}_{N^{(i)}_j}$	$b^{(i)}_{N^{(i)}_j}$	$a^{(i)}_{N^{(i)}_j}$	$b^{(i)}_{N^{(i)}_j}$	$a^{(i)}_{N^{(i)}_j}$	$b^{(i)}_{N^{(i)}_j}$	$a^{(i)}_{N^{(i)}_j}$	$b^{(i)}_{N^{(i)}_j}$
1	1	1	1.39350	−0.15472	1.20517	−0.83158	1.07394	−0.79372	0.98981	−0.75881	0.93084	−0.73083
	2	2	−0.75455	0.33586	−0.00961	0.05816	0.00164	0.01809	0.00274	0.00730	0.00248	0.00336
	3	3	1.39350	−0.15472	−0.18179	0.43148	−0.11408	0.32205	−0.07893	0.25062	−0.05879	0.20359
2	1	4			0.11952	0.06356	0.20536	−0.09207	0.21827	−0.13768	0.21668	−0.15328
	2	5			−0.06299	0.13816	−0.01614	0.05087	−0.00487	0.02294	−0.00142	0.01197
	3	6			−0.07030	0.14022	−0.10386	0.25106	−0.08502	0.23365	−0.06712	0.20289
3	1	7					0.01639	0.10237	0.07606	0.00826	0.09758	−0.03684
	2	8					−0.03113	0.07180	−0.01265	0.03685	−0.00555	0.02024
	3	9					−0.03212	0.06953	−0.06420	0.15982	−0.06212	0.17108
4	1	10							−0.00505	0.09211	0.03286	0.03744
	2	11							−0.01806	0.04347	−0.00934	0.02677
	3	12							−0.01808	0.04147	−0.04280	0.10953
5	1	13									−0.01018	0.07756
	2	14									−0.01164	0.02900
	3	15									−0.01148	0.02753

Table 6 Variances, covariances and generalized variances for ranked-set sampling estimators when three order statistics are selected from each set.

N	$\frac{Var(\hat{\mu}_{GR-RSS})}{\sigma^2}$	$\frac{Var(\hat{\sigma}_{GR-RSS})}{\sigma^2}$	$\frac{Cov(\hat{\mu}_{GR-RSS}, \hat{\sigma}_{GR-RSS})}{\sigma^2}$	$\frac{GVar(\hat{\mu}_{GR-RSS}, \hat{\sigma}_{GR-RSS})}{\sigma^4}$	K	$\frac{Var(\hat{\sigma}_{GR-RSS,NN})}{\sigma^2}$
3	0.37765	0.39677	−0.24682	0.08892	1.00000	0.39677
6	0.09507	0.12966	−0.06823	0.00767	0.99999	0.12963
9	0.04107	0.06454	−0.03083	0.00170	0.99996	0.06444
12	0.02243	0.03875	−0.01734	0.00057	0.99998	0.03870
15	0.01397	0.02587	−0.01102	0.00024	1.00011	0.02610

Note: 1 ≤ i ≤ N/3.

Table 7 Variances, covariances, generalized variances for $\hat{\mu}_{BLUE}$ and $\hat{\sigma}_{BLUE}$.

N	$\frac{Var(\hat{\mu}_{BLUE})}{\sigma^2}$	$\frac{Var(\hat{\sigma}_{BLUE})}{\sigma^2}$	$\frac{Cov(\hat{\mu}_{BLUE}, \hat{\sigma}_{BLUE})}{\sigma^2}$	$\frac{GVar(\hat{\mu}_{BLUE}, \hat{\sigma}_{BLUE})}{\sigma^4}$
2	1.00170	0.81616	−0.68028	0.35473
3	0.37765	0.39677	−0.24682	0.08892
4	0.20545	0.25962	−0.13084	0.03622
6	0.09163	0.15210	−0.05648	0.01075
8	0.05253	0.10704	−0.03175	0.00461
9	0.04190	0.09314	−0.02514	0.00327
10	0.03424	0.08240	−0.02042	0.00240
12	0.02416	0.06691	−0.01427	0.00141
15	0.01576	0.05214	−0.00921	0.00074

4. COMPARISONS

In this section comparison has been made between GR-MVUEs, GR-RSSs and BLUEs. Tables 9 and 10 provide the relative efficiencies of the estimators and show that GR-MVUEs are more efficient than ranked-set samples and BLUEs, when two or three order statistics are selected from each sample. Tables 9 and 10 also show that the ranked-set sample estimators are more efficient than the BLUEs.

5. CONCLUSION

The GR-MVUEs and GR-RSSs are both more efficient than the BLUEs. The GR-MVUE are the most efficient estimators. In applications where ranked-set sampling is useful, the cost involved in sampling and ranking cannot be completely ignored. Therefore, the choice of the estimators based on two or three units selected from each set would provide a reasonable alternative to the balanced ranked-set sample.

Table 8 Sample design, variances and relative efficiencies for generalized ranked-set sampling.

N	Design	Number of Order Statistics Selected	Number of Units Selected	$\frac{Var(\hat{\mu}_{GR-RSS})}{\sigma^2}$	$\frac{Var(\hat{\sigma}_{GR-RSS})}{\sigma^2}$	$\frac{Var(\hat{\mu}_{BLUE})}{Var(\hat{\mu}_{GR-RSS})}$	$\frac{Var(\hat{\sigma}_{BLUE})}{Var(\hat{\sigma}_{GR-RSS})}$
4	4 × 1	2	8	0.22636	0.24452	0.90763	1.06175
	2 × 2	1	8	0.88712	0.65808	0.23159	0.39451
6	6 × 1	3	12	0.09507	0.12966	0.96384	1.17307
	3 × 2	1	18	0.26033	0.23421	0.35199	0.64943
	6 × 1	2	18	0.09334	0.11768	0.98170	1.29249
	2 × 3	1	12	0.59141	0.43872	0.15494	0.34669
8	8 × 1	2	32	0.04959	0.06927	1.05933	1.54526
	4 × 2	1	32	0.11867	0.12092	0.44269	0.88525
	2 × 4	1	16	0.44356	0.32904	0.11843	0.32531
	4 × 2	2	16	0.11318	0.12226	0.46415	0.87551
9	9 × 1	3	27	0.04107	0.06454	1.02026	1.44314
	3 × 3	1	27	0.17355	0.15614	0.24144	0.59653
10	10 × 1	2	50	0.03031	0.04562	1.12979	2.12650
	5 × 2	1	50	0.06623	0.07375	0.79318	1.66217
	2 × 5	1	20	0.35485	0.26323	0.14804	1.40044
12	12 × 1	3	48	0.02243	0.03875	1.07704	1.72679
	12 × 1	2	72	0.020254	0.032299	1.19275	2.07167
	6 × 2	1	72	0.04162	0.04958	0.58051	1.34973
	6 × 2	2	36	0.04667	0.05884	0.51763	1.13720
	6 × 2	3	24	0.04754	0.06483	0.50821	1.03213
	4 × 3	1	48	0.07911	0.08061	0.30537	0.83008
	3 × 4	1	36	0.13016	0.11710	0.18560	0.57141
15	15 × 1	3	75	0.01397	0.02587	1.12785	2.01562
	5 × 3	1	75	0.04415	0.04917	0.35685	1.06055
	3 × 5	1	45	0.10413	0.02950	0.15131	1.76759

Table 9 Relative efficiencies for the estimators when two order statistics are selected from each set.

N	$\frac{Var(\hat{\mu}_{BLUE})}{Var(\hat{\mu}_{GR-MVUE})}$	$\frac{Var(\hat{\sigma}_{BLUE})}{Var(\hat{\sigma}_{GR-MVUE})}$	$\frac{\text{GVar}\begin{pmatrix}\hat{\mu}_{BLUE},\\ \hat{\sigma}_{BLUE}\end{pmatrix}}{\text{GVar}\begin{pmatrix}\hat{\sigma}_{GR-MVUE},\\ \hat{\sigma}_{GR-MVUE}\end{pmatrix}}$	$\frac{Var(\hat{\mu}_{GR-RSS})}{Var(\hat{\mu}_{GR-MVUE})}$	$\frac{Var(\hat{\sigma}_{GR-RSS})}{Var(\hat{\sigma}_{GR-MVUE})}$	$\frac{\text{GVar}\begin{pmatrix}\hat{\mu}_{GR-RSS},\\ \hat{\sigma}_{GR-RSS}\end{pmatrix}}{\text{GVar}\begin{pmatrix}\hat{\mu}_{GR-MVUE},\\ \hat{\sigma}_{GR-MVUE}\end{pmatrix}}$	$\frac{Var(\hat{\mu}_{BLUE})}{Var(\hat{\mu}_{GR-RSS})}$	$\frac{Var(\hat{\sigma}_{BLUE})}{Var(\hat{\sigma}_{GR-RSS})}$	$\frac{\text{GVar}\begin{pmatrix}\hat{\mu}_{BLUE},\\ \hat{\sigma}_{BLUE}\end{pmatrix}}{\text{GVar}\begin{pmatrix}\hat{\mu}_{GR-RSS},\\ \hat{\sigma}_{GR-RSS}\end{pmatrix}}$
2	1.00004	1.00000	1.00000	1.00000	1.00000	1.00000	1.00004	1.00000	1.00000
4	1.94242	1.68311	3.34256	2.14012	1.58522	2.55629	0.90763	1.06175	1.30758
6	2.83638	2.20444	6.57832	2.88925	1.70558	3.59919	0.98170	1.29249	1.82772
8	3.85981	2.95275	11.72574	3.64364	1.91084	4.85301	1.05933	1.54526	2.41618
10	4.86296	3.58941	17.80073	4.30430	1.98719	5.84795	1.12979	1.80627	3.04392

Table 10 | Relative efficiencies for the estimators when three order statistics are selected from each set.

N	$\frac{Var(\hat{\mu}_{BLUE})}{Var(\hat{\mu}_{GR-MVUE})}$	$\frac{Var(\hat{\sigma}_{BLUE})}{Var(\hat{\sigma}_{GR-MVUE})}$	$\frac{GVar\begin{pmatrix}\hat{\mu}_{BLUE}, \\ \hat{\sigma}_{BLUE}\end{pmatrix}}{GVar\begin{pmatrix}\hat{\sigma}_{GR-MVUE}, \\ \hat{\sigma}_{GR-MVUE}\end{pmatrix}}$	$\frac{Var(\hat{\mu}_{GR-RSS})}{Var(\hat{\mu}_{GR-MVUE})}$	$\frac{Var(\hat{\sigma}_{GR-RSS})}{Var(\hat{\sigma}_{GR-MVUE})}$	$\frac{GVar\begin{pmatrix}\hat{\mu}_{GR-RSS}, \\ \hat{\sigma}_{GR-RSS}\end{pmatrix}}{GVar\begin{pmatrix}\hat{\mu}_{GR-MVUE}, \\ \hat{\sigma}_{GR-MVUE}\end{pmatrix}}$	$\frac{Var(\hat{\mu}_{BLUE})}{Var(\hat{\mu}_{GR-RSS})}$	$\frac{Var(\hat{\sigma}_{BLUE})}{Var(\hat{\sigma}_{GR-RSS})}$	$\frac{GVar\begin{pmatrix}\hat{\mu}_{BLUE}, \\ \hat{\sigma}_{BLUE}\end{pmatrix}}{GVar\begin{pmatrix}\hat{\mu}_{GR-RSS}, \\ \hat{\sigma}_{GR-RSS}\end{pmatrix}}$
3	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
6	1.98531	1.88017	3.75113	2.05980	1.60278	2.67714	0.96384	1.17307	1.40117
9	2.98426	2.73130	8.15433	2.92501	1.89261	4.23834	1.02026	1.44314	1.92394
12	3.97551	3.55655	14.16312	3.69115	2.05964	5.71417	1.07704	1.72679	2.47860
15	4.95378	4.35259	21.69725	4.39225	2.15943	7.06818	1.12785	2.01562	3.06971

REFERENCES

1. G.A. McIntyre, Aust. J. Agric. Res. 3 (1952), 385–390.
2. L.K. Halls, T.R. Dell, Forest Sci. 12 (1996), 22–26.
3. H. Chen, E.A. Stasny, D.A. Wolfe, Stat. Med. 24 (2005), 3319–3329.
4. B.D. Nussbaum, B.K. Sinha, in Proceedings of the Section on Statistics and the Environment, American Statistical Association, Alexandria, VA, 1997, pp. 83–87.
5. P.H. Kvam, J. Agric. Biol. Environ. Stat. 8 (2003), 271–279.
6. Z. Chen, Environ. Ecol. Stat. 14 (2007), 355–363.
7. R.W. Howard, S.C. Jones, J.K. Mauldin, R.H. Beal, Environ. Entomol. 11 (1982), 1290–1293.
8. D.A. Wolfe, ISRN Probab. Stat. (2012), 1–32.
9. A. Adatia, Comput. Stat. Data Anal. 33 (2000), 1–13.
10. R.W. Nahhas, D.A. Wolfe, H. Chen, Biometrics. 58 (2002), 964–971.
11. Y. Wang, Z. Chen, J. Liu, Biometrics. 60 (2004), 556–561.
12. R.A. Buchanan, L.L. Conquest, J. Courbois, Environmetrics. 16 (2005), 235–256.
13. D.E. Giles, Commun. Stat. Theor. Methods. 41 (2012), 212–222.
14. N. Balakrishnan, S. Puthenpura, J. Stat. Comput. Simul. 20 (1986), 287–309.