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Advances in Economics, Business and Management Research, volume 133
3rd International Conference on Advances in Management Science and Engineering (IC-AMSE 2020)

Second Order Sufficient Optimality Condition for a Class of Major Constraints Vector Optimization Problem

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Keywords: ***vector optimization, major constraint, pareto weakly efficient solution, optimality condition, second order sufficient condition***

Abstract. Optimality condition of second order of major constraints vector optimization is researched. The concept of major constraints local Pareto weakly efficient solution in major constraints vector optimization is given. With the aid of the major constraint set and its structure representation, a sufficient optimality condition of second order of major constraint local Pareto weakly efficient solution is obtained. The result is useful for the further study of vector optimization theory and also is useful for the study of its numerical methods.

Introduction

Vector optimization problem is an important part of decision science. When a practical vector optimization model is established, the inequality constraints are given by decision-makers usually according to their different needs. So the inequality constraints in the established model sometimes appear incompatible. For this reason, we consider a kind of vector optimization model called major constrains (MC) vector optimization. Its major inequality constraints are compatible.

Hu [1] first introduced the concept of major order, and then presented MC programming problem. Later, Hu and Zhou [2] defined MC optimal solution, and gave two first order(FO) necessary optimality conditions and one FO sufficient optimality condition of MC optimal solution by using the representation of MC set structure. Zhou [3] then extended the results of [2] to the MC vector optimization problem.

The study of optimality conditions has always been a central subject of mathematical programming theory, and the study of its FO and second-order(SO) as well as higher-order(HO) optimality conditions have received extensive attention [4]-[16]. In this paper, we give the concept of MC local Pareto weakly effective solution(ES). With the aid of the MC set and its structural representation, we obtain a SO sufficient optimality condition for MC local Pareto weakly ES.

Preliminaries

Let f be a nonlinear vector function from R^n to R^m and g be also a nonlinear vector function from R^n to R^p. The general nonlinear vector optimization problem is

$$\begin{cases} \min & f(x), \\ s.t. & g(x) \le 0. \end{cases} \qquad \text{(NMP)}$$

Denote $g(x) = (g_1(x), \cdots, g_p(x))^T$, then the inequality constraints of (NMP) are

$$g_i(x) \le 0, \qquad i = 1, \cdots, p.$$

While an actual vector optimization model is established, the various inequality constraints sometimes appear incompatible. Thus with the aid of major cone, a class of MC optimization problem was presented in [1].

Let $v = (v_1, \cdots, v_p)^T \in R^p$, and let $|v|_+$, $|v|_-$ and $|v|_0$ be definited as in [2].

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Let $H=\{v\in R^p \mid |v|_+\geq |v|_-\}$ be the major cone of R^p, clH be the closure of the major cone H in R^p. Obviously, $clH=\{v\in R^p \mid |v|_+ + |v|_0\geq [\frac{p+1}{2}]\}$. If

$K_{i_1,\cdots,i_{[\frac{p+1}{2}]}}=\{v\in R^p \mid v_{i_1}\geq 0,\cdots,v_{i_{[\frac{p+1}{2}]}}\geq 0\}$,

then $clH=\bigcup K_{i_1,\cdots,i_{[\frac{p+1}{2}]}}$ for all $i_1,\cdots,i_{[\frac{p+1}{2}]}\in\{1,\cdots,p\}$.

Consider the following MC vector optimization problem:

$$\begin{cases}\min & f(x)\\ s.t. & g(x)=(g_1(x),\cdots,g_p(x))\in -clH.\end{cases} \tag{MCMP}$$

The MC set of (MCMP) is

$$X_M=\{x\in R^n \mid g(x)\in -clH\}.$$

If $X_{i_1,\cdots,i_{[\frac{p+1}{2}]}}=\{x\in R^n \mid g(x)\in -K_{i_1,\cdots,i_{[\frac{p+1}{2}]}}\}=\{x\in R^n \mid g_{i_1}(x)\leq 0,\cdots,g_{i_{[\frac{p+1}{2}]}}(x)\leq 0\}$, then

$$X_M=\bigcup X_{i_1,\cdots,i_{[\frac{p+1}{2}]}}$$

for all $i_1,\cdots,i_{[\frac{p+1}{2}]}\in\{1,\cdots,p\}$.

$x\in X_M$ means that there exist at least $[\frac{p+1}{2}]$ numbers $i_1,\cdots,i_{[\frac{p+1}{2}]}\in\{1,\cdots,p\}$ such that x meets the inequality system $g_{i_1}(x)\leq 0,\cdots,g_{i_{[\frac{p+1}{2}]}}(x)\leq 0$.

Definition1 Let f be a nonlinear vector function from R^n to R^m and g be also a nonlinear vector function from R^n to R^p $(p\geq 3)$. If $x^*\in X_M$ and there exists an $\delta>0$ such that there is not $x\in X_M\cap N_\delta(x^*)$ meeting

$$f_k(x)<f_k(x^*)(k=1,\cdots,m),$$

where $N_\delta(x^*)=\{x\in R^n \mid \|x-x^*\|<\delta\}$ is a neighborhood of x^*, then x^* is said to be a MC local Pareto weakly ES. The set of all MC local Pareto weakly ES is denoted by X_M^*.

Optimality Condition

Now we give a SO sufficient condition for the MC local Pareto weakly ES of (MCMP).

Let $x^*\in X_M$. For any $[\frac{p+1}{2}]$ numbers $i_1,\cdots,i_{[\frac{p+1}{2}]}\in\{1,\cdots,p\}$, if $x^*\in X_{i_1,\cdots,i_{[\frac{p+1}{2}]}}$, then there exist $\tilde{\mu}_{i_1,\cdots,i_{[\frac{p+1}{2}]}}=(\tilde{\mu}_1^{i_1,\cdots,i_{[\frac{p+1}{2}]}},\cdots,\tilde{\mu}_m^{i_1,\cdots,i_{[\frac{p+1}{2}]}})^T$ belong to R_+^m, $\tilde{\lambda}_{i_1,\cdots,i_{[\frac{p+1}{2}]}}=(\tilde{\lambda}_{i_1}^{j_1,\cdots,i_{[\frac{p+1}{2}]}},\cdots,\tilde{\lambda}_{i_{[\frac{p+1}{2}]}}^{j_1,\cdots,i_{[\frac{p+1}{2}]}})$ belong to $R^{[\frac{p+1}{2}]}$ such that

$$\begin{aligned}&\tilde{\mu}^T_{i_1,\cdots,i_{[\frac{p+1}{2}]}}\nabla f(x^*)+\sum_{m=1}^{[\frac{p+1}{2}]}\tilde{\lambda}_{i_m}^{j_1,\cdots,i_{[\frac{p+1}{2}]}}\nabla g_{i_k}(x^*)=0,\\ &\tilde{\lambda}_{i_m}^{j_1,\cdots,i_{[\frac{p+1}{2}]}}g_{i_m}(x^*)=0,\quad m=1,2,\cdots,[\frac{p+1}{2}].\end{aligned} \tag{1}$$

Define

$$I_{i_1,\cdots,i_{[\frac{p+1}{2}]}} = \{i_m \in \{i_1,\cdots,i_{[\frac{p+1}{2}]}\} \mid g_{i_m}(x^*) = 0\}$$

$$G_{i_1,\cdots,i_{[\frac{p+1}{2}]}} = \{d \in R^n \mid \begin{matrix} \nabla g_{i_m}(x^*)^T d = 0, i_m \in I_{i_1,\cdots,i_{[\frac{p+1}{2}]}}, \tilde{\lambda}_{i_m}^{j_1,\cdots,i_{[\frac{p+1}{2}]}} > 0 \\ \nabla g_{i_m}(x^*)^T d \le 0, i_m \in I_{i_1,\cdots,i_{[\frac{p+1}{2}]}}, \tilde{\lambda}_{i_m}^{j_1,\cdots,i_{[\frac{p+1}{2}]}} = 0 \end{matrix} \}.$$

Theorem1 Let f from R^n to R^m and g from R^n to R^p be twice differentiable at $x^* \in X_M$. For any $i_1,\cdots,i_{[\frac{p+1}{2}]} \in \{1,\cdots,p\}$, if $x^* \in X_{i_1,\cdots,i_{[\frac{p+1}{2}]}}$, then there exist

$\tilde{\mu}_{i_1,\cdots,i_{[\frac{p+1}{2}]}} = (\tilde{\mu}_1^{i_1,\cdots,i_{[\frac{p+1}{2}]}},\cdots,\tilde{\mu}_m^{i_1,\cdots,i_{[\frac{p+1}{2}]}})^T$ belong to R_+^m, $\tilde{\lambda}_{i_1,\cdots,i_{[\frac{p+1}{2}]}} = (\tilde{\lambda}_{i_1}^{j_1,\cdots,i_{[\frac{p+1}{2}]}},\cdots,\tilde{\lambda}_{i_{[\frac{p+1}{2}]}}^{j_1,\cdots,i_{[\frac{p+1}{2}]}})$ belong to $R^{[\frac{p+1}{2}]}$ such that (1) is true, and

$$d^T \nabla_x^2 L(x^*, \tilde{\lambda}_{i_1,\cdots,i_{[\frac{p+1}{2}]}}) d > 0 \tag{2}$$

for any vector $d \in G_{i_1,\cdots,i_{[\frac{p+1}{2}]}} \setminus \{0\}$, then x^* is a MC local Pareto weakly ES, that is $x \in X_M^*$, where

$$L(x, \tilde{\lambda}_{i_1,\cdots,i_{[\frac{p+1}{2}]}}) = \tilde{\mu}_{i_1,\cdots,i_{[\frac{p+1}{2}]}}^T f(x) + \sum_{m=1}^{[\frac{p+1}{2}]} \tilde{\lambda}_{i_m}^{j_1,\cdots,i_{[\frac{p+1}{2}]}} g_{i_k}(x).$$

Proof By contradiction. Suppose that x^* is not a local Pareto weakly ES, that is $x \notin X_M^*$, then there exist $x^{(k)} \in X_M, x^{(k)} \to x^*$ such that $f_v(x^{(k)}) < f_v(x^*)(v = 1,\cdots,m)$. Since $X_M = \bigcup_{i_1,\cdots,i_{[\frac{p+1}{2}]} \in \{1,\cdots,p\}} X_{i_1,\cdots,i_{[\frac{p+1}{2}]}}$, there exist $[\frac{p+1}{2}]$ numbers $i_1,\cdots,i_{[\frac{p+1}{2}]} \in \{1,\cdots,p\}$ and a subsequence of $\{x^{(k)}\}$, without loss of generality, which is denoted also as $\{x^{(k)}\}$ such that $\{x^{(k)}\} \subset X_{i_1,\cdots,i_{[\frac{p+1}{2}]}}$. By the continuity of $g(x)$ and $x^{(k)} \to x^*$, we have $x^* \in X_{i_1,\cdots,i_{[\frac{p+1}{2}]}}$. By the assumption of this theorem, there exist here exist

$\tilde{\mu}_{i_1,\cdots,i_{[\frac{p+1}{2}]}} = (\tilde{\mu}_1^{i_1,\cdots,i_{[\frac{p+1}{2}]}},\cdots,\tilde{\mu}_m^{i_1,\cdots,i_{[\frac{p+1}{2}]}})^T$ belong to R_+^m, $\tilde{\lambda}_{i_1,\cdots,i_{[\frac{p+1}{2}]}} = (\tilde{\lambda}_{i_1}^{j_1,\cdots,i_{[\frac{p+1}{2}]}},\cdots,\tilde{\lambda}_{i_{[\frac{p+1}{2}]}}^{j_1,\cdots,i_{[\frac{p+1}{2}]}})$ belong to $R^{[\frac{p+1}{2}]}$

such that (1)is true. Denote $d^{(k)} = \dfrac{x^{(k)} - x^*}{\| x^{(k)} - x^* \|}$,then $x^{(k)} = x^* + t_k d^{(k)}$ where $t_k = \| x^{(k)} - x^* \|$.

Since $\{d^{(k)}\}$ is a bounded sequence, there exists a convergent subsequence $\{d^{(k_j)}\}$, and its limit is written by $d^{(0)}$. Expanding $g_{i_m}(x)$ at x^* and taking $x = x^{(k_j)}$, we obtain

$$g_{i_m}(x^{(k_j)}) = g(x^* + t_{k_j} d^{(k_j)}) = g_{i_m}(x^*) + t_{k_j} \nabla g_{i_m}(x^*)^T d^{(k_j)} + o(t_{k_j}) \tag{3}$$

Since $g_{i_m}(x^*) = 0$ for $i_m \in I_{i_1,\cdots,i_{[\frac{p+1}{2}]}}$, and $\{x^{(k_j)}\} \subset X_{i_1,\cdots,i_{[\frac{p+1}{2}]}}$ means $g_{i_m}(x^{(k_j)}) \le 0$, it follows that from (3)

$$t_{k_j} \nabla g_{i_m}(x^*)^T d^{(k_j)} + o(t_{k_j}) \le 0 \tag{4}$$

Dividing both sides of the above equation by t_{k_j} and as $j \to \infty$,we obtain

$$\nabla g_{i_m}(x^*)^T d^{(0)} \le 0, i_m \in I_{i_1,\cdots,i_{[\frac{p+1}{2}]}}.$$

In a similar way, we get

$$\nabla f_v(x^*)^T d^{(0)} \le 0 (v = 1,\cdots,m). \tag{5}$$

Now we prove $d^{(0)} \in G_{i_1,\cdots,i_{[\frac{p+1}{2}]}}$. If $d^{(0)} \notin G_{i_1,\cdots,i_{[\frac{p+1}{2}]}}$, then there must exist a subscript $i_m \in I_{i_1,\cdots,i_{[\frac{p+1}{2}]}}$ such that $\tilde{\lambda}_{i_m}^{j_1,\cdots,i_{[\frac{p+1}{2}]}} > 0$ and $\nabla g_{i_m}(x^*)^T d^{(0)} < 0$ from the definition of set $G_{i_1,\cdots,i_{[\frac{p+1}{2}]}}$. From(1), we have

$$\tilde{\mu}^T_{i_1,\cdots,i_{[\frac{p+1}{2}]}} \nabla f(x^*)^T d^{(0)} = -\sum_{m=1}^{[\frac{p+1}{2}]} \tilde{\lambda}_{i_m}^{j_1,\cdots,i_{[\frac{p+1}{2}]}} \nabla g_{i_k}(x^*)^T d^{(0)} > 0. \tag{6}$$

By $\tilde{\mu}_{i_1,\cdots,i_{[\frac{p+1}{2}]}} = (\tilde{\mu}_1^{i_1,\cdots,i_{[\frac{p+1}{2}]}},\cdots,\tilde{\mu}_m^{i_1,\cdots,i_{[\frac{p+1}{2}]}})^T \in R_+^m$ and (5), we get $\tilde{\mu}^T_{i_1,\cdots,i_{[\frac{p+1}{2}]}} \nabla f(x^*)^T d^{(0)} \le 0$. This contradicts (6), therefore $d^{(0)} \in G_{i_1,\cdots,i_{[\frac{p+1}{2}]}}$. Now, expanding $L(x,\tilde{\lambda}_{i_1,\cdots,i_{[\frac{p+1}{2}]}})$ at x^* and taking $x = x^{(k_j)}$, we obtain

$$L(x^{(k_j)},\tilde{\lambda}_{i_1,\cdots,i_{[\frac{p+1}{2}]}}) = L(x^*,\tilde{\lambda}_{i_1,\cdots,i_{[\frac{p+1}{2}]}}) + t_{k_j}\nabla_x L(x^*,\tilde{\lambda}_{i_1,\cdots,i_{[\frac{p+1}{2}]}})^T (x^{(k_j)} - x^*)$$

$$+\frac{1}{2} t_{k_j}{}^2 (d^{(k_j)})^T \nabla_x^2 L(x^*,\tilde{\lambda}_{i_1,\cdots,i_{[\frac{p+1}{2}]}}) d^{(k_j)} + o(t_{k_j}{}^2). \tag{7}$$

Since $x^{(k_j)} \in X_{i_1,\cdots,i_{[\frac{p+1}{2}]}}$, it means $g_{i_m}(x^{(k_j)}) \le 0$. Hence, for $\tilde{\lambda}_{i_m}^{j_1,\cdots,i_{[\frac{p+1}{2}]}} \ge 0 (m = 1,\cdots,[\frac{p+1}{2}])$,

$$L(x^{(k_j)},\tilde{\lambda}_{i_1,\cdots,i_{[\frac{p+1}{2}]}}) = \tilde{\mu}^T_{i_1,\cdots,i_{[\frac{p+1}{2}]}} f(x^{(k_j)}) + \sum_{m=1}^{[\frac{p+1}{2}]} \tilde{\lambda}_{i_m}^{j_1,\cdots,i_{[\frac{p+1}{2}]}} g_{i_m}(x^{(k_j)}) \le \tilde{\mu}^T_{i_1,\cdots,i_{[\frac{p+1}{2}]}} f(x^{(k_j)}). \tag{8}$$

Since

$$L(x^*,\tilde{\lambda}_{i_1,\cdots,i_{[\frac{p+1}{2}]}}) = \tilde{\mu}^T_{i_1,\cdots,i_{[\frac{p+1}{2}]}} f(x^*), \tag{9}$$

$$\nabla_x L(x^*,\tilde{\lambda}_{i_1,\cdots,i_{[\frac{p+1}{2}]}}) = 0, \tag{10}$$

and

$$\tilde{\mu}^T_{i_1,\cdots,i_{[\frac{p+1}{2}]}} f(x^{(k_j)}) \le \tilde{\mu}^T_{i_1,\cdots,i_{[\frac{p+1}{2}]}} f(x^*) \tag{11}$$

Substituting(8)(9)(10)(11) into(7), we have

$$\frac{1}{2} t_{k_j}{}^2 (d^{(k_j)})^T \nabla_x^2 L(x^*,\tilde{\lambda}_{i_1,\cdots,i_{[\frac{p+1}{2}]}}) d^{(k_j)} + o(t_{k_j}{}^2) \le 0.$$

Dividing both sides of the above equation by $t_{k_j}{}^2$ and as $j \to \infty$, we have

$$d^T \nabla_x^2 L(x^*, \tilde{\lambda}_{i_1,\cdots,i_{[\frac{p+1}{2}]}})d \le 0 .$$

This result contradicts the hypothesis

$$d^T \nabla_x^2 L(x^*, \tilde{\lambda}_{i_1,\cdots,i_{[\frac{p+1}{2}]}})d > 0\,(d \in G_{i_1,\cdots,i_{[\frac{p+1}{2}]}} \setminus \{0\}).$$

Summary

Here the inequality constraints system of the vector optimization model has no solution, but its major inequality constraints system has solutions. So we research this new kind of vector optimization problem. We call this new type of vector optimization as MC vector optimization.

The establishment of optimality conditions of second order of vector optimization problem is very important. This paper gives the concept of MC local Pareto weakly ES. With the aid of MC set and its structural representation, we obtain a SO sufficient optimality conditions for the MC local Pareto weakly ES.

Then we will further study the stability analysis and the sensitivity analysis, and we will also study how to give its numerical methods and so on.

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