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A Novel Pythagorean Fuzzy LINMAP-Based Compromising Approach for Multiple Criteria Group Decision-Making with Preference Over Alternatives

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ABSTRACT

This paper presents a new compromising approach to multiple criteria group decision-making (MCGDM) for the treatment of uncertainty which is based on Pythagorean fuzzy (PF) sets. The present work intends to propose a novel linear programming technique for multidimensional analysis of preference (LINMAP) by way of some useful concepts related to PF dominance relations, individual consistency and inconsistency levels, and individual fit measurements. The concept of PF scalar function-based dominance measures is defined to conduct intracriterion comparisons concerning uncertain evaluation information based on Pythagorean fuzziness; moreover, several valuable properties are also investigated to demonstrate its effectiveness. For the assessment of overall dominance of alternatives, this paper provides a synthetic index, named a comprehensive dominance measure, which is the aggregation of the weighted dominance measures by combining unknown weight information and PF dominance measures of various criteria. For each decision-maker, this paper employs the proposed measures to evaluate the individual levels of rank consistency and rank inconsistency regarding the obtained overall dominance relations and the decision-maker's preference comparisons over paired alternatives. In the framework of individual fit measurements, this paper constructs bi-objective mathematical programming models and then provides their corresponding parametric linear programming models for generating the best compromise alternative. Realistic applications with some comparative analyses concerning railway project investment are implemented to demonstrate the appropriateness and usefulness of the proposed methodology in addressing actual MCGDM problems.

1. INTRODUCTION

Multiple criteria group decision-making (MCGDM) intends to prioritize a set of candidate alternatives based on a set of evaluative criteria regarding subjective preference and judgments of multiple decision-makers. The linear programming technique for multidimensional analysis of preference (LINMAP) initiated by Srinivasan and Shocker [1] provides an efficient decision-making tool to address MCGDM problems with preference information over alternatives and unknown weights of criteria [2–4]. When decision-makers provide preference relations between pairs of alternatives in the MCGDM process, the LINMAP methods can be effectively applied to resolve the objective weights of criteria and specify the best compromise choice by ranking alternatives. According to preference relations through pairwise comparisons over alternatives, a linear programming model that desires to accomplish the minimum inconsistency is established in the LINMAP procedure to determine the optimal weights of criteria objectively and generate the priority ranking results of alternatives [5–7].

However, the classical LINMAP methods cannot be directly employed to handle the decision-making problems involving fuzzy information because of the uncertainty contained in performance information or evaluation values of candidate alternatives in terms of criteria [8–10]. Accordingly, numerous studies have extended the classical LINMAP for conducting multiple criteria decision analysis in a variety of different fuzzy circumstances. For example, Wan and Li [11] emanated from LINMAP to construct an intuitionistic fuzzy programming method for handling heterogeneous MCGDM containing intuitionistic fuzzy truth degrees. Furthermore, Wan and Li [8] considered the hesitancy degrees about pairwise comparisons as interval-valued intuitionistic fuzzy sets to establish a fuzzy LINMAP-based method for conducting a heterogeneous decision analysis. Zhang *et al.* [12] utilized the LINMAP and Shapley values to develop an interval-valued intuitionistic fuzzy mathematical programming model to manage uncertain MCGDM problems. Moreover, Zhang *et al.* [13] established a mathematical programming-based approach to heterogeneous MCGDM involving aspirations and incomplete preference information. Zhang and Xu [10] applied the LINMAP structure to present an interval programming method for handling MCGDM problems

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with hesitant fuzzy ratings of alternatives and pairwise judgments between alternatives using interval numbers. Wan *et al.* [14] developed a hesitant fuzzy mathematical programming model to deal with hybrid MCGDM involving hesitant fuzzy truth degrees and incomplete criteria weight information. Xu *et al.* [15] combined hesitant fuzzy linguistic term sets with LINMAP to provide a hesitant fuzzy linguistic LINMAP method. Liu *et al.* [5] explored a double-hierarchy hesitant fuzzy linguistic mathematical programming technique for MCGDM. To multiple criteria decision analysis, Gou *et al.* [6] proposed a hesitant fuzzy linguistic possibility degree-based linear assignment method and made some comparisons with a hesitant fuzzy LINMAP. Song *et al.* [7] solved a decision-making problem with multi-stage uncertain risk on the grounds of interval grey numbers and an extended LINMAP method. Liao *et al.* [16] established a linear programming model to address decision-making issues, where the decision-maker's preferences over alternatives in connection with criteria are described as probabilistic linguistic term sets. With the assistance of LINMAP, Qin *et al.* [17] constructed some linear programming models to solve an MCGDM problem in interval type-2 fuzzy contexts. Haghighi *et al.* [3] developed a soft computing model involving interval type-2 fuzzy information by the agency of a linear assignment method and LINMAP.

Vagueness and impreciseness are unavoidable uncertainties in human evaluation processes [18]. The concept of Pythagorean fuzzy (PF) sets, initiated by Yager [19–21] and Yager and Abbasov [22], is a powerful tool in handling real-world uncertainty because PF sets slacken the prerequisite in which the sum of membership and nonmembership degrees is less than or equal to one with the square sum is less than or equal to one [23–25]. Accordingly, PF sets allow decision-makers to portray uncertain assessment data agilely and conveniently during the MCGDM process.

The LINMAP methodology has been successfully extended to diverse uncertain representations for conducting multiple criteria decision analysis or handling MCGDM issues under a variety of distinct fuzzy environments. Nonetheless, most of the current LINMAP-based techniques are not applicable to the decision environments under complex uncertainty of Pythagorean fuzziness. In this regard, Wan *et al.* [9] and Xue *et al.* [26] put forward new LINMAP-based techniques in PF uncertain circumstances. Wan *et al.* [9] utilized the information entropy to derive individual subjective criterion weight vectors for multiple decision-makers and synthesized them into a collective one via a cross-entropy optimization model. Subsequently, Wan *et al.* [9] constructed a PF mathematical programming model for addressing MCGDM problems based on the PF truth degree. Xue *et al.* [26] defined new PF entropy measures and exploited a PF LINMAP method for solving MCGDM problems. These two studies have made significant contributions to enrich the theory of PF sets and have been applied to diverse fields of green supplier selection of a smelting equipment [9] and the selection of railway project investment [26]. In contrast to abovementioned information entropy-based techniques, this paper adopts the perspective of PF representation to construct the core concept in the PF LINMAP framework. Different from Wan *et al.* [9] and Xue *et al.* [26], this paper intends to employ the concept of PF scalar functions from fuzzy rule bases and develop a novel PF LINMAP-based compromising approach to deal with MCGDM problems within PF decision environments.

It is worthwhile to mention that the concept of PF scalar functions presented by Yager [19,20] and Yager and Abbasov [22] can be utilized to facilitate comparisons among complicated PF information. Yager [20] and Yager and Abbasov [22] explored the relationship between complex numbers and Pythagorean membership grades; then they demonstrated that Pythagorean membership grades are regarded as a subclass of complex numbers called $\Pi{-}i$ numbers. Moreover, they introduced a mapping, i.e., a PF scalar function, from the $\Pi{-}i$ numbers to the unit intervals to enhance the usage of PF sets in the decision-making field. Liu and Zhang [27] further demonstrated several useful properties of PF scalar functions and obtained some partial orders on $\Pi{-}i$ numbers based on such functions. Zeng *et al.* [28] introduced the following four commonly used approaches to comparing and ranking Pythagorean membership grades in PF sets: the approach via score functions [29], the approach via score functions and accuracy functions [30], the approach via closeness indices [31], and the approach via PF scalar functions [20]. Zeng *et al.* [28] investigated several comparative examples and indicated that the approach by virtue of PF scalar functions is more helpful than the other existing comparison approaches. Consistent with Zeng *et al.*'s findings, Li and Zeng [32] also validated the effectiveness and superiority of PF scalar functions in comparing complex PF information. Moreover, Li and Zeng [32] and Zeng *et al.* [28] mentioned that the PF scalar function can thoroughly consider the influence of the fundamental characteristics of PF information. On the grounds of the PF scalar function in terms of certain anchor points of reference, Chen [33] introduced the measurement of PF precedence indices and precedence-based preference functions and further established a new PF preference ranking organization method for enrichment evaluations (PROMETHEE). Chen demonstrated the usefulness and effectiveness of the PF scalar function because its based PF precedence index can simplify the data-processing procedure and avoid the loss of high-order uncertain information. The previous studies indicated the merits of PF scalar functions; it would be desirable to apply the PF scalar function-based method to address intricate decision-making problems. Nevertheless, the concept of PF scalar functions has not employed in the current LINMAP-based methods within PF decision environments. Considering the benefits of the PF scalar function in measuring the magnitudes of Pythagorean membership grades as well as the usefulness of comparisons of PF information, this paper utilizes the PF scalar functions to present PF scalar function-based dominance measures that form the PF LINMAP basis for building a strong foundation of valuable concepts related to PF dominance relations, individual consistency and inconsistency levels, and individual fit measurements.

The primary purpose of this paper is to exploit an innovative PF LINMAP-based compromising approach for coping with MCGDM problems within PF environment. More fundamentally, this paper fully utilizes the essential characteristics of PF sets and provides a PF scalar function-based approach for determining PF dominance relations and building a core structure of the PF LINMAP procedure. First, the idea of PF scalar functions is fully employed to construct two types of PF scalar function-based dominance measures with respective to displaced and fixed reference points. Next, Types I and II comprehensive dominance measures are established for the sake of determining PF dominance relations. Some essential and important properties of the proposed measures are investigated to validate the practical usefulness of the proposed measures in synthetic comparisons for PF information. This paper utilizes the PF scalar function-based approach to specify the overall dominance relations and contrasts the obtained results with the decision-makers' paired preference relations. Based on the outcomes, this paper identifies the individual levels of rank consistency and rank inconsistency for acquiring individual fit measurements, i.e., individual goodness of fit and individual poorness of fit. Based on Types I and II dominance measures, two bi-objective mathematical programming models are formulated with the intention of maximal total group comprehensive dominance measure and minimal the group poorness of fit. To improve the computation efficiency, the bi-objective models are transformed into single-objective parametric linear programming models for simplicity. Two algorithmic procedures of the developed PF LINMAP models are presented to address MCGDM problems in PF contexts. The optimal collective weights of criteria and the individual degrees of violation yielded by the proposed models can facilitate generating the overall priority ranking orders and identifying the best compromise alternative. To examine the feasibility and effectiveness of the developed PF LINMAP approaches in practice, this paper investigates an MCGDM problem involving the railway project investment and conducts some sensitivity analyses and comparative studies.

The significant contributions of this paper are highlighted as four aspects: (i) incorporation of PF scalar functions into the core procedure for enriching the LINMAP-based methodology; (ii) exploitation of useful concepts such as PF scalar function-based (comprehensive) dominance measures and fit measurements; (iii) development of an innovative PF LINMAP-based compromising approach for manipulating subjective preferences over alternatives; and (iv) construction of applicable and effective parametric optimization models to assist multiple decision-makers in tackling MCGDM problems.

The remainder of this article is organized as follows: Section 2 concisely presents several essential concepts related to PF sets, Pythagorean membership grades, and PF scalar functions. Section 3 formulates the MCGDM problem in the uncertain context based on PF sets. Section 4 exploits novel PF scalar function-based dominance measures and explores their useful properties. Next, this section provides comprehensive dominance measures for the sake of acquiring overall dominance of alternatives. Section 5 develops some useful concepts, such as individual levels of rank consistency and inconsistency, individual fit measurements, and group comprehensive dominance measures. Based on the proposed concepts, this section employs the PF scalar function-based approach to originate a novel PF LINMAP methodology via effective parametric optimization models. Section 6 presents a practical decision-making case concerning a railway investment issue to show the implementation procedure of the developed approaches. Section 7 conducts a sensitivity analysis and makes some comparative discussions to explore the influences of relevant parameters and to evaluate the usefulness and strengths of the developed models and techniques. Lastly, Section 8 delivers the conclusions.

2. PRELIMINARY

This section describes preliminary definitions and concepts of PF sets, such as Pythagorean membership grades and PF scalar functions, all of which are necessary for the subsequent study.

Yager [19–21] and Yager and Abbasov [22] initiated a Pythagorean membership grade that belongs to a class of nonstandard membership grades. Furthermore, Li and Zeng [32] and Zeng *et al.* [28] revealed that a Pythagorean membership grade can be characterized by five parameters comprising the degrees of membership, nonmembership, and indeterminacy, the strength of commitment about membership, and the direction of commitment. Nonetheless, Chen [34] indicated that the degree of indeterminacy and the strength of commitment are dual concepts. Due to the duality of the two parameters, Chen [34,35] suggested the employment of the four parameters consisting of membership, nonmembership, strength, and direction for PF characterization. Zhang and Xu [29] presented a useful mathematical representation of PF sets. Motivated by Zhang and Xu's expressions [29], Chen [34,35] provided an effective definition and contributed a new operationally mathematical representation of PF sets involving Pythagorean membership grades.

Definition 1. [20,29,34,35] Let X be a finite universe of discourse. A PF set P is represented by the collection of a Pythagorean membership grade p toward each element $x \in X$ as follows:

$$P = \left\{ \langle x, p \rangle \,|\, x \in X \right\}. \tag{1}$$

Here, p is characterized by a series of ordered parameters consisting of the degree of membership $\mu_P(x)$, the degree of nonmembership $\nu_P(x)$, the strength of commitment $r_P(x)$, and the direction of commitment $d_P(x)$. They are defined as follows:

$$p = \left(\mu_P(x), \nu_P(x); r_P(x), d_P(x) \right), \tag{2}$$

where $\mu_P(x), \nu_P(x), r_P(x), d_P(x) : X \to [0,1]$ and for each $x \in X$:

$$0 \le \left(\mu_P(x)\right)^2 + \left(\nu_P(x)\right)^2 \le 1. \tag{3}$$

Definition 2. [19–22] Let P be a PF set in the universal set X. Let $p = \left(\mu_P(x), \nu_P(x); r_P(x), d_P(x) \right)$ represent a Pythagorean membership grade of an element $x \in X$ belonging to the set P. Let $\theta_P(x)$ be described in radians in the range $[0, \pi/2]$. The parameters $\mu_P(x)$, $\nu_P(x)$, $r_P(x)$, and $d_P(x)$ are derived as follows:

$$\mu_P(x) = r_P(x) \cdot \cos\left(\theta_P(x)\right), \tag{4}$$

$$\nu_P(x) = r_P(x) \cdot sin\left(\theta_P(x)\right), \tag{5}$$

$$r_P(x) = \sqrt{\left(\mu_P(x)\right)^2 + \left(\nu_P(x)\right)^2}, \tag{6}$$

$$d_P(x) = \frac{\pi - 2 \cdot \theta_P(x)}{\pi}, \tag{7}$$

where $\mu_P(x), \nu_P(x), r_P(x), d_P(x) \in [0,1]$.

Definition 3. [29,34,35] Let p be a Pythagorean membership grade contained by a PF set P in the universal set X. The degree of indeterminacy $\tau_P(x) : X \to [0,1]$ is given by

$$\tau_P(x) = \sqrt{1 - \left(\mu_P(x)\right)^2 - \left(\nu_P(x)\right)^2}, \tag{8}$$

This equation supports the duality property of $\tau_P(x)$ and $r_P(x)$ by virtue of $\left(\tau_P(x)\right)^2 + \left(r_P(x)\right)^2 = 1$.

For simplicity, let an ordered pair $(\mu_P(x), \nu_P(x))$ in two-dimensional Cartesian coordinates depict a geometrical illustration of the space of a Pythagorean membership grade p, as demonstrated in Figure 1. The horizontal axis of the two-dimensional plot in this figure denotes the membership degree $\mu_P(x)$, while the vertical axis denotes the nonmembership degree $\nu_P(x)$.

It is noted that the larger the value of $r_P(x)$, the stronger the commitment regarding membership at point x to the PF set P and the lower the uncertainty. The Pythagorean membership grade p allows lack of commitment and uncertainty in assigning the degrees of membership and nonmembership. Specifically, the degree $\mu_P(x)$ represents the support for the membership of x in P; in contrast, the degree $\nu_P(x)$ represents the support against the membership of x in P. The direction of $r_P(x)$ is completely toward membership if $d_P(x) = 1$ and nonmembership if $d_P(x) = 0$, whereas intermediate values of $d_P(x)$ indicate partial support for membership and nonmembership.

In particular, when $d_P(x) = 1$ (i.e., a favorable direction of commitment), it is obtained that $\theta_P(x) = 0$ because $(\pi - 2 \cdot \theta_P(x))/\pi = 1$. It follows that $\mu_P(x) = r_P(x)$ and $\nu_P(x) = 0$ by reason of $\cos(\theta_P(x)) = 1$ and $\sin(\theta_P(x)) = 0$, respectively. When $d_P(x) = 0.5$ (i.e., a neutral direction of commitment), it is derived that $\theta_P(x) = \pi/4$ and $\cos(\theta_P(x)) = \sin(\theta_P(x)) = \sqrt{2}2$, which render $\mu_P(x) = \nu_P(x) = \left((r_P(x))^2/2\right)^{0.5}$. When $d_P(x) = 0$ (i.e., a unfavorable direction of commitment), it is acquired that $\theta_P(x) = \pi/2$, $\cos(\theta_P(x)) = 0$, and $sin(\theta_P(x)) = 1$, which yield $\mu_P(x) = 0$ and $\nu_P(x) = r_P(x)$. Therefore, the closer $d_P(x)$ is to 1, the closer $\theta_P(x)$ is to 0, and the stronger the commitment $r_P(x)$ is to supporting the membership of x in the PF set P. On the contrary, the closer $d_P(x)$ is to 0, the closer $\theta_P(x)$ is to $\pi/2$, and the stronger the commitment $r_P(x)$ is to disapproving the membership of x in P.

It is worthwhile to mention that the membership degree $\mu_P(x)$ and nonmembership degree $\nu_P(x)$ are related via Pythagorean complements with respect to the strength of commitment $r_P(x)$ owing to the fact that $(\mu_P(x))^2 = (r_P(x))^2 - (\nu_P(x))^2$ and $(\nu_P(x))^2 = (r_P(x))^2 - (\mu_P(x))^2$. Next, let $p^c = (\mu_{P^c}(x), \nu_{P^c}(x); r_{P^c}(x), d_{P^c}(x))$ denote the complement of a Pythagorean membership grade p. As depicted in Figure 1, one has $\mu_{P^c}(x) = \nu_P(x)$ and $\nu_{P^c}(x) = \mu_P(x)$ by cause of the Pythagorean complement between $\mu_P(x)$ and $\nu_P(x)$ in terms of $r_P(x)$. Moreover, it follows that $r_{P^c}(x) = r_P(x)$, $d_{P^c}(x) = 1 - d_P(x)$, and $\theta_{P^c}(x) = \pi/2 - \theta_P(x)$. The complement p^c of p is specified as below:

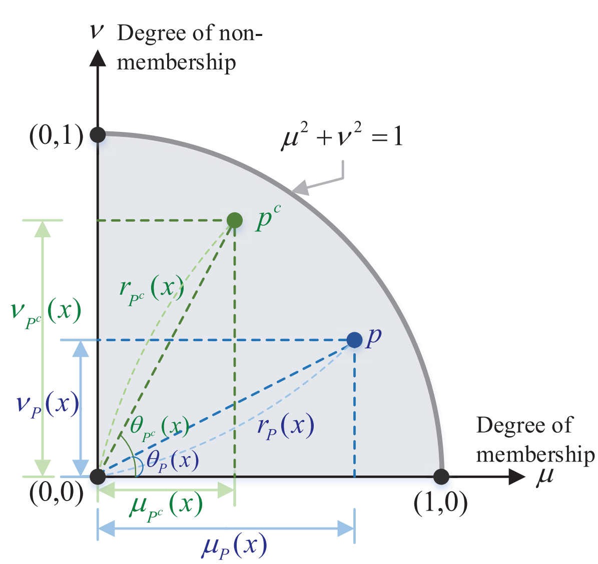


Figure 1 | Geometrical interpretation concerning the space of a Pythagorean membership grade in Pythagorean fuzzy (PF) contexts.

$$p^c = (\nu_P(x), \mu_P(x); r_P(x), 1 - d_P(x)). \tag{9}$$

Yager [19,20] employed the Takagi–Sugeno approach to propose a scalar function from fuzzy rule bases. Chen [33] indicated that Yager's scalar function can provide a representative value associated with each PF information and possess several useful and desirable properties. Liu and Zhang [27] investigated essential properties of Yager's function and studied some partial orders on Π−i numbers.

Definition 4. [19,20,33] Let $p = (\mu_P(x), \nu_P(x); r_P(x), d_P(x))$ denote a Pythagorean membership grade of an element $x \in X$ to a PF set P. The PF scalar function $V(p)$ of p is derived as follows:

$$V(p) = \frac{1}{2} + r_P(x)\left(\frac{1}{2} - \frac{2 \cdot \theta_P(x)}{\pi}\right). \tag{10}$$

Theorem 1. *For a PF set P, the PF scalar function V(p) of a Pythagorean membership grade* $p = (\mu_P(x), \nu_P(x);\ r_P(x), d_P(x)$ *with the radians* $\theta_P(x)$ *satisfies the following properties:*

(T1.1) $0 \le V(p) \le 1$;

(T1.2) $V(p) = 0$ *if and only if* $r_P(x) = 1$ *and* $\theta_P(x) = \pi/2$;

(T1.3) $V(p) = 1$ *if and only if* $r_P(x) = 1$ *and* $\theta_P(x) = 0$;

(T1.4) *V(p) decreases as* $\theta_P(x)$ *increases if* $r_P(x) > 0$;

(T1.5) *V(p) decreases as* $r_P(x)$ *increases if* $\theta_P(x) > \pi/4$;

(T1.6) *V(p) increases as* $r_P(x)$ *increases if* $\theta_P(x) < \pi/4$;

(T1.7) *V(p) is not a one-to-one mapping.*

Proof. The proofs have been given in Chen [33] and Liu and Zhang [27].

3. MCGDM PROBLEM UNDER PF UNCERTAINTY

This section intends to formulate an MCGDM problem under complex uncertainty of Pythagorean fizziness.

Consider an MCGDM problem based on PF sets. Let $Z = \{z_1, z_2, \cdots, z_m\}$ represent a discrete set of m $(m \ge 2)$ candidate alternatives. Let $C = \{c_1, c_2, \cdots, c_n\}$ be a finite set of n $(n \ge 2)$ evaluative criteria with the weight vector $\mathrm{w} = (w_1, w_2, \cdots, w_n)$, such that the weight $w_j \in [0, 1]$ for all $j \in \{1, 2, \cdots, n\}$ and $\sum_{j=1}^{n} w_j = 1$. Because all of the n criteria belong to salient attributes during an MCGDM process, nonnegative boundary conditions should be imposed into the feasible ranges of the weights. Specifically, this paper designates that $w_j \ge \varepsilon_j$ for all $c_j \in C$, where ε_j is a sufficiently small nonnegative value. Note that the weight vector w is unknown and required to be resolved, in which the weight w_j satisfies $\sum_{j=1}^{n} w_j = 1$ and

$w_j \geq \varepsilon_j$ ($\varepsilon_j \in [0,1]$) for all $j \in \{1,2,\cdots,n\}$. Let $E = \{e_1, e_2, \cdots, e_K\}$ represent the set of decision-makers participated in the MCGDM process.

Let μ_{ij}^k and ν_{ij}^k indicates the degrees of satisfaction and dissatisfaction, respectively, of the decision-maker $e_k \in E$ for the alternative $z_i \in Z$ in connection with a criterion $c_j \in C$. They are subject to $0 \leq \left(\mu_{ij}^k\right)^2 + \left(\nu_{ij}^k\right)^2 \leq 1$. The PF evaluative rating of z_i in terms of c_j provided by the decision-maker e_k is represented as a Pythagorean membership grade p_{ij}^k, in which $\mu_{ij}^k = r_{ij}^k \cdot \cos\left(\theta_{ij}^k\right)$, $\nu_{ij}^k = r_{ij}^k \cdot \sin\left(\theta_{ij}^k\right)$, $r_{ij}^k = \left(\left(\mu_{ij}^k\right)^2 + \left(\nu_{ij}^k\right)^2\right)^{0.5}$, and $d_{ij}^k = \left(\pi - 2\cdot\theta_{ij}^k\right)/\pi$ for $\theta_{ij}^k \in [0, \pi/2]$; p_{ij}^k is expressed as follows:

$$p_{ij}^k = \left(\mu_{ij}^k, \nu_{ij}^k; r_{ij}^k, d_{ij}^k\right), \tag{11}$$

where $i \in \{1,2,\cdots,m\}$, $j \in \{1,2,\cdots,n\}$, and $k \in \{1,2,\cdots,K\}$. Moreover, the degree of indeterminacy associated with each p_{ij}^k is $\tau_{ij}^k = \left(1 - \left(\mu_{ij}^k\right)^2 - \left(\nu_{ij}^k\right)^2\right)^{0.5}$.

Let P^k denote the decision-maker e_k's PF decision matrix in an MCGDM problem within the PF environment; P^k is concisely expressed as follows:

$$\mathrm{P}^k = \left[p_{ij}^k\right]_{m\times n} = \begin{array}{c} \\ z_1 \\ z_2 \\ \vdots \\ z_m \end{array} \begin{array}{c} \begin{array}{cccc} c_1 & c_2 & \cdots & c_n \end{array} \\ \begin{bmatrix} p_{11}^k & p_{12}^k & \cdots & p_{1n}^k \\ p_{21}^k & p_{22}^k & \cdots & p_{2n}^k \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1}^k & p_{m2}^k & \cdots & p_{mn}^k \end{bmatrix} \end{array} \tag{12}$$

$$= \begin{array}{c} z_1 \\ z_2 \\ \vdots \\ z_m \end{array} \begin{bmatrix} \left(\mu_{11}^k, \nu_{11}^k; r_{11}^k, d_{11}^k\right) & \left(\mu_{12}^k, \nu_{12}^k; r_{12}^k, d_{12}^k\right) & \cdots & \left(\mu_{1n}^k, \nu_{1n}^k; r_{1n}^k, d_{1n}^k\right) \\ \left(\mu_{21}^k, \nu_{21}^k; r_{21}^k, d_{21}^k\right) & \left(\mu_{22}^k, \nu_{22}^k; r_{22}^k, d_{22}^k\right) & \cdots & \left(\mu_{2n}^k, \nu_{2n}^k; r_{2n}^k, d_{2n}^k\right) \\ \vdots & \vdots & \ddots & \vdots \\ \left(\mu_{m1}^k, \nu_{m1}^k; r_{m1}^k, d_{m1}^k\right) & \left(\mu_{m2}^k, \nu_{m2}^k; r_{m2}^k, d_{m2}^k\right) & \cdots & \left(\mu_{mn}^k, \nu_{mn}^k; r_{mn}^k, d_{mn}^k\right) \end{bmatrix}.$$

It is worth noting that the PF evaluative ratings can be conveniently acquired through the medium of applicable linguistic rating scales. Consider a nine-point rating scale as an example. Table 1 depicts some practical and beneficial approaches to evaluate the alternatives by use of nine-point linguistic scales. To be specific, Gündoğdu [36] introduced a useful nine-point linguistic scale for facilitating the construction of the evaluation matrix based on PF sets and spherical fuzzy sets. Mete [37], Oz *et al.* [38], and Pérez-Domínguez *et al.* [39] employed an identical nine-point scale for expressing linguistic evaluations in PF settings. Rani *et al.* [40] introduced a nine-point linguistic scale for assisting decision-makers' judgments concerning the performance evaluations of alternatives in terms of criteria. Seker and Aydin [41] provided nine-point linguistic terms and their corresponding interval-valued PF numbers that can be used for the evaluation of alternatives with criteria. By applying a middle-point approach, these interval-valued PF numbers can be converted into Pythagorean membership grades, and the obtained results are shown in Table 1.

Decision-makers can express the linguistic evaluation values of alternatives toward each criterion. Afterward, these linguistic terms are transformed into PF evaluative ratings by the agency of the linguistic rating system in Table 1. Accordingly, the PF decision matrices from decision-makers can be determined using an appropriate linguistic rating system (e.g., the linguistic terms given in Table 1).

The decision-maker e_k provides individual preference relations between the alternatives in the set Z with the preference set Ω^k in which Ω^k denotes a set of ordered pairs (ϕ, φ) and is defined as follows:

$$\Omega^k = \left\{(\phi, \varphi)\,|z_\phi \succeq z_\varphi,\ \phi, \varphi \in \{1,2,\cdots,m\}\right\}, \tag{13}$$

for $k \in \{1,2,\cdots,K\}$, where the preference relation $z_\phi \succeq z_\varphi$ signifies that the alternative z_ϕ is noninferior to the alternative z_φ. Namely, either z_ϕ is preferred to z_φ or z_ϕ and z_φ are evenly preferred for the decision-maker e_k.

Table 1 Pythagorean fuzzy (PF) evaluative ratings in relation to linguistic scales.

Source	Linguistic Term	PF Evaluative Rating
Gündoğdu [36]	Absolutely high	(0.90, 0.10; 0.91, 0.93)
	Very high	(0.80, 0.20; 0.82, 0.84)
	High	(0.70, 0.30; 0.76, 0.74)
	Slightly high	(0.60, 0.40; 0.72, 0.63)
	Fair	(0.50, 0.50; 0.71, 0.50)
	Slightly low	(0.40, 0.60; 0.72, 0.37)
	Low	(0.30, 0.70; 0.76, 0.26)
	Very low	(0.20, 0.80; 0.82, 0.16)
	Absolutely low	(0.10, 0.90; 0.91, 0.07)
Mete [37]; Oz *et al.* [38]; Pérez-Domínguez [39]	Absolutely high	(1.00, 0.10; 1.00, 0.94)
	Very high	(0.80, 0.44; 0.91, 0.68)
	High	(0.70, 0.60; 0.92, 0.55)
	Moderately high	(0.60, 0.71; 0.93, 0.45)
	Medium	(0.50, 0.80; 0.94, 0.36)
	Moderately low	(0.40, 0.87; 0.96, 0.27)
	Low	(0.25, 0.92; 0.95, 0.17)
	Very low	(0.10, 0.97; 0.98, 0.07)
	Absolutely low	(0.10, 0.99; 1.00, 0.06)
Rani *et al.* [40]	Absolutely high	(0.98, 0.20; 1.00, 0.87)
	Very high	(0.87, 0.35; 0.94, 0.76)
	High	(0.70, 0.40; 0.81, 0.67)
	Medium high	(0.65, 0.45; 0.79, 0.61)
	Average	(0.50, 0.55; 0.74, 0.47)
	Medium low	(0.40, 0.70; 0.81, 0.33)
	Low	(0.36, 0.80; 0.88, 0.27)
	Very low	(0.25, 0.87; 0.91, 0.18)
	Very very low	(0.20, 0.98; 1.00, 0.13)
Seker and Aydin [41]	Absolutely good	(0.81, 0.10; 0.82, 0.92)
	Very good	(0.72, 0.19; 0.74, 0.84)
	Good	(0.63, 0.28; 0.69, 0.73)
	Medium good	(0.54, 0.37; 0.65, 0.62)
	Fair	(0.45, 0.45; 0.64, 0.50)
	Medium bad	(0.37, 0.54; 0.65, 0.38)
	Bad	(0.28, 0.63; 0.69, 0.27)
	Very bad	(0.19, 0.72; 0.74, 0.16)
	Absolutely bad	(0.10, 0.81; 0.82, 0.08)

Theoretically, the preference set Ω^k contains at most $m(m-1)/2$ paired comparison judgments. Nonetheless, the decision-makers frequently express incomplete preference judgments toward two alternatives in real-world situations. That is, only partial preference relations are filled in by the decision-makers' subjective judgments. For instance, there are five candidate alternatives in Z, i.e., $Z = \{z_1, z_2, \cdots, z_5\}$. Suppose that the decision-maker e_k provides his/her confirmed preference relations $z_3 \succeq z_1$, $z_2 \succeq z_4$, and $z_3 \succeq z_4$, i.e., $\Omega^k = \{(3,1), (2,4), (3,4)\}$. In this case, there are only three paired comparison judgments in Ω^k. Another issue may be faced with preference conflicts regarding some decision-makers' ordered pairs. For example, the preference relation (3, 5) (i.e., $z_3 \succeq z_5$) in the set Ω^1 is filled in by the decision-maker e_1's subjective judgments, while the inconsistent relation (5, 3) (i.e., $z_5 \succeq z_3$) in the set Ω^2 is filled in by the decision-maker e_2's subjective judgments. In this regard, this paper would like to propose an effective procedure in the PF LINMAP method for the sake of overcoming the difficulty about incomplete and inconsistent information.

To deal with partial preference relations based on human subjective judgments, this paper devises a scheme that comes as close as possible to meet most of the decision-makers' paired preference relations in developing the PF LINMAP methods. In particular, this paper defines a useful synthetic index, named a comprehensive dominance measure, to evaluate the overall dominance of competing alternatives. The obtained overall dominance relations are contrasted with paired preference relations in the preference sets Ω^k for all $k \in \{1, 2, \cdots, K\}$. No errors are attributed to the consistency between paired preference relations and overall dominance relations; whereas errors are to be minimized through the subsequent PF LINMAP models. Moreover, this paper presents new consistency and inconsistency measurements for handling incomplete and inconsistent information among the sets Ω^k in the proposed PF LINMAP procedure.

4. DOMINANCE MEASURE VIA PF SCALAR FUNCTIONS

This section introduces different points of reference for PF information and establishes two types of PF scalar function-based dominance measures for facilitating intracriterion comparisons of PF evaluative ratings.

In general, different points of reference might have distinct influences on the contrast of various PF evaluative ratings, which would result in the change of the decision-makers' preference intensities. In this respect, this paper considers two types of reference points to define the PF scalar function-based dominance measures. First, the displaced reference points (i.e., the positive-ideal and negative-ideal reference points) under anchored judgments are utilized to construct the Type I dominance measure. Second, this paper regard the largest Pythagorean membership grade (1, 0; 1, 1) and the smallest grade (0, 1; 1, 0) as the fixed reference points for presenting the Type II dominance measure.

Definition 5. (Displaced reference points) For each decision-maker e_k's PF decision matrix $\mathrm{P}^k = [p_{ij}^k]_{m\times n}$, the positive-ideal reference point p_{*j}^k and the negative-ideal reference point $p_{\#j}^k$ in connection with each criterion $c_j \in C$ are derived as follows:

$$p_{*j}^k = \left(\mu_{*j}^k, \nu_{*j}^k; r_{*j}^k, d_{*j}^k\right) = \left(\max_{i=1}^m \mu_{ij}^k, \min_{i=1}^m \nu_{ij}^k; \sqrt{\left(\max_{i=1}^m \mu_{ij}^k\right)^2 + \left(\min_{i=1}^m \nu_{ij}^k\right)^2}, \frac{\pi - 2\cdot\theta_{*j}^k}{\pi}\right), \tag{14}$$

$$p_{\#j}^k = \left(\mu_{\#j}^k, \nu_{\#j}^k; r_{\#j}^k, d_{\#j}^k\right) = \left(\min_{i=1}^m \mu_{ij}^k, \max_{i=1}^m \nu_{ij}^k; \sqrt{\left(\min_{i=1}^m \mu_{ij}^k\right)^2 + \left(\max_{i=1}^m \nu_{ij}^k\right)^2}, \frac{\pi - 2\cdot\theta_{\#j}^k}{\pi}\right), \tag{15}$$

where $\theta_{*j}^k = \cos^{-1}\left(\mu_{*j}^k / r_{*j}^k\right) = \sin^{-1}\left(\nu_{*j}^k / r_{*j}^k\right)$ and $\theta_{\#j}^k = \cos^{-1}\left(\mu_{\#j}^k / r_{\#j}^k\right) = \sin^{-1}\left(\nu_{\#j}^k / r_{\#j}^k\right)$.

Definition 6. (Fixed reference points) For each decision-maker e_k, the largest fixed reference point p_{+j}^k and the smallest fixed reference point p_{-j}^k in connection with each criterion $c_j \in C$ are given by

$$p_{+j}^k = \left(\mu_{+j}^k, \nu_{+j}^k; r_{+j}^k, d_{+j}^k\right) = (1, 0; 1; 1), \tag{16}$$

$$p_{-j}^k = \left(\mu_{-j}^k, \nu_{-j}^k; r_{-j}^k, d_{-j}^k\right) = (0, 1; 1, 0), \tag{17}$$

where $\theta_{+j}^k = \cos^{-1}(1/1) = sin^{-1}(0/1) = 0$ and $\theta_{-j}^k = \cos^{-1}(0/1) = sin^{-1}(1/1) = 1.5708$.

Based on Definition 2, the radians θ_{ij}^k associated with a PF evaluative rating p_{ij}^k can be acquired by $\theta_{ij}^k = (\pi/2)\cdot\left(1 - d_{ij}^k\right)$. Accordingly, the PF scalar function $V\left(p_{ij}^k\right)$ can be determined as follows:

$$V\left(p_{ij}^k\right) = \frac{1}{2} + r_{ij}^k\left(d_{ij}^k - \frac{1}{2}\right). \tag{18}$$

Theorem 2. *The PF scalar function $V\left(p_{ij}^k\right)$ meets the following properties:*

(T2.1) $0 \le V\left(p_{ij}^k\right) \le 1$;

(T2.2) $V\left(p_{ij}^k\right) = 0$ if and only if $r_{ij}^k = 1$ and $d_{ij}^k = 0$;

(T2.3) $V\left(p_{ij}^k\right) = 0.5$ if and only if either $r_{ij}^k = 0$ or $d_{ij}^k = 0.5$;

(T2.4) $V\left(p_{ij}^k\right) = 1$ if and only if $r_{ij}^k = d_{ij}^k = 1$;

(T2.5) $V\left(p_{ij}^k\right)$ increases as d_{ij}^k increases if $r_{ij}^k > 0$;

(T2.6) $V\left(p_{ij}^k\right)$ increases as r_{ij}^k increases if $d_{ij}^k > 0.5$;

(T2.7) $V\left(p_{ij}^k\right)$ decreases as r_{ij}^k increases if $d_{ij}^k < 0.5$.

Proof. (T2.1) is obvious from (T1.1). For the necessity of (T2.2), if $V\left(p_{ij}^k\right) = 0$, the condition $r_{ij}^k\left(d_{ij}^k - 0.5\right) = -0.5$ must be fulfilled.

This indicates that $r_{ij}^k = 1$ and $d_{ij}^k = 0$ because $r_{ij}^k, d_{ij}^k \in [0,1]$. For the sufficiency of (T2.2), if $r_{ij}^k = 1$ and $d_{ij}^k = 0$, then $V\left(p_{ij}^k\right) = 0$. Thus, (T2.2) is correct. For the necessity of (T2.3), if $V\left(p_{ij}^k\right) = 0.5$, then $r_{ij}^k\left(d_{ij}^k - 0.5\right) = 0$ must hold, which follows that either $r_{ij}^k = 0$ or $d_{ij}^k = 0.5$. For the sufficiency of (T2.3), either $r_{ij}^k = 0$ or $d_{ij}^k = 0.5$ will result in $V\left(p_{ij}^k\right) = 0.5$, i.e., (T2.3) is valid. For the necessity of (T2.4), if $V\left(p_{ij}^k\right) = 1$, the condition $r_{ij}^k\left(d_{ij}^k - 0.5\right) = 0.5$ must be satisfied, which implies that $r_{ij}^k = d_{ij}^k = 1$. For the sufficiency of (T2.4), it is easy to see $V\left(p_{ij}^k\right) = 1$ if $r_{ij}^k = d_{ij}^k = 1$. Thus, (T2.4) is valid. For (T2.5), the partial derivative of $V\left(p_{ij}^k\right)$ with respect to d_{ij}^k is obtained as follows:

$$\frac{\partial V\left(p_{ij}^k\right)}{\partial d_{ij}^k} = \frac{\partial\left(0.5 + r_{ij}^k\left(d_{ij}^k - 0.5\right)\right)}{\partial d_{ij}^k} = r_{ij}^k.$$

Note that $r_{ij}^k \in [0,1]$. It can be concluded that $V\left(p_{ij}^k\right)$ increases as d_{ij}^k increases from 0 to 1 if $r_{ij}^k > 0$. For (T2.6) and (T2.7), the partial derivative of $V\left(p_{ij}^k\right)$ with respect to r_{ij}^k is derived as follows:

$$\frac{\partial V\left(p_{ij}^k\right)}{\partial r_{ij}^k} = \frac{\partial\left(0.5 + r_{ij}^k\left(d_{ij}^k - 0.5\right)\right)}{\partial r_{ij}^k} = d_{ij}^k - 0.5,$$

where $d_{ij}^k \in [0,1]$. It is easy to see that $\partial V\left(p_{ij}^k\right)/\partial r_{ij}^k > 0$ and $\partial V\left(p_{ij}^k\right)/\partial r_{ij}^k < 0$ in the cases of $d_{ij}^k > 0.5$ and $d_{ij}^k < 0.5$, respectively. Thus, $V\left(p_{ij}^k\right)$ increases as r_{ij}^k increases if $d_{ij}^k > 0.5$. In contrast, $V\left(p_{ij}^k\right)$ decreases as r_{ij}^k increases if $d_{ij}^k < 0.5$. Therefore, (T2.6) and (T2.7) are valid. This completes the proof.

Definition 7. Regarding a PF evaluative rating p_{ij}^k, the PF scalar function-based dominance measure when anchoring p_{*j}^k and $p_{\#j}^k$, denoted by Type I, is defined as follows:

$$M^{\mathrm{I}}\left(p_{ij}^k\right) = \frac{\left|V\left(p_{ij}^k\right) - V\left(p_{\#j}^k\right)\right|}{\left|V\left(p_{ij}^k\right) - V\left(p_{*j}^k\right)\right| + \left|V\left(p_{ij}^k\right) - V\left(p_{\#j}^k\right)\right|}. \quad (19)$$

The PF scalar function-based dominance measure of p_{ij}^k when anchoring p_{+j}^k and p_{-j}^k, denoted by Type II, is defined as follows:

$$M^{\mathrm{II}}\left(p_{ij}^k\right) = \frac{\left|V\left(p_{ij}^k\right) - V\left(p_{-j}^k\right)\right|}{\left|V\left(p_{ij}^k\right) - V\left(p_{+j}^k\right)\right| + \left|V\left(p_{ij}^k\right) - V\left(p_{-j}^k\right)\right|}. \quad (20)$$

Theorem 3. *The Type I dominance measure $M^I\left(p_{ij}^k\right)$ corresponding to a PF evaluative rating p_{ij}^k meets the following properties:*

(T3.1) $0 \le M^I\left(p_{ij}^k\right) \le 1$;

(T3.2) If $p_{ij}^k = p_{\#j}^k$, then $M^I\left(p_{ij}^k\right) = 0$;

(T3.3) If $p_{ij}^k = p_{*j}^k$, then $M^I\left(p_{ij}^k\right) = 1$.

Proof. For (T3.1), it is known that the PF scalar functions $0 \le V\left(p_{ij}^k\right), V\left(p_{*j}^k\right), V\left(p_{\#j}^k\right) \le 1$ from (T1.1). Thus, it can be verified $0 \le M^{\mathrm{I}}\left(p_{ij}^k\right) \le 1$. For (T3.2), the condition $p_{ij}^k = p_{\#j}^k$ leads to $\left|V\left(p_{ij}^k\right) - V\left(p_{\#j}^k\right)\right| = 0$, which indicates that $M^{\mathrm{I}}\left(p_{ij}^k\right) = 0$. For (T3.3), the condition $p_{ij}^k = p_{*j}^k$ leads to $\left|V\left(p_{ij}^k\right) - V\left(p_{*j}^k\right)\right| = 0$. It follows that $M^{\mathrm{I}}\left(p_{ij}^k\right) = \left|V\left(p_{ij}^k\right) - V\left(p_{\#j}^k\right)\right| / \left|V\left(p_{ij}^k\right) - V\left(p_{\#j}^k\right)\right| = 1$. This completes the proof.

Theorem 4. *The Type II dominance measure $M^{II}\left(p_{ij}^k\right)$ corresponding to a PF evaluative rating p_{ij}^k meets the following properties:*

(T4.1) $0 \le M^{II}\left(p_{ij}^k\right) \le 1$;

(T4.2) If $p_{ij}^k = p_{-j}^k$, then $M^{II}\left(p_{ij}^k\right) = 0$;

(T4.3) If $p_{ij}^k = p_{+j}^k$, then $M^{II}\left(p_{ij}^k\right) = 1$.

Proof. The proofs of (T4.1)–(T4.3) are similar to those of (T3.1)–(T3.3), respectively.

Theorem 5. *The PF scalar function-based dominance measures $M^I\left(p_{ij}^k\right)$ and $M^{II}\left(p_{ij}^k\right)$ of a PF evaluative rating p_{ij}^k can be determined as follows:*

$$M^I\left(p_{ij}^k\right) = \frac{V\left(p_{ij}^k\right) - V\left(p_{\#j}^k\right)}{V\left(p_{*j}^k\right) - V\left(p_{\#j}^k\right)}, \quad (21)$$

$$M^{II}\left(p_{ij}^k\right) = V\left(p_{ij}^k\right). \quad (22)$$

Proof. From Definition 5, it is easily observed that $\mu_{ij}^k \ge \mu_{\#j}^k$ and $\nu_{ij}^k \le \nu_{\#j}^k$ because $\mu_{\#j}^k = \min_{i=1}^m \mu_{ij}^k$ and $\nu_{\#j}^k = \max_{i=1}^m \nu_{ij}^k$, respectively. The directions of commitments d_{ij}^k and $d_{\#j}^k$ indicate on a scale from 0 to 1 how fully the strengths r_{ij}^k and $r_{\#j}^k$, respectively, are pointing toward membership. Hence, it can be inferred that $d_{ij}^k \ge d_{\#j}^k$, which implies that $V\left(p_{ij}^k\right) \ge V\left(p_{\#j}^k\right)$ based on (T2.5). Analogously, one has $\mu_{ij}^k \le \mu_{*j}^k$ and $\nu_{ij}^k \ge \nu_{*j}^k$ based on $\mu_{*j}^k = \max_{i=1}^m \mu_{ij}^k$ and $\nu_{*j}^k = \min_{i=1}^m \nu_{ij}^k$, respectively. It follows that

$V\left(p_{ij}^k\right) \le V\left(p_{*j}^k\right)$. Thus, Eq. (19) becomes

$$M^I\left(p_{ij}^k\right) = \frac{V\left(p_{ij}^k\right) - V\left(p_{\#j}^k\right)}{\left(V\left(p_{*j}^k\right) - V\left(p_{ij}^k\right)\right) + \left(V\left(p_{ij}^k\right) - V\left(p_{\#j}^k\right)\right)} = \frac{V\left(p_{ij}^k\right) - V\left(p_{\#j}^k\right)}{V\left(p_{*j}^k\right) - V\left(p_{\#j}^k\right)},$$

which indicates that Eq. (21) is correct. Next, it can be easily obtained that $V\left(p_{-j}^k\right) = 0$ and $V\left(p_{+j}^k\right) = 1$ because $p_{-j}^k = (0,1;1,0)$ and $p_{+j}^k = (1,0;1;1)$, respectively. Thus, one has $M^{\mathrm{II}}\left(p_{ij}^k\right) e) = \left|V\left(p_{ij}^k\right) - 0\right| / \left(\left|V\left(p_{ij}^k\right) - 1\right| + \left|V\left(p_{ij}^k\right) - 0\right|\right) = V\left(p_{ij}^k\right)$, i.e., Eq. (22) is correct. This completes the proof.

Theorem 6. *For two PF evaluative ratings p_{ij}^k and $p_{i'j}^k$ in terms of a specific criterion $c_j \in C$, if $p_{ij}^k \le p_{i'j}^k$ (i.e., $\mu_{ij}^k \le \mu_{i'j}^k$ and $\nu_{ij}^k \ge \nu_{i'j}^k$), then $M^I\left(p_{ij}^k\right) \le M^I\left(p_{i'j}^k\right)$ and $M^{II}\left(p_{ij}^k\right) \le M^{II}\left(p_{i'j}^k\right)$.*

Proof. According to a natural quasi-ordering on the space of Pythagorean membership grades [20,31], $p_{ij}^k \le p_{i'j}^k$ if and only if $\mu_{ij}^k \le \mu_{i'j}^k$ and $\nu_{ij}^k \ge \nu_{i'j}^k$. Meanwhile, the directions of commitments d_{ij}^k and $d_{i'j}^k$ represent how the strengths r_{ij}^k and $r_{i'j}^k$, respectively, are pointing toward membership, which follows that $d_{ij}^k \le d_{i'j}^k$. Hence, it can be inferred that $V\left(p_{ij}^k\right) \le V\left(p_{i'j}^k\right)$ based on (T2.5). For the decision-maker e_k, the displaced reference points are the same for the alternatives in the set Z, More concretely, the $V\left(p_{\#j}^k\right)$ values are identical for the alternatives z_i and $z_{i'}$ because z_i and $z_{i'}$ have the same negative-ideal reference point $p_{\#j}^k$. Analogously, the $V\left(p_{*j}^k\right)$ values are identical for z_i and $z_{i'}$ because of the same positive-ideal reference point p_{*j}^k. Moreover, it is known that $V\left(p_{ij}^k\right) - V\left(p_{\#j}^k\right) \le V\left(p_{i'j}^k\right) - V\left(p_{\#j}^k\right)$ because $V\left(p_{ij}^k\right) \le V\left(p_{i'j}^k\right)$. Therefore, it can be proven that $M^{\mathrm{I}}\left(p_{ij}^k\right) \le M^{\mathrm{I}}\left(p_{i'j}^k\right)$ and $M^{\mathrm{II}}\left(p_{ij}^k\right) \le M^{\mathrm{II}}\left(p_{i'j}^k\right)$. This completes the proof.

As demonstrated in Theorems 3–6, the PF scalar function-based dominance measures $M^{\mathrm{I}}\left(p_{ij}^k\right)$ and $M^{\mathrm{II}}\left(p_{ij}^k\right)$ possess several important and useful properties. Accordingly, they can be thoroughly employed to identify the PF dominance relationships about PF evaluative ratings because of the practical usefulness in intracriterion comparisons. To further assess the overall dominance of alternatives, this paper incorporates the criterion weights into Types I and II dominance measures to propose a synthetic index, named a comprehensive dominance measure.

Definition 8. Let $w = (w_1, w_2, \cdots, w_n)$ be the weight vector of n criteria. Based on Types I and II dominance measures $M^{\mathrm{I}}\left(p_{ij}^k\right)$ and $M^{\mathrm{II}}\left(p_{ij}^k\right)$, the comprehensive dominance measures of the alternative $z_i \in Z$ for the decision-maker $e_k \in E$ are defined as follows:

$$\mathbb{C}_i^{\mathrm{I},k} = \sum_{j=1}^n w_j \cdot M^{\mathrm{I}}\left(p_{ij}^k\right), \tag{23}$$

$$\mathbb{C}_i^{\mathrm{II},k} = \sum_{j=1}^n w_j \cdot M^{\mathrm{II}}\left(p_{ij}^k\right). \tag{24}$$

Theorem 7. *The Type I comprehensive dominance measure $\mathbb{C}_i^{I,k}$ meets the following properties:*

(T7.1) $0 \le \mathbb{C}_i^{I,k} \le 1$;

(T7.2) If $p_{ij}^k = p_{\#j}^k$ for all $j \in \{1, 2, \cdots, n\}$, then $\mathbb{C}_i^{I,k} = 0$;

(T7.3) If $p_{ij}^k = p_{*j}^k$ for all $j \in \{1, 2, \cdots, n\}$, then $\mathbb{C}_i^{I,k} = 1$;

(T7.4) If $p_{ij}^k \le p_{i'j}^k$ for all $j \in \{1, 2, \cdots, n\}$, then $\mathbb{C}_i^{I,k} \le \mathbb{C}_{i'}^{I,k}$.

Proof. For (T7.1), it is obvious that $0 \le \mathbb{C}_i^{I,k} \le 1$ according to the normalization conditions of criterion weights (i.e., $0 \le w_j \le 1$ for all $j \in \{1, 2, \cdots, n\}$ and $\sum_{j=1}^n w_j = 1$) and the property in (T3.1) (i.e., $0 \le M^{\mathrm{I}}\left(p_{ij}^k\right) \le 1$). For (T7.2), based on (T3.2), it is known that $M^{\mathrm{I}}\left(p_{ij}^k\right) = 0$ because of the premise condition $p_{ij}^k = p_{\#j}^k$ for all j. It indicates that $\mathbb{C}_i^{\mathrm{I},k} = 0$. For (T7.3), one can easily conclude that $\mathbb{C}_i^{\mathrm{I},k} = 1$ because $M^{\mathrm{I}}\left(p_{ij}^k\right) = 1$ from (T3.3) according to the condition $p_{ij}^k = p_{*j}^k$ for all j. For (T7.4), by applying Theorem 6, the premise condition $p_{ij}^k \le p_{i'j}^k$ indicates that $M^{\mathrm{I}}\left(p_{ij}^k\right) \le M^{\mathrm{I}}\left(p_{i'j}^k\right)$. Thus, it can be inferred that $\mathbb{C}_i^{\mathrm{I},k} \le \mathbb{C}_{i'}^{\mathrm{I},k}$ because $w_j \cdot M^{\mathrm{I}}\left(p_{ij}^k\right) \le w_j \cdot M^{\mathrm{I}}\left(p_{i'j}^k\right)$ for all j. This completes the proof.

Theorem 8. *The Type II comprehensive dominance measure $\mathbb{C}_i^{II,k}$ satisfies the following properties:*

(T8.1) $\mathbb{C}_i^{II,k} = \sum_{j=1}^n w_j \cdot V\left(p_{ij}^k\right)$;

(T8.2) $0 \le \mathbb{C}_i^{II,k} \le 1$;

(T8.3) If $p_{ij}^k = p_{-j}^k$ for all $j \in \{1, 2, \cdots, n\}$, then $\mathbb{C}_i^{II,k} = 0$;

(T8.4) If $p_{ij}^k = p_{+j}^k$ for all $j \in \{1, 2, \cdots, n\}$, then $\mathbb{C}_i^{II,k} = 1$;

(T8.5) If $p_{ij}^k \le p_{i'j}^k$ for all $j \in \{1, 2, \cdots, n\}$, then $\mathbb{C}_i^{II,k} \le \mathbb{C}_{i'}^{II,k}$.

Proof. (T8.1) can be easily checked based on Theorem 5. The proofs of (T8.2)–(T8.5) are analogous to those of (T7.1)–(T7.4), respectively.

5. PROPOSED PF LINMAP METHODOLOGY

This section utilizes the PF scalar function-based appoach to bring forward a novel PF LINMAP methodology. Moreover, two effective algorithmic procedures based on Types I and II dominance measures are also provided to facilitate solving MCGDM problems in PF contexts.

The proposed comprehensive dominance measures can be utilized to specify the overall dominance relations among candidate alternatives. Recall that the preference set Ω^k is composed of the stated ordered pairs that represent the subjective preference relations between alternatives provided by the decision-maker e_k. Based on the PF scalar function-based approach, this paper determines the overall dominance relations yielded by Type I or Type II comprehensive dominance measures. These results are contrasted with preference information provided by all decision-makers. More concretely, no error can be attributed to the paired preference relation between alternatives z_ϕ and z_ϕ for the ordered pair $(\phi,\varphi)\in\Omega^k$ if $\mathbb{C}_\phi^{\mathrm{I},k}\geq\mathbb{C}_\varphi^{\mathrm{I},k}$ and $\mathbb{C}_\phi^{\mathrm{II},k}\geq\mathbb{C}_\varphi^{\mathrm{II},k}$ based on Types I and II dominance measures, respectively, whereas errors can be found if $\mathbb{C}_\phi^{\mathrm{I},k}<\mathbb{C}_\varphi^{\mathrm{I},k}$ or $\mathbb{C}_\phi^{\mathrm{II},k}<\mathbb{C}_\varphi^{\mathrm{II},k}$.

It is noteworthy that the decision-makers may indicate partial preference relations for the alternatives. Moreover, preference conflicts may exist among the decision-makers' subjective judgments. Accordingly, the comparison results between the stated subjective preferences and the overall dominance relations are somewhat conflicting. This paper employs the proposed comprehensive dominance measures to define the individual levels of rank consistency and rank inconsistency regarding the obtained overall dominance relations and the decision-maker's paired comparison judgments about alternatives. Based on the Type I comprehensive dominance measure, the individual level of rank consistency between the preorders of the alternatives z_ϕ and z_ϕ for the ordered pair $(\phi,\varphi)\in\Omega^k$ $(k\in\{1,2,\cdots,K\})$ is determined as follows:

$$\left(\mathbb{C}_\phi^{\mathrm{I},k}-\mathbb{C}_\varphi^{\mathrm{I},k}\right)^+=\begin{cases}\mathbb{C}_\phi^{\mathrm{I},k}-\mathbb{C}_\varphi^{\mathrm{I},k} & \text{if } \mathbb{C}_\phi^{\mathrm{I},k}\geq\mathbb{C}_\varphi^{\mathrm{I},k},\\ 0 & \text{if } \mathbb{C}_\phi^{\mathrm{I},k}<\mathbb{C}_\varphi^{\mathrm{I},k},\end{cases}$$
$$=\max\left\{0,\mathbb{C}_\phi^{\mathrm{I},k}-\mathbb{C}_\varphi^{\mathrm{I},k}\right\}. \tag{25}$$

Moreover, based on the Type II comprehensive dominance measure, the individual level of rank consistency for each $(\phi,\varphi)\in\Omega^k$ is defined as follows:

$$\left(\mathbb{C}_\phi^{\mathrm{II},k}-\mathbb{C}_\varphi^{\mathrm{II},k}\right)^+=\begin{cases}\mathbb{C}_\phi^{\mathrm{II},k}-\mathbb{C}_\varphi^{\mathrm{II},k} & \text{if } \mathbb{C}_\phi^{\mathrm{II},k}\geq\mathbb{C}_\varphi^{\mathrm{II},k},\\ 0 & \text{if } \mathbb{C}_\phi^{\mathrm{II},k}<\mathbb{C}_\varphi^{\mathrm{II},k},\end{cases}$$
$$=\max\left\{0,\mathbb{C}_\phi^{\mathrm{II},k}-\mathbb{C}_\varphi^{\mathrm{II},k}\right\}. \tag{26}$$

Note that $\left(\mathbb{C}_\phi^{\mathrm{I},k}-\mathbb{C}_\varphi^{\mathrm{I},k}\right)^+\geq 0$ and $\left(\mathbb{C}_\phi^{\mathrm{II},k}-\mathbb{C}_\varphi^{\mathrm{II},k}\right)^+\geq 0$.

In contrast, this paper utilizes the Type I comprehensive dominance measure to define the individual level of rank inconsistency between the preorders of z_ϕ and z_ϕ for each $(\phi,\varphi)\in\Omega^k$ as follows:

$$\left(\mathbb{C}_\phi^{\mathrm{I},k}-\mathbb{C}_\varphi^{\mathrm{I},k}\right)^-=\begin{cases}\mathbb{C}_\varphi^{\mathrm{I},k}-\mathbb{C}_\phi^{\mathrm{I},k} & \text{if } \mathbb{C}_\phi^{\mathrm{I},k}<\mathbb{C}_\varphi^{\mathrm{I},k},\\ 0 & \text{if } \mathbb{C}_\phi^{\mathrm{I},k}\geq\mathbb{C}_\varphi^{\mathrm{I},k},\end{cases}$$
$$=\max\left\{0,\mathbb{C}_\varphi^{\mathrm{I},k}-\mathbb{C}_\phi^{\mathrm{I},k}\right\}. \tag{27}$$

Additionally, based on the Type II comprehensive dominance measure, the individual level of rank inconsistency for each $(\phi,\varphi)\in\Omega^k$ is defined as follows:

$$\left(\mathbb{C}_\phi^{\mathrm{II},k}-\mathbb{C}_\varphi^{\mathrm{II},k}\right)^-=\begin{cases}\mathbb{C}_\varphi^{\mathrm{II},k}-\mathbb{C}_\phi^{\mathrm{II},k} & \text{if } \mathbb{C}_\phi^{\mathrm{II},k}<\mathbb{C}_\varphi^{\mathrm{II},k},\\ 0 & \text{if } \mathbb{C}_\phi^{\mathrm{II},k}\geq\mathbb{C}_\varphi^{\mathrm{II},k},\end{cases}$$
$$=\max\left\{0,\mathbb{C}_\varphi^{\mathrm{II},k}-\mathbb{C}_\phi^{\mathrm{II},k}\right\}. \tag{28}$$

Note that $\left(\mathbb{C}_\phi^{\mathrm{I},k}-\mathbb{C}_\varphi^{\mathrm{I},k}\right)^-\geq 0$ and $\left(\mathbb{C}_\phi^{\mathrm{II},k}-\mathbb{C}_\varphi^{\mathrm{II},k}\right)^-\geq 0$. Furthermore, it is easily proved that

$$\left(\mathbb{C}_\phi^{\mathrm{I},k}-\mathbb{C}_\varphi^{\mathrm{I},k}\right)^+-\left(\mathbb{C}_\phi^{\mathrm{I},k}-\mathbb{C}_\varphi^{\mathrm{I},k}\right)^-=\mathbb{C}_\phi^{\mathrm{I},k}-\mathbb{C}_\varphi^{\mathrm{I},k}, \tag{29}$$

$$\left(\mathbb{C}_\phi^{\mathrm{II},k}-\mathbb{C}_\varphi^{\mathrm{II},k}\right)^+-\left(\mathbb{C}_\phi^{\mathrm{II},k}-\mathbb{C}_\varphi^{\mathrm{II},k}\right)^-=\mathbb{C}_\phi^{\mathrm{II},k}-\mathbb{C}_\varphi^{\mathrm{II},k}. \tag{30}$$

Next, this paper defines individual fit measurements for each decision-maker. This paper ascertains goodness of fit by integrating individual levels of rank consistency for all ordered pairs. Moreover, this paper figures out poorness of fit by combining individual levels of rank inconsistency. More specifically, for each decision-maker $e_k\in E$, this paper aggregates the individual levels of rank consistency $\left(\mathbb{C}_\phi^{\mathrm{I},k}-\mathbb{C}_\varphi^{\mathrm{I},k}\right)^+$ for all $(\phi,\varphi)\in\Omega^k$ to identify the Type I individual goodness of fit $G^{\mathrm{I},k}$; moreover, the individual levels $\left(\mathbb{C}_\phi^{\mathrm{II},k}-\mathbb{C}_\varphi^{\mathrm{II},k}\right)^+$ are aggregated to identify the Type II individual goodness of fit $G^{\mathrm{II},k}$, as follows:

$$G^{\mathrm{I},k}=\sum_{(\phi,\varphi)\in\Omega^k}\left(\mathbb{C}_\phi^{\mathrm{I},k}-\mathbb{C}_\varphi^{\mathrm{I},k}\right)^+, \tag{31}$$

$$G^{\mathrm{II},k}=\sum_{(\phi,\varphi)\in\Omega^k}\left(\mathbb{C}_\phi^{\mathrm{II},k}-\mathbb{C}_\varphi^{\mathrm{II},k}\right)^+. \tag{32}$$

It is worthy to note that $G^{\mathrm{I},k}\geq 0$ and $G^{\mathrm{II},k}\geq 0$ because $\left(\mathbb{C}_\phi^{\mathrm{I},k}-\mathbb{C}_\varphi^{\mathrm{I},k}\right)^+\geq 0$ and $\left(\mathbb{C}_\phi^{\mathrm{II},k}-\mathbb{C}_\varphi^{\mathrm{II},k}\right)^+\geq 0$, respectively.

In the same way, this paper sums the individual levels of rank inconsistency $\left(\mathbb{C}_\phi^{\mathrm{I},k}-\mathbb{C}_\varphi^{\mathrm{I},k}\right)^-$ for all $(\phi,\varphi)\in\Omega^k$ to determine the Type I individual poorness of fit $B^{\mathrm{I},k}$; additionally, the individual levels $\left(\mathbb{C}_\phi^{\mathrm{II},k}-\mathbb{C}_\varphi^{\mathrm{II},k}\right)^-$ are summed to acquire the Type II individual poorness of fit $B^{\mathrm{II},k}$, as follows:

$$B^{\mathrm{I},k}=\sum_{(\phi,\varphi)\in\Omega^k}\left(\mathbb{C}_\phi^{\mathrm{I},k}-\mathbb{C}_\varphi^{\mathrm{I},k}\right)^-, \tag{33}$$

$$B^{\mathrm{II},k}=\sum_{(\phi,\varphi)\in\Omega^k}\left(\mathbb{C}_\phi^{\mathrm{II},k}-\mathbb{C}_\varphi^{\mathrm{II},k}\right)^-. \tag{34}$$

Because $\left(\mathbb{C}_\phi^{\mathrm{I},k}-\mathbb{C}_\varphi^{\mathrm{I},k}\right)^-\geq 0$ and $\left(\mathbb{C}_\phi^{\mathrm{II},k}-\mathbb{C}_\varphi^{\mathrm{II},k}\right)^-\geq 0$, one can obtain that $B^{\mathrm{I},k}\geq 0$ and $B^{\mathrm{II},k}\geq 0$, respectively.

In general, the decision-makers are conceived to prefer an alternative that has the highest group comprehensive dominance measure. Accordingly, it is reasonable to designate the total sum of group comprehensive dominance measures as a maximal objective. In particular, the total group comprehensive dominance measure is defined as the total sum of $\mathbb{C}_i^{\mathrm{I},k}$ (or $\mathbb{C}_i^{\mathrm{II},k}$) for all $z_i \in Z$ and $e_k \in E$. On the other side, a lower individual poorness of fit $B^{\mathrm{I},k}$ (or $B^{\mathrm{II},k}$) represents a small extent of violation associated with the preference set Ω^k. To determine a set of collective criterion weights for which the total group comprehensive dominance measure $\sum_{k=1}^{k}\sum_{i=1}^{m}\mathbb{C}_i^{\mathrm{I},k}$ (or $\sum_{k=1}^{k}\sum_{i=1}^{m}\mathbb{C}_i^{\mathrm{II},k}$) is maximal and the group poorness of fit $\sum_{k=1}^{k}B^{\mathrm{I},k}$ (or $\sum_{k=1}^{k}B^{\mathrm{II},k}$) is minimal, this paper constructs a bi-objective mathematical programming model as the basic model in the PF LINMAP approach.

In addition, it is reasonable to presume that the decision-maker e_k would like to acquire a solution with a higher individual goodness of fit $G^{\mathrm{I},k}$ (or $G^{\mathrm{II},k}$) and a lower individual poorness of fit $B^{\mathrm{I},k}$ (or $B^{\mathrm{II},k}$). This implies that the condition of $G^{\mathrm{I},k} \geq B^{\mathrm{I},k}$ (or $G^{\mathrm{II},k} \geq B^{\mathrm{II},k}$) should be incorporated into the basic optimization model. Let $\hbar^k$ be a nonnegative number that is used to represent the lowest acceptable level concerning the deviation between $G^{\mathrm{I},k}$ (or $G^{\mathrm{II},k}$) and $B^{\mathrm{I},k}$ (or $B^{\mathrm{II},k}$). More concretely, the individual goodness of fit $G^{\mathrm{I},k}$ (or $G^{\mathrm{II},k}$) is not smaller than individual poorness of fit $B^{\mathrm{I},k}$ (or $B^{\mathrm{II},k}$) by $\hbar^k$ for each decision-maker e_k. That is, the basic optimization model should be formulated under the consideration of $G^{\mathrm{I},k} - B^{\mathrm{I},k} \geq \hbar^k$ (or $G^{\mathrm{II},k} - B^{\mathrm{II},k} \geq \hbar^k$).

Based on Type I dominance measures, to maximize the total group comprehensive dominance measure $\sum_{k=1}^{k}\sum_{i=1}^{m}\mathbb{C}_i^{\mathrm{I},k}$ and to minimize the group poorness of fit $\sum_{k=1}^{k}B^{\mathrm{I},k}$, the following bi-objective mathematical programming model is established:

$$\textbf{Model 1 (I)}.\ \max\left\{\sum_{k=1}^{K}\sum_{i=1}^{m}\mathbb{C}_i^{\mathrm{I},k}\right\}$$
$$\min\left\{\sum_{k=1}^{K}B^{\mathrm{I},k}\right\} \tag{35}$$
$$\text{s.t.}\begin{cases} G^{\mathrm{I},k} - B^{\mathrm{I},k} \geq \hbar^k & \text{for all } k,\\ \sum_{j=1}^{n} w_j = 1,\ w_j \geq \varepsilon_j & \text{for all } j.\end{cases}$$

Based on Type II dominance measures, the bi-objective (i.e., maximizing the $\sum_{k=1}^{k}\sum_{i=1}^{m}\mathbb{C}_i^{\mathrm{II},k}$ value and minimizing the $\sum_{k=1}^{k}B^{\mathrm{II},k}$ value) mathematical programming model is constructed as follows:

$$\textbf{Model 1 (II)}.\ \max\left\{\sum_{k=1}^{K}\sum_{i=1}^{m}\mathbb{C}_i^{\mathrm{II},k}\right\}$$
$$\min\left\{\sum_{k=1}^{K}B^{\mathrm{II},k}\right\} \tag{36}$$
$$\text{s.t.}\begin{cases} G^{\mathrm{II},k} - B^{\mathrm{II},k} \geq \hbar^k & \text{for all } k,\\ \sum_{j=1}^{n} w_j = 1,\ w_j \geq \varepsilon_j & \text{for all } j.\end{cases}$$

It is worth mentioning that the relative importance among multiple decision-makers can be appropriately reflected by means of the lowest acceptable level concerning the deviation between individual goodness of fit and individual poorness of fit. For example, if the importance of the decision-maker e_k is higher than the other decision-makers, a relatively high $\hbar^k$ value can be designated for convenience. Namely, there is no need to determine the importance weights of multiple decision-makers, which can facilitate the implementation efficiency of the developed PF LINMAP techniques.

Let $\Gamma_{\phi\varphi}^{\mathrm{I},k}$ and $\Gamma_{\phi\varphi}^{\mathrm{II},k}$ denote the individual degrees of violation with respect to the ordered pair $(\phi,\varphi) \in \Omega^k$ based on Types I and II dominance measures, respectively. They are derived using the maximal value of 0 and $\mathbb{C}_\varphi^{\mathrm{I},k} - \mathbb{C}_\phi^{\mathrm{I},k}$ (or $\mathbb{C}_\varphi^{\mathrm{II},k} - \mathbb{C}_\phi^{\mathrm{II},k}$) for each (ϕ,φ), as follows:

$$\Gamma_{\phi\varphi}^{\mathrm{I},k} = \max\left\{0, \mathbb{C}_\varphi^{\mathrm{I},k} - \mathbb{C}_\phi^{\mathrm{I},k}\right\} = \max\left\{0, \sum_{j=1}^{n}\left(M^{\mathrm{I}}\left(p_{\varphi j}^k\right) - M^{\mathrm{I}}\left(p_{\phi j}^k\right)\right)\cdot w_j\right\}, \tag{37}$$

$$\Gamma_{\phi\varphi}^{\mathrm{II},k} = \max\left\{0, \mathbb{C}_\varphi^{\mathrm{II},k} - \mathbb{C}_\phi^{\mathrm{II},k}\right\} = \max\left\{0, \sum_{j=1}^{n}\left(M^{\mathrm{II}}\left(p_{\varphi j}^k\right) - M^{\mathrm{II}}\left(p_{\phi j}^k\right)\right)\cdot w_j\right\}. \tag{38}$$

It is easy to see that the following conditions hold: $\Gamma_{\phi\varphi}^{\mathrm{I},k} \geq 0$, $\Gamma_{\phi\varphi}^{\mathrm{I},k} \geq \sum_{j=1}^{n}\left(M^{\mathrm{I}}\left(p_{\varphi j}^k\right) - M^{\mathrm{I}}\left(p_{\phi j}^k\right)\right)\cdot w_j$, $\Gamma_{\phi\varphi}^{\mathrm{II},k} \geq 0$, and $\Gamma_{\phi\varphi}^{\mathrm{II},k} \geq \sum_{j=1}^{n}\left(M^{\mathrm{II}}\left(p_{\varphi j}^k\right) - M^{\mathrm{II}}\left(p_{\phi j}^k\right)\right)\cdot w_j$ for all $(\phi,\varphi) \in \Omega^k$. It can be obtained that

$$\sum_{j=1}^{n}\left(M^{\mathrm{I}}\left(p_{\phi j}^k\right) - M^{\mathrm{I}}\left(p_{\varphi j}^k\right)\right)\cdot w_j + \Gamma_{\phi\varphi}^{\mathrm{I},k} \geq 0, \tag{39}$$

$$\sum_{j=1}^{n}\left(M^{\mathrm{II}}\left(p_{\phi j}^k\right) - M^{\mathrm{II}}\left(p_{\varphi j}^k\right)\right)\cdot w_j + \Gamma_{\phi\varphi}^{\mathrm{II},k} \geq 0. \tag{40}$$

Furthermore, the group poorness of fit $\sum_{k=1}^{k}B^{\mathrm{I},k}$ and $\sum_{k=1}^{k}B^{\mathrm{II},k}$ can be expressed as follows:

$$\sum_{k=1}^{K}B^{\mathrm{I},k} = \sum_{k=1}^{K}\sum_{(\phi,\varphi)\in\Omega^k}\max\{0, \mathbb{C}_\varphi^{\mathrm{I},k} - \mathbb{C}_\phi^{\mathrm{I},k}\} = \sum_{k=1}^{K}\sum_{(\phi,\varphi)\in\Omega^k}\Gamma_{\phi\varphi}^{\mathrm{I},k}, \tag{41}$$

$$\sum_{k=1}^{K}B^{\mathrm{II},k} = \sum_{k=1}^{K}\sum_{(\phi,\varphi)\in\Omega^k}\max\{0, \mathbb{C}_\varphi^{\mathrm{II},k} - \mathbb{C}_\phi^{\mathrm{II},k}\} = \sum_{k=1}^{K}\sum_{(\phi,\varphi)\in\Omega^k}\Gamma_{\phi\varphi}^{\mathrm{II},k}. \tag{42}$$

The differences $G^{\mathrm{I},k} - B^{\mathrm{I},k}$ and $G^{\mathrm{II},k} - B^{\mathrm{II},k}$ become

$$G^{\mathrm{I},k} - B^{\mathrm{I},k} = \sum_{(\phi,\varphi)\in\Omega^k}\left[(\mathbb{C}_\phi^{\mathrm{I},k} - \mathbb{C}_\varphi^{\mathrm{I},k})^+ - (\mathbb{C}_\phi^{\mathrm{I},k} - \mathbb{C}_\varphi^{\mathrm{I},k})^-\right] = \sum_{(\phi,\varphi)\in\Omega^k}\sum_{j=1}^{n}\left(M^{\mathrm{I}}\left(p_{\phi j}^k\right) - M^{\mathrm{I}}\left(p_{\varphi j}^k\right)\right)\cdot w_j, \tag{43}$$

$$G^{\mathrm{II},k} - B^{\mathrm{II},k} = \sum_{(\phi,\varphi)\in\Omega^k}\left[(\mathbb{C}_\phi^{\mathrm{II},k} - \mathbb{C}_\varphi^{\mathrm{II},k})^+ - (\mathbb{C}_\phi^{\mathrm{II},k} - \mathbb{C}_\varphi^{\mathrm{II},k})^-\right] = \sum_{(\phi,\varphi)\in\Omega^k}\sum_{j=1}^{n}\left(M^{\mathrm{II}}\left(p_{\phi j}^k\right) - M^{\mathrm{II}}\left(p_{\varphi j}^k\right)\right)\cdot w_j. \tag{44}$$

From Definition 8, one has $\mathbb{C}_i^{\mathrm{I},k} = \sum_{j=1}^{n} M^{\mathrm{I}}\left(p_{ij}^k\right) w_j$ and $\mathbb{C}_i^{\mathrm{II},k} = \sum_{j=1}^{n} M^{\mathrm{II}}\left(p_{ij}^k\right) w_j$ Based on the above results, the basic Models 1(I) and 1(II) can be equivalently represented via the following bi-objective optimization models:

$$\textbf{Model 2 (I)}.\ \max\left\{\sum_{k=1}^{K}\sum_{i=1}^{m}\mathbb{C}_i^{\mathrm{I},k} = \sum_{k=1}^{K}\sum_{i=1}^{m}\sum_{j=1}^{n} M^{\mathrm{I}}\left(p_{ij}^k\right) w_j\right\}$$

$$\min\left\{\sum_{k=1}^{K} B^{\mathrm{I},k} = \sum_{k=1}^{K}\sum_{(\phi,\varphi)\in\Omega^k}\Gamma_{\phi\varphi}^{\mathrm{I},k}\right\}$$

$$\text{s.t.}\begin{cases}\sum_{(\phi,\varphi)\in\Omega^k}\sum_{j=1}^{n}\left(M^{\mathrm{I}}\left(p_{\phi j}^k\right) - M^{\mathrm{I}}\left(p_{\varphi j}^k\right)\right) w_j \geq \hbar^k \text{ for all } k,\\ \sum_{j=1}^{n}\left(M^{\mathrm{I}}\left(p_{\phi j}^k\right) - M^{\mathrm{I}}\left(p_{\varphi j}^k\right)\right)\cdot w_j + \Gamma_{\phi\varphi}^{\mathrm{I},k} \geq 0\\ \text{and } \Gamma_{\phi\varphi}^{\mathrm{I},k} \geq 0 \text{ for } (\phi,\varphi)\in\Omega^k,\\ \sum_{j=1}^{n} w_j = 1, w_j \geq \varepsilon_j \quad \text{for all } j;\end{cases} \tag{45}$$

$$\textbf{Model 2 (II)}.\ \max\left\{\sum_{k=1}^{K}\sum_{i=1}^{m}\mathbb{C}_i^{\mathrm{II},k} = \sum_{k=1}^{K}\sum_{i=1}^{m}\sum_{j=1}^{n} M^{\mathrm{II}}\left(p_{ij}^k\right) w_j\right\}$$

$$\min\left\{\sum_{k=1}^{K} B^{\mathrm{II},k} = \sum_{k=1}^{K}\sum_{(\phi,\varphi)\in\Omega^k}\Gamma_{\phi\varphi}^{\mathrm{II},k}\right\} \tag{46}$$

$$\text{s.t.}\begin{cases}\sum_{(\phi,\varphi)\in\Omega^k}\sum_{j=1}^{n}\left(M^{\mathrm{II}}\left(p_{\phi j}^k\right) - M^{\mathrm{II}}\left(p_{\varphi j}^k\right)\right) w_j \geq \hbar^k \text{ for all } k,\\ \sum_{j=1}^{n}\left(M^{\mathrm{II}}\left(p_{\phi j}^k\right) - M^{\mathrm{II}}\left(p_{\varphi j}^k\right)\right)\cdot w_j + \Gamma_{\phi\varphi}^{\mathrm{II},k} \geq 0\\ \text{and } \Gamma_{\phi\varphi}^{\mathrm{II},k} \geq 0 \text{ for } (\phi,\varphi)\in\Omega^k,\\ \sum_{j=1}^{n} w_j = 1, w_j \geq \varepsilon_j \text{ for all } j.\end{cases}$$

The bi-objective models in Models 2(I) and 2(II) can be further transformed into simple linear programming models for enhancing implementation efficiency. The minimal objectives $\sum_{k=1}^{k} B^{\mathrm{I},k}$ in Model 2(I) and $\sum_{k=1}^{k} B^{\mathrm{II},k}$ in Model 2(II) are equivalent to the maximal objectives $-\sum_{k=1}^{k} B^{\mathrm{I},k}$ and $-\sum_{k=1}^{k} B^{\mathrm{II},k}$, respectively. It follows that $\min\left\{\sum_{k=1}^{k}\sum_{(\phi,\varphi)\in\Omega}\Gamma_{\phi\varphi}^{\mathrm{I},k}\right\} = \max\left\{-\sum_{k=1}^{k}\sum_{(\phi,\varphi)\in\Omega}\Gamma_{\phi\varphi}^{\mathrm{I},k}\right\}$ and $\min\left\{\sum_{k=1}^{K}\sum_{(\phi,\varphi)\in\Omega}\Gamma_{\phi\varphi}^{\mathrm{II},k}\right\} = \max\left\{-\sum_{k=1}^{K}\sum_{(\phi,\varphi)\in\Omega}\Gamma_{\phi\varphi}^{\mathrm{II},k}\right\}$. Let a parameter η represent the weight in connection with the objective of "total group comprehensive dominance"; let $1-\eta$ represent the weight in connection with the objective of "group poorness of fit," in which $\eta \in [0,1]$. The use of the parameter η can effectively coordinate the two objectives in Models 2(I) and 2(II) and transform bi-objective models into single-objective linear programming models. Based on Types I and II dominance measures, the proposed PF LINMAP methodology can be represented by the following parametric linear programming models:

Type I PF LINMAP Model.

$$\max\left\{\eta\sum_{k=1}^{K}\sum_{i=1}^{m}\sum_{j=1}^{n} M^{\mathrm{I}}\left(p_{ij}^k\right) w_j - (1-\eta)\sum_{k=1}^{K}\sum_{(\phi,\varphi)\in\Omega^k}\Gamma_{\phi\varphi}^{\mathrm{I},k}\right\}$$

$$\text{s.t.}\begin{cases}\sum_{(\phi,\varphi)\in\Omega^k}\sum_{j=1}^{n}\left(M^{\mathrm{I}}\left(p_{\phi j}^k\right) - M^{\mathrm{I}}\left(p_{\varphi j}^k\right)\right) w_j \geq \hbar^k \text{ for all } k,\\ \sum_{j=1}^{n}\left(M^{\mathrm{I}}\left(p_{\phi j}^k\right) - M^{\mathrm{I}}\left(p_{\varphi j}^k\right)\right)\cdot w_j + \Gamma_{\phi\varphi}^{\mathrm{I},k} \geq 0\\ \text{and } \Gamma_{\phi\varphi}^{\mathrm{I},k} \geq 0 \text{ for } (\phi,\varphi)\in\Omega^k,\\ \sum_{j=1}^{n} w_j = 1,\ w_j \geq \varepsilon_j \text{ for all } j;\end{cases} \tag{47}$$

Type II PF LINMAP Model.

$$\max\left\{\eta\sum_{k=1}^{K}\sum_{i=1}^{m}\sum_{j=1}^{n} M^{\mathrm{II}}\left(p_{ij}^k\right) w_j - (1-\eta)\sum_{k=1}^{K}\sum_{(\phi,\varphi)\in\Omega^k}\Gamma_{\phi\varphi}^{\mathrm{II},k}\right\}$$

$$\text{s.t.}\begin{cases}\sum_{(\phi,\varphi)\in\Omega^k}\sum_{j=1}^{n}\left(M^{\mathrm{II}}\left(p_{\phi j}^k\right) - M^{\mathrm{II}}\left(p_{\varphi j}^k\right)\right) w_j \geq \hbar^k \text{ for all } k,\\ \sum_{j=1}^{n}\left(M^{\mathrm{II}}\left(p_{\phi j}^k\right) - M^{\mathrm{II}}\left(p_{\varphi j}^k\right)\right)\cdot w_j + \Gamma_{\phi\varphi}^{\mathrm{II},k} \geq 0\\ \text{and } \Gamma_{\phi\varphi}^{\mathrm{II},k} \geq 0 \text{ for } (\phi,\varphi)\in\Omega^k,\\ \sum_{j=1}^{n} w_j = 1,\ w_j \geq \varepsilon_j \text{ for all } j.\end{cases} \tag{48}$$

The optimal solutions of the collective weight $\overline{w}_j = (\overline{w}_1, \overline{w}_2, \cdots, \overline{w}_n)$ for each criterion $c_j \in C$ and the individual violation $\overline{\Gamma}_{\phi\varphi}^{\mathrm{I},k}$ (or $\overline{\Gamma}_{\phi\varphi}^{\mathrm{II},k}$) for each ordered pair $(\phi,\varphi)\in\Omega^k$ can be determined by solving the Type I or Type II PF LINMAP models. The corresponding group comprehensive dominance measure of the alternative $z_i \in Z$ is calculated as below:

$$\sum_{k=1}^{K}\overline{\mathbb{C}}_i^{\mathrm{I},k} = \sum_{k=1}^{K}\sum_{j=1}^{n}\overline{w}_j\cdot M^{\mathrm{I}}\left(p_{ij}^k\right), \tag{49}$$

$$\sum_{k=1}^{K}\overline{\mathbb{C}}_i^{\mathrm{II},k} = \sum_{k=1}^{K}\sum_{j=1}^{n}\overline{w}_j\cdot M^{\mathrm{II}}\left(p_{ij}^k\right). \tag{50}$$

Lastly, the overall priority ranking results of candidate alternatives is acquired in conformity with the decreasing order of the values of $\sum_{k=1}^{k}\overline{\mathbb{C}}_i^{\mathrm{I},k}$ (or $\sum_{k=1}^{k}\overline{\mathbb{C}}_i^{\mathrm{II},k}$). Moreover, the best compromise solution is ranked the best by virtue of the values of $\sum_{k=1}^{k}\overline{\mathbb{C}}_i^{\mathrm{I},k}$ (or $\sum_{k=1}^{k}\overline{\mathbb{C}}_i^{\mathrm{II},k}$) among all $z_i \in Z$.

Based on Types I and II dominance measures, the algorithmic procedures of the proposed PF LINMAP methodology can be summarized as follows:

Algorithm 1 Type I PF LINMAP method

- Step I.1: Formulate an MCGDM problem involving the set of alternatives $Z = \{z_1, z_2, \cdots, z_m\}$, the set of criteria $C = \{c_1, c_2, \cdots, c_n\}$, and the set of decision-makers $E = \{e_1, e_2, \cdots, e_K\}$.
- Step I.2: Request each decision-maker e_k to provide the paired preference relations over alternatives to form the preference set $\Omega^k = \left\{(\phi, \varphi) \mid z_\phi \geq z_\varphi, \phi, \varphi \in \{1, 2, \cdots, m\}\right.$.
- Step I.3: Establish the PF evaluative rating p_{ij}^k of each alternative $z_i \in Z$ in regard to criterion $c_j \in C$ for the decision-maker $e_k \in E$. Form the PF decision matrix $\mathrm{P}^k = [p_{ij}^k]_{m \times n}$.
- Step I.4: Identify the displaced reference points, i.e., the positive-ideal reference point p_{*j}^k and the negative-ideal reference point $p_{\#j}^k$, with respect to each criterion c_j based on the P^k.
- Step I.5: Derive the PF scalar function $V\left(p_{ij}^k\right)$ for each p_{ij}^k in P^k. Calculate the PF scalar functions $V\left(p_{*j}^k\right)$ and $V\left(p_{\#j}^k\right)$ for each $c_j \in C$ and $e_k \in E$.
- Step I.6: Compute the PF scalar function-based dominance measure when anchoring p_{*j}^k and $p_{\#j}^k$, i.e., Type I dominance measure $M^{\mathrm{I}}\left(p_{ij}^k\right)$, of each p_{ij}^k in P^k.
- Step I.7: Set the parameter values. Specify the lowest acceptable level $\hbar^k$ for all k and the nonnegative boundary condition ε_j for all j, where $\hbar^k \geq 0$ and $\varepsilon_j \in [0, 1]$. Assign the relative importance η of the "total group comprehensive dominance" objective over the "group poorness of fit" objective, where $\eta \in [0, 1]$.
- Step I.8: Denote $\mathrm{w} = \left(w_1, w_2, \cdots, w_n\right)$ as the weight vector of the n criteria such that $\sum_{j=1}^n w_j = 1$ and $w_j \geq \varepsilon_j$ for all j. Let $\Gamma_{\phi\varphi}^{\mathrm{I},k}$ denote the individual degrees of violation with respect to the ordered pair $(\phi, \varphi) \in \Omega^k$.
- Step I.9: Construct the linear programming problem using the Type I PF LINMAP model to resolve the optimal collective weight $\overline{w}_j$ and the individual violation $\overline{\Gamma}_{\phi\varphi}^{\mathrm{I},k}$.
- Step I.10: Obtain the optimal group comprehensive dominance measure $\sum_{k=1}^k \overline{\mathbb{C}}_i^{\mathrm{I},k}$ for each z_i. Identify the overall priority ranking orders of the m alternatives and the best compromise solution in line with the $\sum_{k=1}^k \overline{\mathbb{C}}_i^{\mathrm{I},k}$ values.

Algorithm 2 Type II PF LINMAP method

- Steps II.1–II.3: see Steps I.1–I.3 of Algorithm I.
- Step II.4: Calculate the PF scalar function $V\left(p_{ij}^k\right)$ for each p_{ij}^k to acquire the PF scalar function-based dominance measure when anchoring p_{+j}^k and p_{-j}^k, i.e., Type II dominance measure $M^{\mathrm{II}}\left(p_{ij}^k\right)$, of each p_{ij}^k in P^k.
- Step II.5: See Step I.7 of Algorithm I.
- Step II.6: Denote $\mathrm{w} = \left(w_1, w_2, \cdots, w_n\right)$ as the weight vector of the n criteria such that $\sum_{j=1}^n w_j = 1$ and $w_j \geq \varepsilon_j$ for all j. Let $\Gamma_{\phi\varphi}^{\mathrm{II},k}$ denote the individual degrees of violation with respect to the ordered pair $(\phi, \varphi) \in \Omega^k$.
- Step II.7: Construct the linear programming problem using the Type II PF LINMAP model to resolve the optimal collective weight $\overline{w}_j$ and the individual violation $\overline{\Gamma}_{\phi\varphi}^{\mathrm{II},k}$.
- Step II.8: Obtain the optimal group comprehensive dominance measure $\sum_{k=1}^k \overline{\mathbb{C}}_i^{\mathrm{II},k}$ for each z_i. Identify the overall priority ranking orders of the m alternatives and the best compromise solution in line with the $\sum_{k=1}^k \overline{\mathbb{C}}_i^{\mathrm{II},k}$ values.

6. Practical Application

This section employs the PF LINMAP techniques to address an MCGDM problem concerning railway project investment introduced by Xue *et al.* [26] to validate the practicability and effectiveness of the developed methodology in realistic situations.

Xue *et al.* [26] utilized PF entropy measures to develop a novel PF LINMAP method and applied it to manage an MCGDM problem regarding the railway project selection in China's Belt and Road Initiative (BRI). Till now, BRI involves infrastructure development and investments in countries and international organizations in Asia, Europe, Africa, the Middle East, and the Americas. Consider the importance of railway projects in the infrastructure investment of BRI, Xue *et al.* took Germany (z_1), Russia (z_2), Singapore (z_3), and Malaysia (z_4) as the candidate alternatives in the set Z, because these countries have intentions to cooperate with the railway project. Moreover, Xue *et al.* explored an indicator system involving financial and noneconomic evaluations. Their proposed indicator system involves six aspects consisting of financial internal rate of return (c_1), net present value (c_2), investment recovery period (c_3), debt ratio and current ratio (c_4), repayment period of loan (c_5), and public benefit and diplomatic influence (c_6). The six aspects were regarded as the evaluative criteria in the set C. Three experts participated in a group decision-making process in this case. On the basis of the above, in Steps I.1 and II.1, the MCGDM problem under study was formulated with $Z = \{z_1, z_2, z_3, z_4\}$, $C = \{c_1, c_2, \cdots, c_6\}$, and $E = \{e_1, e_2, e_3\}$, in which $m = 4$, $n = 6$, and $K = 3$.

First, this paper employed the proposed Algorithm I (i.e., Type I PF LINMAP method) to solve the above MCGDM problem. In Step I.2, in reference to the investigated data in Xue *et al.* [26], the three decision-makers provided some paired preference relations over alternatives. The decision-makers' preference sets were constructed as follows: Ω^1 ={(3, 2), (4, 1), (3, 1)}, Ω^2 ={(2, 1), (4, 3), (2, 4), (3, 1)}, and Ω^3 ={(3, 1), (3, 4), (2, 3), (2, 4)}.

In Step I.3, the three decision-makers evaluated the four alternatives based on the six criteria; they provided the results of the degree of satisfaction μ_{ij}^k and the degree of dissatisfaction ν_{ij}^k of alternative $z_i \in Z$ with respect to criterion $c_j \in C$, as shown in Xue *et al.* [26]. For mathematical representation using Pythagorean membership grades, this paper computed the values of r_{ij}^k, d_{ij}^k, and

θ_{ij}^k to establish the PF evaluative rating $p_{ij}^k = \left(\mu_{ij}^k, \nu_{ij}^k; r_{ij}^k, d_{ij}^k\right)$ in Eq. (11) for each $e_k \in E$. The data of PF evaluative ratings for the decision-makers e_1, e_2, and e_3 are shown in Tables 2–4, respectively, including the values of μ_{ij}^k, ν_{ij}^k, r_{ij}^k, and d_{ij}^k in p_{ij}^k, as well as the radians θ_{ij}^k. Applying Eq. (12), the three PF decision matrices P^1 $\left(=[p_{ij}^1]_{4\times6}\right)$, P^2 $\left(=[p_{ij}^2]_{4\times6}\right)$, and P^3 $\left(=[p_{ij}^3]_{4\times6}\right)$ can be established based on the data (i.e., μ_{ij}^k, ν_{ij}^k, r_{ij}^k, and d_{ij}^k) in Tables 2–4, respectively.

In Step I.4, this paper employed Eqs. (14) and (15) to identify the displaced reference points p_{*j}^k and $p_{\#j}^k$, respectively, in connection with $c_j \in C$ for each e_k. The determination results of the positive- and negative-ideal reference points are revealed in Tables 5 and 6, respectively.

In Step I.5, this paper utilized Eq. (18) to compute the PF scalar functions $V\left(p_{ij}^k\right)$, $V\left(p_{*j}^k\right)$, and $V\left(p_{\#j}^k\right)$. The results of $V\left(p_{ij}^k\right)$ for

Table 2 Relevant data associated with the decision-maker e_1.

z_i	c_j	μ_{ij}^1	ν_{ij}^1	r_{ij}^1	d_{ij}^1	θ_{ij}^1	$V\left(p_{ij}^1\right)$
z_1	c_1	0.70	0.60	0.9220	0.5489	0.7086	0.5451
	c_2	0.80	0.60	1.0000	0.5903	0.6435	0.5903
	c_3	0.50	0.50	0.7071	0.5000	0.7854	0.5000
	c_4	0.40	0.70	0.8062	0.3305	1.0517	0.3633
	c_5	0.90	0.40	0.9849	0.7338	0.4182	0.7302
	c_6	0.40	0.90	0.9849	0.2662	1.1526	0.2698
z_2	c_1	0.90	0.40	0.9849	0.7338	0.4182	0.7302
	c_2	0.80	0.60	1.0000	0.5903	0.6435	0.5903
	c_3	0.70	0.70	0.9899	0.5000	0.7854	0.5000
	c_4	0.90	0.30	0.9487	0.7952	0.3218	0.7800
	c_5	0.80	0.20	0.8246	0.8440	0.2450	0.7837
	c_6	0.90	0.40	0.9849	0.7338	0.4182	0.7302
z_3	c_1	0.70	0.50	0.8602	0.6051	0.6202	0.5904
	c_2	0.40	0.30	0.5000	0.5903	0.6435	0.5452
	c_3	0.90	0.10	0.9055	0.9296	0.1107	0.8890
	c_4	0.80	0.20	0.8246	0.8440	0.2450	0.7837
	c_5	0.70	0.40	0.8062	0.6695	0.5191	0.6367
	c_6	0.60	0.60	0.8485	0.5000	0.7854	0.5000
z_4	c_1	0.90	0.30	0.9487	0.7952	0.3218	0.7800
	c_2	0.70	0.20	0.7280	0.8228	0.2783	0.7350
	c_3	0.40	0.30	0.5000	0.5903	0.6435	0.5452
	c_4	0.90	0.40	0.9849	0.7338	0.4182	0.7302
	c_5	0.50	0.40	0.6403	0.5704	0.6747	0.5451
	c_6	0.60	0.70	0.9220	0.4511	0.8622	0.4549

Table 3 Relevant data associated with the decision-maker e_2.

z_i	c_j	μ_{ij}^2	ν_{ij}^2	r_{ij}^2	d_{ij}^2	θ_{ij}^2	$V\left(p_{ij}^2\right)$
z_1	c_1	0.50	0.40	0.6403	0.5704	0.6747	0.5451
	c_2	0.30	0.90	0.9487	0.2048	1.2490	0.2200
	c_3	0.40	0.30	0.5000	0.5903	0.6435	0.5452
	c_4	0.90	0.40	0.9849	0.7338	0.4182	0.7302
	c_5	0.30	0.70	0.7616	0.2578	1.1659	0.3155
	c_6	0.40	0.50	0.6403	0.4296	0.8961	0.4549
z_2	c_1	0.90	0.30	0.9487	0.7952	0.3218	0.7800
	c_2	0.80	0.50	0.9434	0.6444	0.5586	0.6362
	c_3	0.70	0.60	0.9220	0.5489	0.7086	0.5451
	c_4	0.90	0.20	0.9220	0.8608	0.2187	0.8326
	c_5	0.90	0.20	0.9220	0.8608	0.2187	0.8326
	c_6	0.80	0.30	0.8544	0.7716	0.3588	0.7321
z_3	c_1	0.70	0.60	0.9220	0.5489	0.7086	0.5451
	c_2	0.50	0.30	0.5831	0.6560	0.5404	0.5909
	c_3	0.80	0.10	0.8062	0.9208	0.1244	0.8393
	c_4	0.90	0.10	0.9055	0.9296	0.1107	0.8890
	c_5	0.70	0.30	0.7616	0.7422	0.4049	0.6845
	c_6	0.50	0.50	0.7071	0.5000	0.7854	0.5000
z_4	c_1	0.30	0.90	0.9487	0.2048	1.2490	0.2200
	c_2	0.20	0.30	0.3606	0.3743	0.9828	0.4547
	c_3	0.30	0.70	0.7616	0.2578	1.1659	0.3155
	c_4	0.50	0.80	0.9434	0.3556	1.0122	0.3638
	c_5	0.70	0.60	0.9220	0.5489	0.7086	0.5451
	c_6	0.40	0.70	0.8062	0.3305	1.0517	0.3633

Table 4 Relevant data associated with the decision-maker e_3.

z_i	c_j	μ_{ij}^3	ν_{ij}^3	r_{ij}^3	d_{ij}^3	θ_{ij}^3	$V\left(p_{ij}^3\right)$
z_1	c_1	0.40	0.30	0.5000	0.5903	0.6435	0.5452
	c_2	0.30	0.40	0.5000	0.4097	0.9273	0.4548
	c_3	0.50	0.40	0.6403	0.5704	0.6747	0.5451
	c_4	0.60	0.60	0.8485	0.5000	0.7854	0.5000
	c_5	0.70	0.70	0.9899	0.5000	0.7854	0.5000
	c_6	0.30	0.70	0.7616	0.2578	1.1659	0.3155
z_2	c_1	0.80	0.30	0.8544	0.7716	0.3588	0.7321
	c_2	0.70	0.10	0.7071	0.9097	0.1419	0.7897
	c_3	0.80	0.20	0.8246	0.8440	0.2450	0.7837
	c_4	0.90	0.10	0.9055	0.9296	0.1107	0.8890
	c_5	0.80	0.30	0.8544	0.7716	0.3588	0.7321
	c_6	0.80	0.10	0.8062	0.9208	0.1244	0.8393
z_3	c_1	0.20	0.50	0.5385	0.2422	1.1903	0.3612
	c_2	0.30	0.40	0.5000	0.4097	0.9273	0.4548
	c_3	0.80	0.60	1.0000	0.5903	0.6435	0.5903
	c_4	0.90	0.10	0.9055	0.9296	0.1107	0.8890
	c_5	0.30	0.10	0.3162	0.7952	0.3218	0.5933
	c_6	0.60	0.40	0.7211	0.6257	0.5880	0.5906
z_4	c_1	0.60	0.80	1.0000	0.4097	0.9273	0.4097
	c_2	0.40	0.10	0.4123	0.8440	0.2450	0.6419
	c_3	0.20	0.40	0.4472	0.2952	1.1071	0.4084
	c_4	0.70	0.10	0.7071	0.9097	0.1419	0.7897
	c_5	0.60	0.20	0.6325	0.7952	0.3218	0.6867
	c_6	0.80	0.10	0.8062	0.9208	0.1244	0.8393

Table 5 Results of the positive-ideal reference point p_{*j}^k.

e_k	c_j	μ_{*j}^k	ν_{*j}^k	r_{*j}^k	d_{*j}^k	θ_{*j}^k	$V\left(p_{*j}^k\right)$
e_1	c_1	0.90	0.30	0.9487	0.7952	0.3218	0.7800
	c_2	0.80	0.20	0.8246	0.8440	0.2450	0.7837
	c_3	0.90	0.10	0.9055	0.9296	0.1107	0.8890
	c_4	0.90	0.20	0.9220	0.8608	0.2187	0.8326
	c_5	0.90	0.20	0.9220	0.8608	0.2187	0.8326
	c_6	0.90	0.40	0.9849	0.7338	0.4182	0.7302
e_2	c_1	0.90	0.30	0.9487	0.7952	0.3218	0.7800
	c_2	0.80	0.30	0.8544	0.7716	0.3588	0.7321
	c_3	0.80	0.10	0.8062	0.9208	0.1244	0.8393
	c_4	0.90	0.10	0.9055	0.9296	0.1107	0.8890
	c_5	0.90	0.20	0.9220	0.8608	0.2187	0.8326
	c_6	0.80	0.30	0.8544	0.7716	0.3588	0.7321
e_3	c_1	0.80	0.30	0.8544	0.7716	0.3588	0.7321
	c_2	0.70	0.10	0.7071	0.9097	0.1419	0.7897
	c_3	0.80	0.20	0.8246	0.8440	0.2450	0.7837
	c_4	0.90	0.10	0.9055	0.9296	0.1107	0.8890
	c_5	0.80	0.10	0.8062	0.9208	0.1244	0.8393
	c_6	0.80	0.10	0.8062	0.9208	0.1244	0.8393

each p_{ij}^{k} are listed in the last columns of Tables 2–4. The results of the PF scalar functions $V\left(p_{*j}^{k}\right)$ and $V\left(p_{\#j}^{k}\right)$ are indicated in the last columns of Tables 5 and 6, respectively.

In Step I.6, this paper employed the PF scalar functions $V\left(p_{ij}^{k}\right)$, $V\left(p_{*j}^{k}\right)$, and $V\left(p_{\#j}^{k}\right)$ to determine the Type I dominance measure $M^{\mathrm{I}}\left(p_{ij}^{k}\right)$ using Eq. (21) based on Theorem 5. The obtained results are presented in Table 7.

In Step I.7, this paper regarded the three decision-makers of equal importance to designate the lowest acceptable level $\hbar^{k}=0.3$ for $k\in\{1,2,3\}$. Moreover, this paper specified the nonnegative boundary condition ε_j =0.025 for $j\in\{1,2,\cdots,6\}$. The relative importance η was set as 0.2. Namely, the weights of the two objectives of "total group comprehensive dominance" and "group poorness of fit" were 0.2 and 0.8, respectively.

Table 6 Results of the negative-ideal reference point $p_{\#j}^{k}$.

e_k	c_j	$\mu_{\#j}^{k}$	$\nu_{\#j}^{k}$	$r_{\#j}^{k}$	$d_{\#j}^{k}$	$\theta_{\#j}^{k}$	$V\left(p_{\#j}^{k}\right)$
e_1	c_1	0.70	0.60	0.9220	0.5489	0.7086	0.5451
	c_2	0.40	0.60	0.7211	0.3743	0.9828	0.4094
	c_3	0.40	0.70	0.8062	0.3305	1.0517	0.3633
	c_4	0.40	0.70	0.8062	0.3305	1.0517	0.3633
	c_5	0.50	0.40	0.6403	0.5704	0.6747	0.5451
	c_6	0.40	0.90	0.9849	0.2662	1.1526	0.2698
e_2	c_1	0.30	0.90	0.9487	0.2048	1.2490	0.2200
	c_2	0.20	0.90	0.9220	0.1392	1.3521	0.1674
	c_3	0.30	0.70	0.7616	0.2578	1.1659	0.3155
	c_4	0.50	0.80	0.9434	0.3556	1.0122	0.3638
	c_5	0.30	0.70	0.7616	0.2578	1.1659	0.3155
	c_6	0.40	0.70	0.8062	0.3305	1.0517	0.3633
e_3	c_1	0.20	0.80	0.8246	0.1560	1.3258	0.2163
	c_2	0.30	0.40	0.5000	0.4097	0.9273	0.4548
	c_3	0.20	0.60	0.6325	0.2048	1.2490	0.3133
	c_4	0.60	0.60	0.8485	0.5000	0.7854	0.5000
	c_5	0.30	0.70	0.7616	0.2578	1.1659	0.3155
	c_6	0.30	0.70	0.7616	0.2578	1.1659	0.3155

Table 7 Results of the Type I dominance measure $M^{\mathrm{I}}\left(p_{ij}^{k}\right)$.

e_k	c_j	$M^{\mathrm{I}}\left(p_{1j}^{k}\right)$	$M^{\mathrm{I}}\left(p_{2j}^{k}\right)$	$M^{\mathrm{I}}\left(p_{3j}^{k}\right)$	$M^{\mathrm{I}}\left(p_{4j}^{k}\right)$
e_1	c_1	0.0000	0.7880	0.1931	1.0000
	c_2	0.4834	0.4834	0.3628	0.8699
	c_3	0.2600	0.2600	1.0000	0.3459
	c_4	0.0000	0.8879	0.8957	0.7818
	c_5	0.6438	0.8298	0.3184	0.0000
	c_6	0.0000	1.0000	0.5000	0.4021
e_2	c_1	0.5805	1.0000	0.5805	0.0000
	c_2	0.0932	0.8303	0.7501	0.5088
	c_3	0.4385	0.4383	1.0000	0.0000
	c_4	0.6977	0.8927	1.0000	0.0000
	c_5	0.0000	1.0000	0.7135	0.4439
	c_6	0.2483	1.0000	0.3706	0.0000
e_3	c_1	0.6376	1.0000	0.2809	0.3749
	c_2	0.0000	1.0000	0.0000	0.5585
	c_3	0.4928	1.0000	0.5889	0.2021
	c_4	0.0000	1.0000	1.0000	0.7447
	c_5	0.3522	0.7953	0.5304	0.7086
	c_6	0.0000	1.0000	0.5252	1.0000

In Step I.8, let $\mathrm{w}=(w_1,w_2,\cdots,w_6)$ denote the weight vector of the six criteria such that $\sum_{j=1}^{6}w_j=1$ and $w_j\geq\varepsilon_j=0.025$ for all j. According to the decision-makers' preference sets, the individual degrees of violation based on Type I dominance measures were denoted as $\Gamma_{32}^{\mathrm{I},1}$, $\Gamma_{41}^{\mathrm{I},1}$, and $\Gamma_{31}^{\mathrm{I},1}$ for Ω^1, $\Gamma_{21}^{\mathrm{I},2}$, $\Gamma_{43}^{\mathrm{I},2}$, $\Gamma_{24}^{\mathrm{I},2}$, and $\Gamma_{31}^{\mathrm{I},2}$ for Ω^2, and $\Gamma_{31}^{\mathrm{I},3}$, $\Gamma_{34}^{\mathrm{I},3}$, $\Gamma_{23}^{\mathrm{I},3}$, and $\Gamma_{24}^{\mathrm{I},3}$ for Ω^3.

In Step I.9, this paper applied the Type I PF LINMAP model in Eq. (47) to construct a parametric linear programming model, as shown in the Appendix. The model was solved to obtain the optimal collective weight vector and individual degrees of violation as follows: $\overline{w}=\left(\overline{w}_1,\overline{w}_2,\cdots,\overline{w}_6\right)$ =(0.0250, 0.2825, 0.0946, 0.5479, 0.0250, 0.0250), $\overline{\Gamma}_{43}^{\mathrm{I},2}=0.7412$, and $\overline{\Gamma}_{32}^{\mathrm{I},1}=\overline{\Gamma}_{41}^{\mathrm{I},1}=\overline{\Gamma}_{31}^{\mathrm{I},1}=\overline{\Gamma}_{21}^{\mathrm{I},2}=\overline{\Gamma}_{24}^{\mathrm{I},2}=\overline{\Gamma}_{31}^{\mathrm{I},2}=\overline{\Gamma}_{31}^{\mathrm{I},3}=\overline{\Gamma}_{34}^{\mathrm{I},3}=\overline{\Gamma}_{23}^{\mathrm{I},3}=\overline{\Gamma}_{24}^{\mathrm{I},3}=0$.

In Step I.10, the optimal group comprehensive dominance measures were derived using Eq. (49) as follows: $\sum_{k=1}^{3}\overline{\mathbb{C}}_1^{\mathrm{I},k}=0.7194$ for z_1, $\sum_{k=1}^{3}\overline{\mathbb{C}}_2^{\mathrm{I},k}=2.5481$ for z_2, $\sum_{k=1}^{3}\overline{\mathbb{C}}_3^{\mathrm{I},k}=2.2461$ for z_3, and $\sum_{k=1}^{3}\overline{\mathbb{C}}_4^{\mathrm{I},k}=1.5337$ for z_4. Therefore, the overall priority ranking of the four alternatives was $z_2\succ z_3\succ z_4\succ z_1$; moreover, the best compromise solution was z_2. The obtained results via Algorithm I are concordant with those yielded by Xue *et al.*'s PF entropy-based LINMAP method [26].

Next, this paper employed the developed Algorithm II (i.e., Type II PF LINMAP method) to handle the above MCGDM problem. It is noted that Steps II.1–II.3 and II.5 of Algorithm II are the same as Steps I.1–I.3 and I.7 of Algorithm I, respectively.

In Step II.4, this paper computed the PF scalar function $V\left(p_{ij}^{k}\right)$ to acquire the Type II dominance measure $M^{\mathrm{II}}\left(p_{ij}^{k}\right)$ of each p_{ij}^{k} according to Eq. (22) in Theorem 5. That is, $M^{\mathrm{II}}\left(p_{ij}^{1}\right)=V\left(p_{ij}^{1}\right)$, $M^{\mathrm{II}}\left(p_{ij}^{2}\right)=V\left(p_{ij}^{2}\right)$, and $M^{\mathrm{II}}\left(p_{ij}^{3}\right)=V\left(p_{ij}^{3}\right)$ for all $z_i\in Z$ and $c_j\in C$, and the results were indicated in the last columns of Tables 2–4.

In Step II.6, let the weight vector of criteria $\mathrm{w}=(w_1,w_2,\cdots,w_6)$. Furthermore, the individual degrees of violation based on Type II dominance measures were denoted as $\Gamma_{32}^{\mathrm{II},1}$, $\Gamma_{41}^{\mathrm{II},1}$, and $\Gamma_{31}^{\mathrm{II},1}$ for Ω^1, $\Gamma_{21}^{\mathrm{II},2}$, $\Gamma_{43}^{\mathrm{II},2}$, $\Gamma_{24}^{\mathrm{II},2}$, and $\Gamma_{31}^{\mathrm{II},2}$ for Ω^2, and $\Gamma_{31}^{\mathrm{II},3}$, $\Gamma_{34}^{\mathrm{II},3}$, $\Gamma_{23}^{\mathrm{II},3}$, and $\Gamma_{24}^{\mathrm{II},3}$ for Ω^3.

In Step II.7, this paper employed the Type II PF LINMAP model in Eq. (48) to set up a parametric linear programming model. Solving the model, the optimal collective weight vector $\overline{w}=\left(\overline{w}_1,\overline{w}_2,\cdots,\overline{w}_6\right)$ =(0.0250, 0.0250, 0.0535, 0.7810, 0.0905, 0.0250), and the optimal individual degrees of violation were as follows: $\overline{\Gamma}_{43}^{\mathrm{II},2}=0.4657$ and $\overline{\Gamma}_{32}^{\mathrm{II},1}=\overline{\Gamma}_{41}^{\mathrm{II},1}=\overline{\Gamma}_{31}^{\mathrm{II},1}=\overline{\Gamma}_{21}^{\mathrm{II},2}=\overline{\Gamma}_{24}^{\mathrm{II},2}=\overline{\Gamma}_{31}^{\mathrm{II},2}=\overline{\Gamma}_{31}^{\mathrm{II},3}=\overline{\Gamma}_{34}^{\mathrm{II},3}=\overline{\Gamma}_{23}^{\mathrm{II},3}=\overline{\Gamma}_{24}^{\mathrm{II},3}=0$.

In Step II.8, the optimal group comprehensive dominance measures were obtained using Eq. (50) as follows: $\sum_{k=1}^{3}\overline{\mathbb{C}}_1^{\mathrm{II},k}=1.5680$ for z_1, $\sum_{k=1}^{3}\overline{\mathbb{C}}_2^{\mathrm{II},k}=2.4281$ for z_2, $\sum_{k=1}^{3}\overline{\mathbb{C}}_3^{\mathrm{II},k}=2.4149$ for z_3, and $\sum_{k=1}^{3}\overline{\mathbb{C}}_4^{\mathrm{II},k}=1.8224$ for z_4. The overall priority ranking of the alternatives was $z_2 \succ z_3 \succ z_4 \succ z_1$, along with the best compromise solution z_2. The obtained results using Algorithm II are in conformity with those yielded by the proposed Algorithm I and Xue *et al.*'s approach [26].

7. COMPARATIVE ANALYSIS AND DISCUSSIONS

This section intends to implement a sensitivity analysis and conduct some comparative studies to validate the workableness and attractiveness of the proposed methodology.

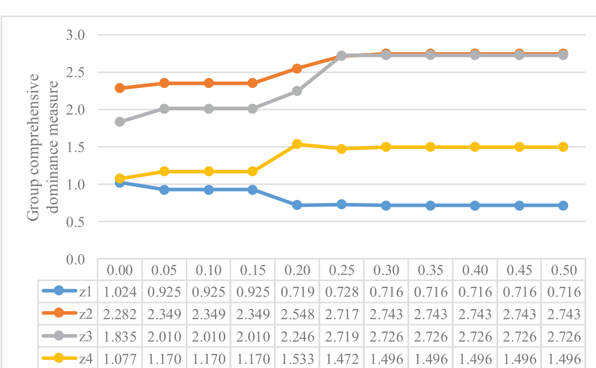
Based on the Types I and II PF LINMAP models in Eqs. (47) and (48), the parameter η can adjust the proportion of the "total group comprehensive dominance" part and the "group poorness of fit" part. The proposed methodology, which originate from classical LINMAP, should initiate ideas that are least violated in regard to the decision-makers' paired preference relations over alternatives. This implies that the proportion of the "group poorness of fit" part (with the weight $1-\eta$) should be larger than the other part. To inherit the merits of classical LINMAP, $0 \leq \eta \leq 0.5$ is recommended for practical application. Accordingly, the following sensitivity analysis would explore the application results concerning the MCGDM problem of railway project investment with different values of η varying from 0 to 0.5 under the settings of $\hbar^k = 0.3$ for $k \in \{1,2,3\}$ and $\varepsilon_j = 0.025$ for $j \in \{1,2,\cdots,6\}$.

The first sensitivity analysis investigates the influences of various values of the parameter η on the solution results using the Type I PF LINMAP model. Consider eleven cases in which $\eta = 0.00, 0.05, \cdots, 0.50$. Employing Algorithm I, the comparison results of the optimal group comprehensive dominance measure $\sum_{k=1}^{3} \overline{\mathbb{C}}_i^{\mathrm{I},k}$ of the four alternatives are depicted in Figure 2.

According to the comparison results in Figure 2, it is easy to observe that the Type I PF LINMAP model has high robustness because the identical ranking results were obtained in most cases. As shown in this figure, the Type I PF LINMAP model rendered the overall priority ranking $z_2 \succ z_3 \succ z_4 \succ z_1$ in the ten cases of $\eta = 0.00, 0.05, \cdots, 0.20$ and $\eta = 0.30, 0.35, \cdots, 0.50$. In these cases, the same best compromise solution z_2 was obtained as well. Nonetheless, a different ranking $z_3 \succ z_2 \succ z_4 \succ z_1$ was acquired in the case of $\eta = 0.25$. Moreover, the best compromise solution becomes z_3 in this case. As a whole, the optimal group comprehensive dominance measures $\sum_{k=1}^{3} \overline{\mathbb{C}}_2^{\mathrm{I},k}$, $\sum_{k=1}^{3} \overline{\mathbb{C}}_3^{\mathrm{I},k}$, and $\sum_{k=1}^{3} \overline{\mathbb{C}}_4^{\mathrm{I},k}$ for alternatives z_2, z_3, and z_4, respectively, progressively increase when the values of η is from 0 to 0.5. In contrast, for z_1, the results of $\sum_{k=1}^{3} \overline{\mathbb{C}}_1^{\mathrm{I},k}$ decrease with η varying from 0 to 0.5.

Additionally, the contrast results of the optimal Type I comprehensive dominance measure $\overline{\mathbb{C}}_i^{\mathrm{I},k}$ for the decision-makers e_1, e_2, and e_3 are revealed in Figures 3–5, respectively. With reference to the results in these figures, the comparisons among the optimal Type I comprehensive dominance measures $\overline{\mathbb{C}}_1^{\mathrm{I},k}$, $\overline{\mathbb{C}}_2^{\mathrm{I},k}$, $\overline{\mathbb{C}}_3^{\mathrm{I},k}$, and $\overline{\mathbb{C}}_4^{\mathrm{I},k}$ are obvious for the decision-maker $e_k \in E$.

The priority rankings of the four alternatives rendered by the Type I PF LINMAP model are demonstrated in Figure 6, consisting of the overall priority ranking orders for group decision-makers and the individual ranking orders for each decision-maker. As mentioned before, the overall priority ranking $z_2 \succ z_3 \succ z_4 \succ z_1$ was acquired in most cases for group decision-makers. On the other hand, consider the individual ranking results for the decision-makers e_1, e_2,



	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
z1	1.024	0.925	0.925	0.925	0.719	0.728	0.716	0.716	0.716	0.716	0.716
z2	2.282	2.349	2.349	2.349	2.548	2.717	2.743	2.743	2.743	2.743	2.743
z3	1.835	2.010	2.010	2.010	2.246	2.719	2.726	2.726	2.726	2.726	2.726
z4	1.077	1.170	1.170	1.170	1.533	1.472	1.496	1.496	1.496	1.496	1.496

Figure 2 Comparisons of the optimal group comprehensive dominance measure $\sum_{k=1}^{3} \overline{\mathbb{C}}_i^{\mathrm{I},k}$ under various values of η ($\hbar^k = 0.3$ and $\varepsilon_j = 0.025$).

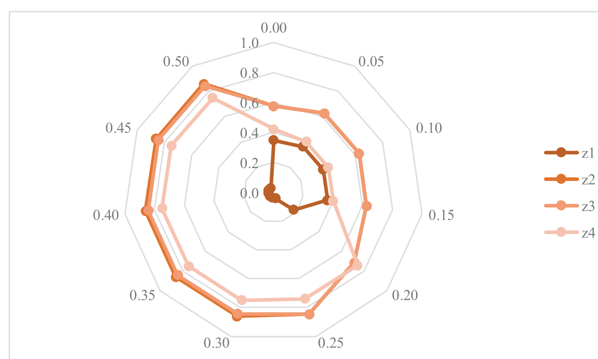


Figure 3 Contrasts of the optimal Type I comprehensive dominance measure $\overline{\mathbb{C}}_i^{\mathrm{I},1}$ under various values of η for the decision-maker e_1.

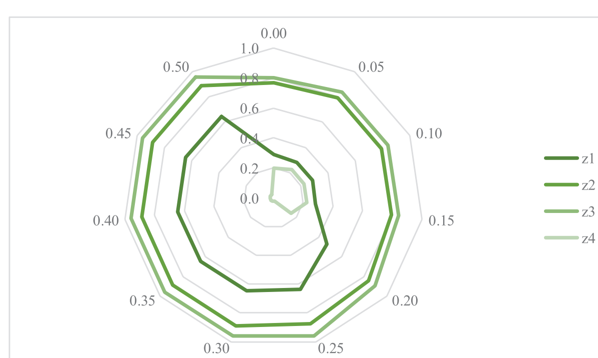


Figure 4 Contrasts of the optimal Type I comprehensive dominance measure $\overline{\mathbb{C}}_i^{\mathrm{I},2}$ under various values of η for the decision-maker e_2.

and e_3. By comparing the optimal Type I comprehensive dominance measure $\overline{\mathbb{C}}_i^{\mathrm{I},1}$ for all $z_i \in Z$, three ranking results $z_3 \succ z_2 \succ z_4 \succ z_1$, $z_4 \succ z_3 \succ z_2 \succ z_1$, and $z_2 \succ z_3 \succ z_4 \succ z_1$ were generated in the cases of $\eta = 0.00, 0.05, 0.10, 0.15, 0.25$, $\eta = 0.20$, and $\eta = 0.30, 0.35, \cdots, 0.50$, respectively, for the decision-maker e_1. Next, based on the $\overline{\mathbb{C}}_i^{\mathrm{I},2}$ values, only one ranking $z_3 \succ z_2 \succ z_1 \succ z_4$ was determined regardless of the η values for the decision-maker e_2. Based on the $\overline{\mathbb{C}}_i^{\mathrm{I},3}$ values, two ranking results $z_2 \succ z_3 \sim z_4 \succ z_1$ and $z_2 \succ z_3 \succ z_4 \succ z_1$ were found in the cases of $\eta = 0.00, 0.05, \cdots, 0.20$ and $\eta = 0.25, 0.30, \cdots, 0.50$, respectively, for the decision-maker e_3.

The second sensitivity analysis explores the effects of different values of the parameter η on the results using the Type II PF LINMAP model. Let $\eta = 0.00, 0.05, \cdots, 0.50$ under the settings of $\hbar^k = 0.3$ and $\varepsilon_j = 0.025$ for all k and j. Applying Algorithm II, the comparison results of the optimal group comprehensive dominance measure $\sum_{k=1}^{3} \overline{\mathbb{C}}_i^{\mathrm{II},k}$ of the alternatives are represented in Figure 7.

From Figure 7, it can be observed that the Type II PF LINMAP model shows strong robustness because the same ranking results were obtained in all of the investigated cases. Specifically, the Type II PF LINMAP model produced the overall priority ranking $z_2 \succ z_3 \succ z_4 \succ z_1$ with different values of η varying from 0 to 0.5. Moreover, the alternative z_2 was the best compromise solution in the MCGDM problem. Overall, the optimal group comprehensive dominance measures $\sum_{k=1}^{3} \overline{\mathbb{C}}_i^{\mathrm{II},k}$ for all $z_i \in Z$ slowly increase as the η value is from 0 to 0.25. When $\eta > 0.25$, the values of $\sum_{k=1}^{3} \overline{\mathbb{C}}_i^{\mathrm{II},k}$ are almost unchangeable for each alternative.

The comparison results of the optimal Type II comprehensive dominance measure $\overline{\mathbb{C}}_i^{\mathrm{II},k}$ for the decision-makers e_1, e_2, and e_3 are sketched in Figures 8–10, respectively. As opposite to Figures 3–5, the contrasts among the optimal Type II comprehensive dominance measures $\overline{\mathbb{C}}_1^{\mathrm{II},k}$, $\overline{\mathbb{C}}_2^{\mathrm{II},k}$, $\overline{\mathbb{C}}_3^{\mathrm{II},k}$, and $\overline{\mathbb{C}}_4^{\mathrm{II},k}$ do not vary dramatically for each $e_k \in E$.

Figure 11 shows the ranking results of the four candidate alternatives produced by the Type II PF LINMAP model. The graphs in this figure consist of the overall priority ranking orders for group decision-makers and the individual ranking orders for each decision-maker. In line with Figure 7, the overall priority ranking $z_2 \succ z_3 \succ z_4 \succ z_1$ was acquired with η varying from 0 to 0.5 for group decision-makers. Next, consider the individual ranking results based on the optimal Type II comprehensive dominance measures $\overline{\mathbb{C}}_i^{\mathrm{II},1}$, $\overline{\mathbb{C}}_i^{\mathrm{II},2}$, and $\overline{\mathbb{C}}_i^{\mathrm{II},3}$ for the decision-makers e_1, e_2, and

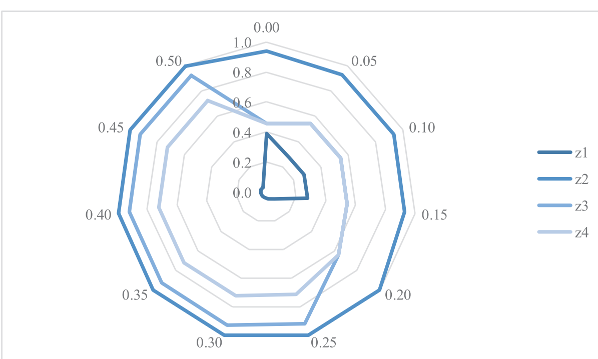


Figure 5 | Contrasts of the optimal Type I comprehensive dominance measure $\overline{\mathbb{C}}_i^{\mathrm{I},3}$ under various values of η for the decision-maker e_3.

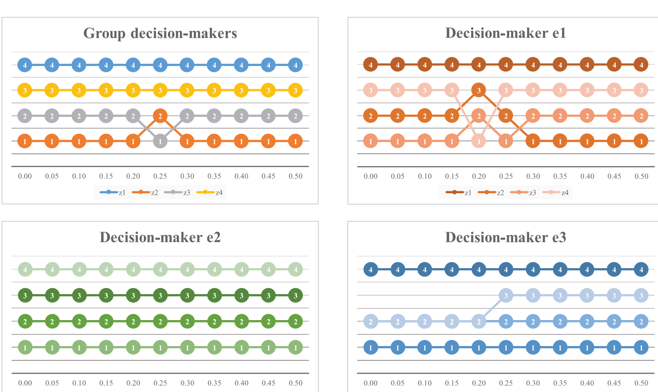
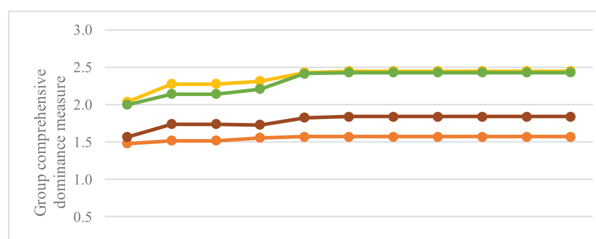


Figure 6 | Results of the priority ranking orders for group and individual decision-makers via the Type I PF linear programming technique for multidimensional analysis of preference (LINMAP) model.



	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
z1	1.477	1.516	1.516	1.554	1.568	1.568	1.568	1.568	1.568	1.568	1.568
z2	2.034	2.273	2.273	2.313	2.428	2.448	2.448	2.448	2.448	2.448	2.448
z3	1.994	2.140	2.140	2.209	2.414	2.426	2.426	2.426	2.426	2.426	2.426
z4	1.566	1.736	1.736	1.726	1.822	1.840	1.840	1.840	1.840	1.840	1.840

Figure 7 | Comparisons of the optimal group comprehensive dominance measure $\sum_{k=1}^{3} \overline{\mathbb{C}}_i^{\mathrm{II},k}$ under various values of η ($\hbar^k = 0.3$ and $\varepsilon_j = 0.025$).

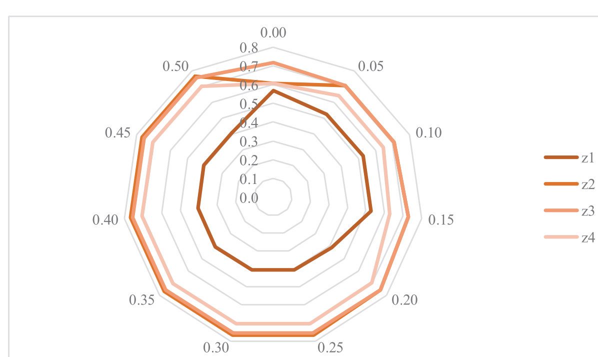


Figure 8 | Contrasts of the optimal Type II comprehensive dominance measure $\overline{\mathbb{C}}_i^{\mathrm{II},1}$ under various values of η for the decision-maker e_1.

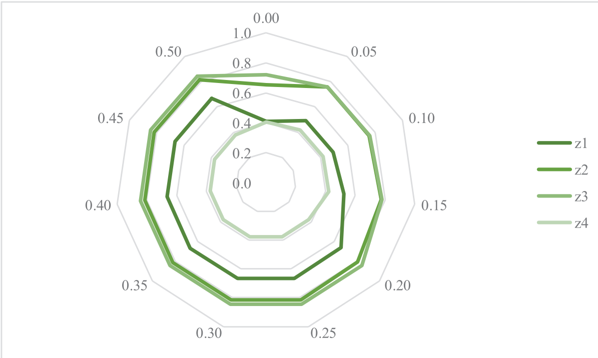


Figure 9 Contrasts of the optimal Type II comprehensive dominance measure $\overline{\mathbb{C}}_i^{\text{II},2}$ under various values of η for the decision-maker e_2.

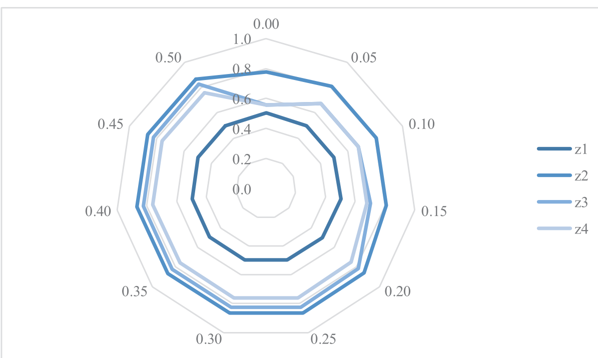


Figure 10 Contrasts of the optimal Type II comprehensive dominance measure $\overline{\mathbb{C}}_i^{\text{II},3}$ under various values of η for the decision-maker e_3.

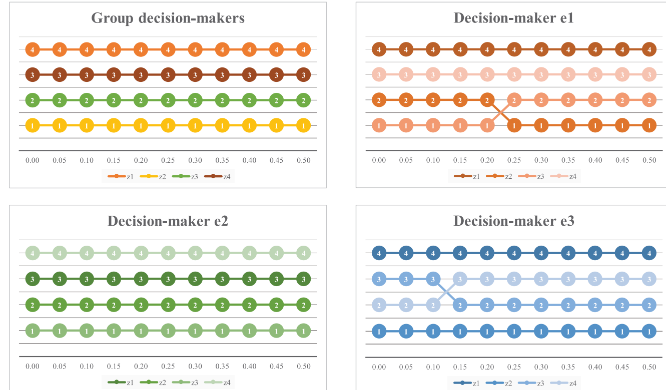


Figure 11 Results of the priority ranking orders for group and individual decision-makers via the Type II PF linear programming technique for multidimensional analysis of preference (LINMAP) model.

e_3, respectively. For the decision-maker e_1, two ranking results $z_3 \succ z_2 \succ z_4 \succ z_1$ and $z_2 \succ z_3 \succ z_4 \succ z_1$ were obtained in the cases of $\eta = 0.00, 0.05, \cdots, 0.20$ and $\eta = 0.25, 0.30, \cdots, 0.50$, respectively, by comparing the $\overline{\mathbb{C}}_i^{\text{II},1}$ values for all $z_i \in Z$. Based on the $\overline{\mathbb{C}}_i^{\text{II},2}$ values, only one ranking $z_3 \succ z_2 \succ z_1 \succ z_4$ was rendered regardless of the η values for the decision-maker e_2. Based on the $\overline{\mathbb{C}}_i^{\text{II},3}$ values, two ranking results $z_2 \succ z_4 \succ z_3 \succ z_1$ and $z_2 \succ z_3 \succ z_4 \succ z_1$ were generated when $\eta = 0.00, 0.05, 0.10$ and $\eta = 0.15, 0.20, \cdots, 0.50$, respectively, for the decision-maker e_3.

Xue *et al.* [26] conducted a comprehensive analysis on the MCGDM problem involving the railway project investment to show the reliability and the sensitivity of their developed PF entropy-based LINMAP method. The three rankings $z_2 \succ z_3 \succ z_4 \succ z_1$, $z_3 \succ z_2 \succ z_1 \succ z_4$, and $z_3 \succ z_2 \succ z_4 \succ z_1$ were acquired according to their obtained results. As a whole, the priority rankings generated via the proposed PF LINMAP methodology are consistent with those using the PF entropy-based LINMAP approach. Moreover, a clear consensus on the MCGDM problem of railway project investment is that alternatives z_2 and z_3 are the better choices, while alternatives z_1 and z_4 are the worse choices. The practicability and effectiveness of the proposed methodology can be verified via the comparative analysis with Xue *et al.*'s approach. Nevertheless, in contrast to the PF entropy measure-based approach, the developed techniques using the PF scalar function-based approach provide an easy way of handling complex PF information. The implementation procedures are much simpler than Xue *et al.*'s approach. Even though, the workableness and advantages of the proposed methodology have been examined by means of the practical application in the railway project investment case. Moreover, the proposed methodology is capable of yielding robust and reliable ranking results according to the sensitivity analysis and comparative discussions.

8. CONCLUSIONS

This paper has extended the core structure of the classical LINMAP methods to the complex uncertainty of Pythagorean fuzziness and has developed a new PF LINMAP-based compromising methodology based on the proposed PF scalar function-based approach for solving MCGDM problems.

The proposed PF LINMAP methodology not only has the merit of the LINMAP in handling preference information over alternatives but also enhance uncertain LINMAP-based approaches by incorporating some novel concepts (e.g., PF scalar function-based dominance measures and Types I and II comprehensive dominance measures) in the core procedure of LINMAP. Two parametric linear programming problems have been formulated for effectively implementing Types I and II PF LINMAP models. Moreover, compared with the Type I PF LINMAP model, the Type II PF LINMAP model has the merits of simplicity and efficiency. The helpfulness and effectiveness of the proposed PF LINMAP-based compromising methods have been verified via the application to the MCGDM problem in regard to railway project investment. Compared to the existing PF LINMAP methods, the developed PF scalar function-based approach is capable of manipulating PF information in a fairly easy and straightforward matter. Moreover, both Types I and II PF LINMAP models have high robustness because of the reasonable and reliable application results via the sensitivity analysis and discussions. Future research can focus on the extension of incorporating the PF scalar function-based approach into other decision-making models. Furthermore, the proposed PF LINMAP-based compromising methodology can be applied to address uncertain MCGDM problems in other application fields.

CONFLICT OF INTEREST

The authors declare that there is no conflict of interest regarding the publication of this paper.

AUTHORS' CONTRIBUTIONS

Jih-Chang Wang: Methodology, Software, Validation, Formal analysis, Data Curation, Writing – Original Draft, Visualization. Ting-Yu Chen: Conceptualization, Methodology, Validation, Formal analysis, Writing – Original Draft, Writing – Review & Editing, Supervision, Funding acquisition.

DATA AVAILABILITY STATEMENT

The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

ETHICAL APPROVAL

This article does not contain any studies with human participants or animals that were performed by the authors.

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APPENDIX

$$\max\Big\{0.2\,(6.4357w_1 + 5.9404w_2 + 6.0264w_3 + 7.9005w_4 +6.3360w_5 + 6.0463w_6) - 0.8\left(\Gamma_{32}^{\mathrm{I},1} + \Gamma_{41}^{\mathrm{I},1} + \Gamma_{31}^{\mathrm{I},1} + \Gamma_{21}^{\mathrm{I},2} +\Gamma_{43}^{\mathrm{I},2} + \Gamma_{24}^{\mathrm{I},2} + \Gamma_{31}^{\mathrm{I},2} + \Gamma_{31}^{\mathrm{I},3} + \Gamma_{34}^{\mathrm{I},3} + \Gamma_{23}^{\mathrm{I},3} + \Gamma_{24}^{\mathrm{I},3}\right)\Big\}$$

subject to:

$$\begin{aligned}
&0.5983w_1 + 0.1452w_2 + 1.5660w_3 + 1.6854w_4 - 1.4806w_5 +0.4021w_6 \geq 0.3,\\
&0.8389w_1 + 1.4742w_2 - 0.0004w_3 + 0.3900w_4 + 2.0000w_5 +1.5034w_6 \geq 0.3,\\
&0.8934w_1 + 0.8829w_2 + 1.6919w_3 + 1.5106w_4 + 0.3515w_5 +0.5252w_6 \geq 0.3,\\
&-0.5949w_1 - 0.1207w_2 + 0.7400w_3 + 0.0078w_4 - 0.5114w_5 -0.5000w_6 + \Gamma_{32}^{\mathrm{I},1} \geq 0,\\
&1.0000w_1 + 0.3865w_2 + 0.0859w_3 + 0.7818w_4 - 0.6438w_5 +0.4021w_6 + \Gamma_{41}^{\mathrm{I},1} \geq 0,\\
&0.1931w_1 - 0.1207w_2 + 0.7400w_3 + 0.8957w_4 - 0.3254w_5 +0.5000w_6 + \Gamma_{31}^{\mathrm{I},1} \geq 0,\\
&0.4195w_1 + 0.7371w_2 - 0.0002w_3 + 0.1950w_4 + 1.0000w_5 +0.7517w_6 + \Gamma_{21}^{\mathrm{I},2} \geq 0,\\
&-0.5805w_1 - 0.2413w_2 - 1.0000w_3 - 1.0000w_4 - 0.2696w_5 -0.3706w_6 + \Gamma_{43}^{\mathrm{I},2} \geq 0,\\
&1.0000w_1 + 0.3215w_2 + 0.4383w_3 + 0.8927w_4 + 0.5561w_5 +1.0000w_6 + \Gamma_{24}^{\mathrm{I},2} \geq 0,\\
&-0.0001w_1 + 0.6569w_2 + 0.5615w_3 + 0.3023w_4 + 0.7135w_5 +0.1223w_6 + \Gamma_{31}^{\mathrm{I},2} \geq 0,\\
&-0.3567w_1 + 0.0000w_2 + 0.0961w_3 + 1.0000w_4 + 0.1782w_5 +0.5252w_6 + \Gamma_{31}^{\mathrm{I},3} \geq 0,\\
&-0.0940w_1 - 0.5585w_2 + 0.3868w_3 + 0.2553w_4 - 0.1782w_5 -0.4748w_6 + \Gamma_{34}^{\mathrm{I},3} \geq 0,\\
&0.7191w_1 + 1.0000w_2 + 0.4111w_3 + 0.0000w_4 + 0.2648w_5 +0.4748w_6 + \Gamma_{23}^{\mathrm{I},3} \geq 0,\\
&0.6251w_1 + 0.4415w_1 + 0.7979w_3 + 0.2553w_4 + 0.0866w_5 +0.0000w_6 + \Gamma_{24}^{\mathrm{I},3} \geq 0,\\
&\Gamma_{32}^{\mathrm{I},1}, \Gamma_{41}^{\mathrm{I},1}, \Gamma_{31}^{\mathrm{I},1}, \Gamma_{21}^{\mathrm{I},2}, \Gamma_{43}^{\mathrm{I},2}, \Gamma_{24}^{\mathrm{I},2}, \Gamma_{31}^{\mathrm{I},2}, \Gamma_{31}^{\mathrm{I},3}, \Gamma_{34}^{\mathrm{I},3}, \Gamma_{23}^{\mathrm{I},3}, \Gamma_{24}^{\mathrm{I},3} \geq 0,\\
&\sum_{j=1}^{6} w_j = 1,\ w_j \geq 0.025 \text{ for } j = 1, 2, \cdots, 6.
\end{aligned}$$