

Clarification of Dependence of the Macaulay Duration on the Period Until Maturity

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Abstract. The article is dedicated to clarification of dependence of the Macaulay duration on the period until maturity by accounting the behavior of this indicator between coupon payments. The problem of dependence of bond duration on the period until maturity accounting the behavior of duration between coupon payments has not been considered before. It has been observed that the Macaulay duration varies linearly during a coupon period and in the end of the period the duration jumps in direct proportion to the period until maturity. The maximum jump for discount obligations is in the region of long periods until maturity. We have received evidence of how the jump depends on the period until maturity. The article contains calculations of how the duration jump depends on the period until maturity. These calculations confirm the proven statements. The obtained results may be useful for long-term investment problems.

1. Introduction

Despite the limited number of conditions when the Macaulay duration may be determined [1, 2], it is widely used in theory and in practice [3-13]. Therefore, analysis of the factors affecting duration is of both theoretical and practical interest. The effect of the primary factors (yield, coupon rate and period until maturity) on duration has been formulated in financial investment literature, e.g., by Fabozzi [14, p. 512]. Publications [15-17] provide proofs of dependence of duration of par, premium and discount bonds on the period until maturity. Proofs have been obtained for the duration of the bonds sold immediately after a coupon payment. It ought to be noted that the duration behavior between coupon payments is not taken into account. The following theorem was proved by the author of publication [17].

Theorem. Let D_n be duration of a bond, the payments whereunder are performed m times a year, and which shall mature after n coupon periods, then the following statements are correct at set values of coupon rate f and yield to maturity r :

- 1) $\lim_{n \rightarrow \infty} \{D_n\} \approx (r + m) / mr$;
- 2) for par or premium bonds ($f \geq r$) the $\{D_n\}$ sequence is ascending, and $\lim_{n \rightarrow \infty} D_n = \sup \{D_n\}$;
- 3) for discount bonds ($f < r$) there is a maximum duration period n_0 , and $\lim_{n \rightarrow \infty} D_n = \inf_{n > n_0} \{D_n\}$.

The theorem was proven for $m = 1$ by comparing members of sequence $\{D_n\}$ and using the method of mathematical induction. We proved that discount bonds feature maximum duration period n_0 and obtained an approximate value thereof:

$$n_0 \approx \frac{1}{r} + \frac{1+r}{r-f} \tag{1}$$

Figs. 1 and 2 demonstrate behavior of members of sequence $\{D_n\}$ for $f \geq r$ and $f < r$.

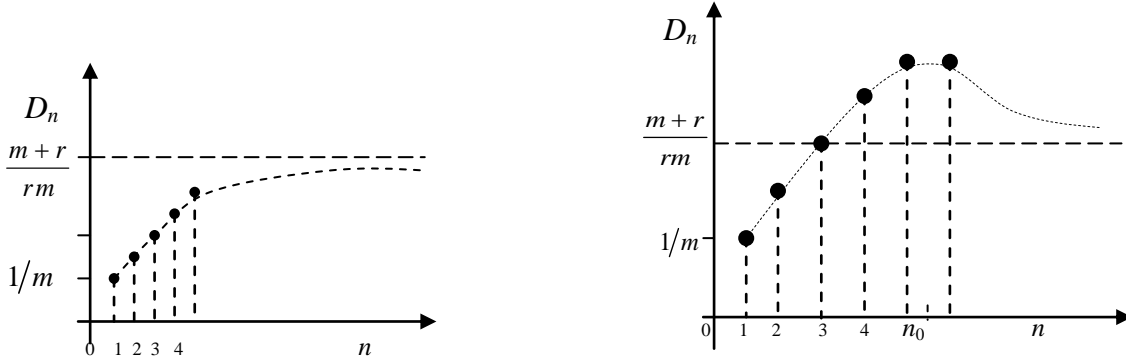


Figure 1. D_n dependence on n ($f \geq r$)

Figure 2. D_n dependence on n ($f < r$)

Heavy points in figs. 1 and 2 - members of sequence D_n immediately after coupon payments. As we have already mentioned, behavior of the duration between coupon payments (in the figures – between heavy points) was not considered in publication [17]. The issue of dependence of bond duration on the period until maturity accounting the behavior of duration between coupon payments has not been considered in the existing literature. That is why the insight into how duration behaves until maturity is not complete. The goal of this article was to take into account behavior of the duration between coupon payments in the problem of dependence of bond duration on the period until maturity.

2. Problem-solving methods and results

We proposed taking behavior of bond duration between coupon payments into account using τ . It is a secondary bond parameter. τ – period of time between the latest coupon payment before the bond is sold and the time of sale. τ follows inequation $0 \leq \tau < 1/m$, where $1/m$ years - coupon period length. Thus, τ concerns time within a coupon period. Beginning of a coupon period – moment immediately after a coupon payment ($\tau = 0$). By the end of a coupon period – nears $1/m$. If a bond is sold after – following a coupon payment when the period until maturity is T years (n coupon payments), the period until the bond's maturity is

$$T = n/m - \tau, \tag{2}$$

where $\tau \in [0, 1/m)$. In the beginning of a coupon period, the period until maturity is n/m years (n coupon periods), and in the end of the coupon period it nears $(n-1)/m$ years. In the beginning of the next coupon period, the period until maturity is $(n-1)/m$ years ($n-1$ coupon periods). Therefore, when – increases from 0 to $1/m$, the period until the bond's maturity decreases by one coupon period.

Let us consider how bond duration behaves throughout a coupon period between coupon payments. Duration of a coupon bond sold after – after a coupon payment, when the number of coupon payments until maturity is n , follows equation

$$D = \frac{\sum_{i=1}^n t_i \frac{fA/m}{(1+r/m)^{t_i}} + t_n \frac{A}{(1+r/m)^{t_n}}}{\sum_{i=1}^n \frac{fA/m}{(1+r/m)^{t_i}} + \frac{A}{(1+r/m)^{t_n}}}, \tag{3}$$

where A – bond's par value, $t_i = \frac{i}{m} - \tau$, $t_n = \frac{n}{m} - \tau$. Hence,

$$D = D_n - \tau, \tag{4}$$

where $\tau \in [0, 1/m)$. Please note that the results in publications [15-17] were obtained for $\tau = 0$. Equations (2) and (4) make up an expression for dependence of bond duration on the period until maturity during a coupon period:

$$D = D_n - n/m + T, \tag{5}$$

where $(n-1)/m < T \leq n/m$, $n = 1, 2, 3, \dots$. According to (5), bond duration is a linear increasing function of period until maturity T at any fixed n value within time $[(n-1)/m, n/m]$, i.e. during a coupon period. On the other hand, according to (2), bond duration is a linear decreasing function of τ at any fixed n value. When τ increases from 0 to $1/m$, bond duration linearly decreases to D_n in the beginning of the period and nears $(D_n - 1/m)$ in the end of the period. In the beginning of the next coupon period, duration is D_{n-1} . Please note that D_1 τ - bond duration one coupon period before maturity, when an obligation becomes a purely discount one. Then $D_1 = 1/m$ years and, therefore, $D_1 - 1/m = 0$. Let us prove the following statement for $n > 1$.

Statement 1. When r and f are set, members of sequence $\{D_n\}$ match the following inequation: $D_n - 1/m < D_{n-1}$, where $n \geq 2$.

Proof. It is evident that inequation $D_n - 1/m < D_{n-1}$ is equivalent to inequation $D_n - D_{n-1} - 1/m < 0$. Let us prove the statement for $m = 1$. Let us use the results obtained by the author of publication [17], where $D_n = \frac{f - fp^n + np^{n-1}a}{f(1-p) + p^{n-1}a}$, $p = \frac{1}{1+r}$, $a = p(1-p)(r-f)$. Let us consider the following remainder:

$$D_n - D_{n-1} - 1 = \frac{(1-p)^2 [f^2(p^{n-1}-1) - p^{n-1}f(r-f)(n-1)(1-p)]}{(f(1-p) + p^{n-1}a)(f(1-p) + p^{n-2}a)}. \tag{6}$$

When $n = 1$, this expression equals zero. Indeed, in that case $D_1 = 1$, D_0 τ bond duration on the day of maturity immediately after the payment of the last coupon. Then $D_0 = 0$ and $D_1 - D_0 - 1 = 1 - 0 - 1 = 0$. When $n > 1$, expression (6) is negative, if $f \leq r$. If $f > r$, let us write the expression written in square parentheses in the numerator in equation (6) as follows:

$$f^2(p^{n-1}-1) - p^{n-1}f(f-r)(1-p) + p^{n-1}f(f-r)n(1-p)$$

and use the following inequation to assess the last positive summand: $a^n - 1 > n(a-1)$, valid for $a > 1$ and $n \geq 2$. Let us assume that $a = 1/p$, as $0 < p < 1$, then $p^{n-1}f(f-r)n(1-p) < (1-p^n)f(f-r)$ and the expression in square parentheses is negative, as its

value is $< -(1 - p^{n-1})fr < 0$. The statement is thus proven. According to statement , as $D_n - 1/m < D_{n-1}$ at $n \geq 2$, then in the end of each coupon period, excluding the last period, when $n = 1$, bond duration jumps up. Let us prove the following statement for jump ΔD_{n-1} .

Statement 2. When r and f are set, the jump of the duration at $(n-1)/m$ is $\Delta D_{n-1} = D_{n-1} - (D_n - 1/m)$. Proof. By definition, the jump of $f(x)$ at x_0 is $|A - B| > 0$, where $A = \lim_{x \rightarrow x_0+0} f(x)$, $B = \lim_{x \rightarrow x_0-0} f(x)$. As

$$\begin{aligned} \lim_{T \rightarrow (n-1)/m+0} D &= \lim_{T \rightarrow (n-1)/m+0} (D_n - n/m + T) = D_n - 1/m, \\ \lim_{T \rightarrow (n-1)/m-0} D &= \lim_{T \rightarrow (n-1)/m-0} (D_{n-1} - (n-1)/m + T) = D_{n-1}, \end{aligned}$$

then $\Delta D_{n-1} = D_{n-1} - (D_n - 1/m)$. Then $\Delta D_{n-1} > 0$, because $D_n - 1/m < D_{n-1}$ at $n \geq 2$, according to statement 1. Please note that $\Delta D_0 = 0$ at $n = 1$. Indeed, $\Delta D_0 = D_0 - (D_1 - 1/m) = 0 - (1/m - 1/m) = 0$. The statement is thus proven.

Note. Expression $\Delta D_{n-1} = D_{n-1} - (D_n - 1/m)$ for the duration jump at $(n-1)/m$ may be formulated as follows:

$$\Delta D_{n-1} = 1/m - (D_n - D_{n-1}). \tag{7}$$

Let us consider how jump ΔD_{n-1} depends on n . Let us prove the following theorem.

Theorem. The following statements are correct when r and f are set:

- 1) sequence $\{\Delta D_{n-1}\}$ is convergent, and $\lim_{n \rightarrow \infty} \Delta D_{n-1} = 1/m$;
- 2) for par or premium bonds ($f \geq r$) the $\{\Delta D_{n-1}\}$ sequence is ascending, and $\lim_{n \rightarrow \infty} \Delta D_{n-1} = \sup\{\Delta D_{n-1}\}$;
- 3) for discount bonds ($f < r$) there is a maximum n_Δ for jump ΔD_{n-1} , and $\lim_{n \rightarrow \infty} \Delta D_{n-1} = \inf_{n > n_\Delta} \{\Delta D_{n-1}\}$.

Proof. 1) According to [17], sequence $\{D_n\}$ is convergent. As $\lim_{n \rightarrow \infty} \{D_n\} = \lim_{n \rightarrow \infty} \{D_{n-1}\}$, then $\lim_{n \rightarrow \infty} (D_n - D_{n-1}) = 0$. Then, according to equation (7), $\lim_{n \rightarrow \infty} \{\Delta D_{n-1}\} = 1/m$. Please note that $1/m$ is the length of a coupon period. The first statement of the theorem is thus proven.

2) In order to prove that if $f \geq r$, sequence $\{\Delta D_{n-1}\}$ is increasing, it is necessary to demonstrate that $\Delta D_n - \Delta D_{n-1} > 0$. As

$$\Delta D_n - \Delta D_{n-1} = (D_n - D_{n-1}) - (D_{n+1} - D_n),$$

the problem amounts to analyzing sequence $\{D_n - D_{n-1}\}$. Using the method of mathematical induction, we get $\Delta D_n - \Delta D_{n-1} > 0$.

3) For discount bonds ($f < r$) there is a maximum duration period n_0 – formula (1). Then, according to equation (7), $\Delta D_{n-1} < 1/m$ if $n < n_0$, near n_0 there is jump $\Delta D_{n-1} \approx 1/m$, and at $n > n_0$ the jump is $\Delta D_{n-1} > 1/m - \Delta D_{n-1}$ increases in direct proportion to n . Let us prove that there is maximum period ΔD_{n-1} . Near n_0 ($n \approx n_0$) the remainder is

$\Delta D_n - \Delta D_{n-1} \approx \Delta D_n - 1/m = 1/m - (D_{n+1} - D_n) - 1/m = D_n - D_{n+1} > 0$, as $n+1 > n_0$, whereas at $n > 2n_0$ $\Delta D_n - \Delta D_{n-1} < 0$. Therefore, there is period n_Δ , when remainder $\Delta D_n - \Delta D_{n-1} \approx 0$, and $n_\Delta > n_0$ – this period n_Δ is in the region of long periods until maturity. The approximate value of maximum period n_Δ of the duration jump is determined under condition that $\Delta D_n - \Delta D_{n-1} \approx 0$:

$$n_\Delta \approx 1 + \frac{2}{r} + \frac{1+r}{r-f} \tag{8}$$

Therefore, jump level ΔD_{n-1} at $f < r$ increases from 0 at $n=1$ to approximately $1/m$ near n_0 with the increase in n . After that, ΔD_{n-1} increases at $n > n_0$, reaches the maximum level near n_Δ , and decreases at the limit to $1/m$. Please note that ΔD_{n-1} tends $1/m$ remaining $> 1/m$, because if period until maturity $n > n_0$, jump $\Delta D_{n-1} > 1/m$. The theorem is thus proven.

Table 1 provides the values of members of sequence $\{D_n\}$ calculated using formula (3) at $\tau = 0$ and jump ΔD_{n-1} and using formula (7) at $m=1$ when $f > r$ or $f < r$.

Table 1. Dependence of D_n and ΔD_{n-1} on n .

n	$f = 0,1; r = 0,08 (f > r)$		$f = 0,05; r = 0,25 (f < r)$	
	D_n	ΔD_{n-1}	D_n	ΔD_{n-1}
1	1.00	0	1	0
2	1.91	0.089	1.90	0.102
3	2.74	0.168	2.68	0.215
4	3.50	0.238	3.35	0.332
6	4.85	0.356	4.34	0.560
7	5.44	0.407	4.68	0.661
12	7.81	0.596	5.34	0.962
13	8.18	0.624	• 5.36	0.987
14	8.53	0.650	5.35	1.005
15	8.86	0.674	5.33	1.016
17	9.44	0.717	5.28	1.027
18	9.71	0.736	5.25	• 1.029
19	9.95	0.754	5.23	1.029
20	10.18	0.770	5.20	1.027
25	11.12	0.836	5.09	1.016
$n \rightarrow \infty$	13.5	1	5	1

Maximums D_n and ΔD_{n-1} are represented in the table with dots. As we can see, calculations confirm the proven statements. Please note that maximum duration periods n_0 and duration jumps n_Δ for bonds with the following parameters: $f = 0,05; r = 0,25; m = 1$, are 13 and 18 years, respectively (according to formula (1), $n_0 \approx 10$ years, according to formula (8), $n_\Delta \approx 15,25$ years).

3. Conclusions

It has been found that the Macaulay duration decreases linearly during a coupon period and jumps up in the end of the period in direct proportion with the increase in the period until maturity for par or premium bonds. The maximum jump for discount obligations is in the region of long periods until maturity.

Consideration of how duration behaves during a coupon period helps to clarify how the Macaulay duration depends on the period until maturity: this dependence is linear during each coupon period and jump-like during the period until maturity.

The upward jump of the duration in the end of a coupon period means a jump of bond price's sensitivity to changes of the market interest rate (bond's interest rate risk), and it also ought to be noted that this sensitivity is decreased immediately prior to a coupon payment.

Please note that these study results were obtained at the same conditions as has been in place when the Macaulay duration was determined, i.e. at the condition of horizontality of the yield curve. In real practice, the yield curve is not horizontal, and its shifts may not be parallel.

As peculiarities of duration behavior (presence/absence and level of jumps) are more noticeable in the region of long periods until maturity, study results may be useful for portfolio and long-term investment problems.

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