

## Special Issue

## On Contradiction and Inclusion Using Functional Degrees

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## ABSTRACT

The notion of inclusion is a cornerstone in set theory and therefore, its generalization in fuzzy set theory is of great interest. The degree of  $f$ -inclusion is one generalization of such a notion that differs from others existing in the literature because the degree of inclusion is considered as a mapping instead of a value in the unit interval. On the other hand, the degree of  $f$ -weak-contradiction was introduced to represent the contradiction between two fuzzy sets via a mapping and its definition has many similarities with the  $f$ -degree of inclusion. This suggests the existence of relations between both  $f$ -degrees. Specifically, following this line, we analyze the relationship between the  $f$ -degree of inclusion and the  $f$ -degree of contradiction via the complement of fuzzy sets and Galois connections.

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## 1. INTRODUCTION

Determining different relationships between fuzzy sets is crucial for the development of different techniques based on fuzzy logic. In set theory, the most important of these relations is, without any doubt, the inclusion (which leads to the equality); however, in spite of the years since the origin of fuzzy sets [1] and its significance, there is no consensus about how to extend inclusion to fuzzy logic. The community of researchers in fuzzy sets and fuzzy logic has focused mainly in the definition of measures which represent the inclusion between fuzzy sets by the assignment of a value in the unit interval. In the literature we can find two main groups of approaches, those presenting constructive measures [2–5] and those based on axiomatic properties [6–9].

On the other hand, contradiction between two statements is usually related to negations, in the sense that if one is true then the other cannot be true. In some cases the negation appears explicitly (“ $A$  is big” vs “ $A$  is not big”) and in others implicitly (“ $A$  is big” vs “ $A$  is small”). Therefore, it is not strange that most of the approaches related to contradiction are built upon negation operators. Our notion of  $f$ -weak contradiction is based on the notion of  $N$ -contradiction [10], which represents the contradiction between two fuzzy sets in terms of a negation operator. Specifically, when substituting the negation operator by a more general mapping (keeping the antitonicity) we obtain the notion of  $f$ -weak contradiction which provides a convenient lattice structure that allows the definition of an  $f$ -degree of contradiction by the use of infimum [11].

One of the remarkable features of the  $f$ -degree of contradiction is that it is determined by taking into account just the information

provided by the fuzzy sets involved. That means that it does not require any preliminary assumption as a residuated lattice structure or negation operator. As a result, we interpret the contradiction between two fuzzy sets simply as “the greater the membership value of one, the lesser the membership value of the other.” This idea was later adapted to the problem of representing the inclusion between two fuzzy sets, as “the greater the membership value of one, the greater the membership value of the other,” generating the notion of  $f$ -inclusion.

The functional approach stated above proposes modelling the degree of inclusion and contradiction by means of certain mappings in the unit interval and leads to the notions of *degree of  $f$ -inclusion* [12,13] and *degree of  $f$ -contradiction* [11]. Using functions in this context is certainly a distinctive feature with respect to previous approaches that can be found in the literature, since most of the studies to both inclusion and contradiction in the fuzzy realm coincide in that the measure of inclusion/contradiction is given by a number in the unit interval.

Interestingly enough, despite the functional character of  $f$ -inclusion, many of the axioms required for measures of inclusion based on real numbers [6–9], after a suitable transformation of real values into mappings [14], also hold in the framework of  $f$ -inclusion which, in fact, can be considered as a kind of Sinha–Dougherty inclusion [7].

As stated previously, the definitions of the  $f$ -degree of contradiction and  $f$ -degree of inclusion are defined similarly, since both share the same roots and also share the same underlying structure: they are both defined as the infimum (or supremum) of a set of mappings which determine crisp relations in terms of an inequality. Such a similarity suggests the existence of some relationship between both notions, and this defines the main contributions of this paper.

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Specifically, we will show that the crisp relationships of  $f$ -inclusion and  $f$ -weak-contradiction can be related to each other in terms of negation operators and isotone Galois connections (also called adjunctions) [15]. Moreover, we show that, in the case of considering a finite universe, the  $f$ -degrees of inclusion and  $f$ -degrees of contradiction form part of two Galois connections, which can be related via the complement of fuzzy sets.

The paper is structured as follows: in Section 2 we recall some basic notions of fuzzy set theory, the notion of  $f$ -inclusion and the notion of  $f$ -weak-contradiction. In Section 3 we study the properties of the  $f$ -degree of contradiction and, then, in Section 4 we introduce the main contribution of the paper, namely, the relation between  $f$ -inclusion and  $f$ -contradiction when a negation operator is explicitly introduced. Finally, in Section 5 we present conclusions and future works.

## 2. PRELIMINARIES

### 2.1. Basic Notions about Fuzzy Sets and Galois Connections

Let us recall that a *fuzzy set*  $A$  is defined on a referential universe  $\mathcal{U}$  (usually omitted) by means of its membership function  $A: \mathcal{U} \rightarrow [0, 1]$ . The standard operations *union* and *intersection* between sets can be extended to operations between fuzzy sets as follows: given two fuzzy sets  $A$  and  $B$ , we define the fuzzy sets  $A \cup B$  and  $A \cap B$  as  $A \cup B(u) = \max\{A(u), B(u)\}$  and  $A \cap B(u) = \min\{A(u), B(u)\}$  for all  $u \in \mathcal{U}$ , respectively. To define the *complement* of a fuzzy set we need to consider negation operators. Let us recall that a *negation operator* is a decreasing mapping  $n: [0, 1] \rightarrow [0, 1]$  such that  $n(0) = 1$  and  $n(1) = 0$ . Given a fixed negation operator,  $n$ , the complement of a fuzzy set  $A$  is defined as  $A^c(u) = n(A(u))$  for all  $u \in \mathcal{U}$ . We say that a negation  $n$  is *involution* if  $n^2(x) = n(n(x)) = x$  for all  $x \in [0, 1]$ . Note that in the case of considering an involutive negation for the definition of the complement, the double complement law holds; i.e.,  $(A^c)^c = A$ .

We will also need the notion of isotone Galois connection (or adjunction). The formal definitions are given below:

**Definition 1.** A pair  $(f, g)$  of mappings  $f, g: [0, 1] \rightarrow [0, 1]$  forms an *isotone Galois connection* if and only if the equivalence below holds for all  $x, y \in [0, 1]$ :

$$f(x) \leq y \iff x \leq g(y) \quad (1)$$

Dually, A pair  $(f, g)$  of mappings  $f, g: [0, 1] \rightarrow [0, 1]$  forms an *anti-tone Galois connection* if and only if the equivalence below holds for all  $x, y \in [0, 1]$ :

$$y \leq f(x) \iff x \leq g(y) \quad (2)$$

In the paper we use the following result concerning isotone Galois connection in  $[0, 1]$ .

**Lemma 1.**

1. If  $f: [0, 1] \rightarrow [0, 1]$  is a mapping such that  $f(0) = 0$  and  $f(\sup(X)) = \sup_{x \in X} \{f(x)\}$  for all set  $X \subseteq [0, 1]$ , then there exists a mapping  $g: [0, 1] \rightarrow [0, 1]$  such that  $(f, g)$  forms an isotone Galois connection.

2. If  $g: [0, 1] \rightarrow [0, 1]$  is a mapping such that  $g(1) = 1$  and  $g(\inf(X)) = \inf_{x \in X} \{g(x)\}$  for all set  $X \subseteq [0, 1]$ , then there exists a mapping  $f: [0, 1] \rightarrow [0, 1]$  such that  $(f, g)$  forms an isotone Galois connection.

### 2.2. The Functional Degree of Inclusion

The main difference of our approach with respect to other approaches dealing with measures of inclusion between fuzzy sets [6–9], is that the degrees used to express the measure are no longer real values, but certain mappings from  $[0, 1]$  to  $[0, 1]$ . This is why hereafter we will often use the particular mappings  $f$  as parameters in the prefixes of several notions. These mappings should satisfy the properties of deflation and monotonicity in order to be considered indexes of inclusion.

**Definition 2.** The set of  *$f$ -degrees of inclusion* (denoted by  $\Omega$ ) is the set of mappings  $f: [0, 1] \rightarrow [0, 1]$  satisfying the following properties for all  $x, y \in [0, 1]$ :

- $f(x) \leq x$ ;
- if  $x \leq y$  then  $f(x) \leq f(y)$

The definition of  $f$ -inclusion [13] is given as follows.

**Definition 3.** Let  $A$  and  $B$  be two fuzzy sets and consider  $f \in \Omega$ . We say that  $A$  is  *$f$ -included in  $B$*  (denoted by  $A \subseteq_f B$ ) if and only if the inequality  $f(A(u)) \leq B(u)$  holds for all  $u \in \mathcal{U}$ .

It is worth remarking that, despite the fact that  $f$ -inclusion is a crisp relation for all  $f \in \Omega$ , we can define a fuzzy-like inclusion relation by considering the whole set of  $f$ -inclusion mappings as possible degrees of inclusion. In other words, different mappings in  $\Omega$  define different relations of  $f$ -inclusion that determine stronger or weaker restrictions.

Another interesting feature of  $\Omega$  that made it suitable to be considered a set of degrees is its lattice structure with the usual pointwise ordering between mappings. We will use  $id$  and  $\perp$  to refer to the greatest and lowest mappings in  $\Omega$ ; i.e.,  $id(x) = x$  and  $\perp(x) = 0$  for all  $x \in [0, 1]$ . Note that  $id$  and  $\perp$  represent the strongest and the weakest degrees of  $f$ -inclusion. Below, we summarize some properties of the notion of  $f$ -inclusion which motivate the consideration of mappings in  $\Omega$  as proper degrees of inclusion [12].

**Theorem 2.** Let  $A, B$  and  $C$  be three fuzzy sets and let  $f, g \in \Omega$ . Then,

1.  $A \subseteq_f A$ ;
2.  $A \subseteq_f B$  and  $B \subseteq_g C$  implies  $A \subseteq_{g \circ f} C$ ;
3.  $A \subseteq_{\perp} B$ ;
4.  $A \subseteq_{id} B \iff A(u) \leq B(u)$  for all  $u \in \mathcal{U} \iff A \subseteq_{f^*} B$  for all  $f^* \in \Omega$ ;
5. If  $f \leq g$  and  $A \subseteq_g B$ , then  $A \subseteq_f B$ .

Moreover, the greater the mapping  $f \in \Omega$  the stronger the restriction imposed by the  $f$ -inclusion; since given  $A \subseteq_f B$ , the value  $f(A(u))$  determines a lower bound of the possible values of  $B(u)$ , for all  $u \in \mathcal{U}$ . In this way, the degree of inclusion of fuzzy set  $A$  into

fuzzy set  $B$  can be defined by choosing the greatest  $f \in \Omega$  such that  $A$  is  $f$ -included in  $B$ ; for more details we refer the reader to [12].

**Definition 4.** Let  $A$  and  $B$  be two fuzzy sets, the  $f$ -degree of inclusion of  $A$  in  $B$ , denoted by  $\text{Inc}(A, B)$ , is defined as follows

$$\text{Inc}(A, B) = \max\{f \in \Omega \mid A \subseteq_f B\}.$$

The analytical expression of the  $f$ -degree of inclusion is given by  $\text{Inc}(A, B) = f_{A,B} \wedge id$ , where

$$f_{A,B}(x) = \inf_{u \in \mathcal{U}} \{B(u) \mid x \leq A(u)\}. \quad (3)$$

The following result summarizes some properties of the  $f$ -degree of inclusion defined above that resemble some axiomatic approaches of measures of inclusion given in the literature [6,7].

**Theorem 3.** Let  $A, B$  and  $C$  be three fuzzy sets, then

1. If  $A$  and  $B$  are two crisp sets, then either  $\text{Inc}(A, B) = id$  or  $\text{Inc}(A, B) = \perp$ ;
2.  $\text{Inc}(A, B) = id$  if and only if  $A(u) \leq B(u)$  for all  $u \in \mathcal{U}$ ;
3.  $\text{Inc}(A, B) = \perp$  if and only if there is a set of elements in the universe  $\{u_i\}_{i \in I} \subseteq \mathcal{U}$  such that  $A(u_i) = 1$  for all  $i \in I$  and  $\bigwedge_{i \in I} B(u_i) = 0$ ;
4. If  $\mathcal{U}$  is finite,  $\text{Inc}(A, B) = \perp$  if and only if there exists  $u \in \mathcal{U}$  such that  $A(u) = 1$  and  $B(u) = 0$ ;
5.  $\text{Inc}(B, C) \circ \text{Inc}(A, B) \leq \text{Inc}(A, C)$ ;
6. If  $B(u) \leq C(u)$  for all  $u \in \mathcal{U}$  then,  $\text{Inc}(A, B) \leq \text{Inc}(A, C)$ ;
7. If  $B(u) \leq C(u)$  for all  $u \in \mathcal{U}$  then,  $\text{Inc}(C, A) \leq \text{Inc}(B, A)$ ;
8.  $\text{Inc}(A, B) = \text{Inc}(T(A), T(B))$  for all transformation<sup>1</sup>  $T: \mathcal{U} \rightarrow \mathcal{U}$  on  $\mathcal{U}$ ;
9.  $\text{Inc}(A, B \cap C) = \text{Inc}(A, B) \wedge \text{Inc}(A, C)$ ;
10.  $\text{Inc}(A \cup B, C) = \text{Inc}(A, C) \wedge \text{Inc}(B, C)$ .

## 2.3. The Notion of $f$ -Weak Contradiction

As with  $f$ -inclusion, the underlying idea here is to measure contradiction not with real values, but with certain mappings from  $[0, 1]$  to  $[0, 1]$ . In this case, those mappings used as degrees of contradiction should satisfy some properties that resemble negation operators.

**Definition 5.** The set of  $f$ -degrees of contradiction (denoted by  $\overline{\Omega}$ ) is the set of mappings  $f: [0, 1] \rightarrow [0, 1]$  satisfying the following properties for all  $x, y \in [0, 1]$ :

- $f(0) = 1$ ;
- if  $x \leq y$  then  $f(y) \leq f(x)$

The notion of  $f$ -weak-contradiction [16,11] is a generalization of the notion of  $N$ -contradiction given by Trillas, Alsina and Jacas [10].

**Definition 6.** Let  $A$  and  $B$  be two fuzzy sets defined over a nonempty universe  $\mathcal{U}$  and let  $f \in \overline{\Omega}$ . We say that  $A$  is  $f$ -weak-contradictory w.r.t.  $B$  if and only if  $A(u) \leq f(B(u))$  holds for all  $u \in \mathcal{U}$ .

As in the case of the  $f$ -inclusion, the  $f$ -weak-contradiction defines crisp relations which determine a certain degree of contradiction between fuzzy sets. Note that  $\overline{\Omega}$  has a structure of complete lattice with the usual pointwise ordering between mappings and that different mappings in  $\overline{\Omega}$  determine different kinds of  $f$ -weak-contradiction relations, i.e. a stronger or weaker restriction. Specifically, fixed  $f \in \overline{\Omega}$  and a value of  $B(u)$  (for some  $u \in \mathcal{U}$ ), the  $f$ -weak-contradiction determines an upper bound on the value of  $A(u)$ .

Note that, as  $f$  is antitonic, the greater the value of  $B(u)$ , the smaller the upper bound, and then also the smaller the value of  $A(u)$ . Therefore, given two fuzzy sets  $A$  and  $B$ , a degree of contradiction between  $A$  and  $B$  can be defined by choosing the least  $f \in \overline{\Omega}$  such that  $A$  is  $f$ -weak-contradictory w.r.t.  $B$ . We denote by  $f^T$  and  $f_\perp$  the greatest and lowest mapping in  $\overline{\Omega}$ ; i.e., the weakest and the strongest degrees of  $f$ -weak-contradiction; for more details we refer the reader to [11].

Below we summarise some properties of the notion of  $f$ -weak-contradiction which motivates the consideration of  $\overline{\Omega}$  as a proper set of degrees of contradiction.

**Theorem 4.** Let  $A, B$  and  $C$  be three fuzzy sets and let  $f, g \in \overline{\Omega}$ . Then,

1.  $A$  is  $f^T$ -weak-contradictory w.r.t.  $B$ ;
2. If  $A$  is  $f_\perp$ -weak-contradictory w.r.t.  $B$  then,  $A$  is  $f$ -weak-contradictory w.r.t.  $B$ ;
3. If  $f \leq g$  and  $A$  is  $f$ -weak-contradictory w.r.t.  $B$  then,  $A$  is  $g$ -weak-contradictory w.r.t.  $B$  as well;
4.  $A$  is  $f_\perp$ -weak-contradictory w.r.t.  $B$  if and only if  $B(u) > 0$  implies  $A(u) = 0$  for all  $u \in \mathcal{U}$ ;
5. If  $A \leq B$  and  $C$  is  $f$ -weak-contradictory w.r.t.  $B$  then,  $C$  is  $f$ -weak-contradictory w.r.t.  $A$ ;
6. If  $A \leq C$  and  $C$  is  $f$ -weak-contradictory w.r.t.  $B$  then,  $A$  is  $f$ -weak-contradictory w.r.t.  $B$ ;
7.  $f^T$ -weak-contradiction is the only  $f$ -weak-contradiction of  $A$  w.r.t.  $B$  if and only if there exists a sequence  $\{u_i\}_{i \in \mathbb{N}} \subseteq \mathcal{U}$  such that  $B(u_i) = 1$  for all  $u_i$  and  $\lim A(u_i) = 1$ ;
8. If  $\mathcal{U}$  is a finite universe,  $f^T$ -weak-contradiction is the only  $f$ -weak-contradiction of  $A$  w.r.t.  $B$  if and only if there exists  $u \in \mathcal{U}$  such that  $A(u) = B(u) = 1$ ;
9. If  $(f, g)$  is an antitone Galois connection then,  $A$  is  $f$ -weak-contradictory w.r.t.  $B$  if and only if  $B$  is  $g$ -weak-contradictory w.r.t.  $A$ .

<sup>1</sup>A transformation is to be interpreted as a bijective mapping.

### 3. REVISITING THE DEGREE OF $f$ -WEAK CONTRADICTION

A measure of contradiction was defined in [11] based on a degree of  $f$ -weak contradiction. However, such a degree has not been thoroughly studied in depth. Let us start by recalling the following definition from [11].

**Definition 7.** Let  $A$  and  $B$  be two fuzzy sets, the  $f$ -degree of contradiction of  $A$  w.r.t.  $B$ , denoted by  $Con(A, B)$ , is given by the least mapping  $f \in \overline{\Omega}$  verifying that  $A$  is  $f$ -weak-contradictory w.r.t.  $B$ .

It is worth noting that, in the original approach, this degree is named “function of weak-contradiction”. We have renamed it here to stress its use as a degree and to highlight its relationship with the  $f$ -degree of inclusion.

Note that it is necessary to prove that the  $f$ -degree of contradiction is well-defined, i.e., that the least mapping  $f \in \overline{\Omega}$  verifying that  $A$  is  $f$ -weak-contradictory w.r.t.  $B$  exists. Such a proof can be found in [11] together with its analytical expression, in which  $Con(A, B)(x)$  is given by

$$\begin{cases} 1 & \text{if } x = 0 \\ \sup_{u \in \mathcal{U}} \{A(u) \mid x \leq B(u)\} & \text{otherwise.} \end{cases} \quad (4)$$

The similarities between the  $f$ -degrees of inclusion and the  $f$ -degree of contradiction are now obvious. Therefore, it makes sense to analyze the  $f$ -degree of contradiction independently from those measures of contradiction defined in [11], as it has been done for the  $f$ -degree of inclusion in [12].

The first result is a direct consequence of Theorem 4 and characterizes the two extreme degrees of contradiction, namely  $f^\perp$  and  $f^\top$  that represents the greatest and lowest degree of contradiction, respectively.

**Corollary 5.** Let  $A$  and  $B$  be two fuzzy sets, then,

1.  $Con(A, B) = f_\perp$  if and only if  $B(u) > 0$  implies  $A(u) = 0$  for all  $u \in \mathcal{U}$ .
2.  $Con(A, B) = f_\top$  if and only if there exists a sequence  $\{u_i\}_{i \in \mathbb{N}} \subseteq \mathcal{U}$  such that  $B(u_i) = 1$  for all  $u_i$  and  $\lim A(u_i) = 1$ .
3. If  $\mathcal{U}$  is a finite universe,  $Con(A, B) = f_\top$  if and only if there exists  $u \in \mathcal{U}$  such that  $A(u) = B(u) = 1$ .

The following result is also a consequence of Theorem 4 and concerns monotonicity of  $Con$ .

**Corollary 6.** Let  $A, B$  and  $C$  be three fuzzy sets, then,

1. if  $B(u) \leq C(u)$  for all  $u \in \mathcal{U}$  then,  $Con(A, B) \leq Con(A, C)$ ;
2. if  $B(u) \leq C(u)$  for all  $u \in \mathcal{U}$  then,  $Con(C, A) \leq Con(B, A)$ .

At this point, it is worth noting that the pointwise ordering between the different  $f$ -degrees of contradiction classifies the  $f$ -degrees inversely to the strength of the contradiction that they represent. That is, the smaller the function  $Con(A, B)$ , the stronger the contradiction of  $A$  w.r.t.  $B$ . The two first novel results about the  $f$ -degree of contradiction concern union and intersection between fuzzy sets.

**Theorem 7.** Let  $A, B$  and  $C$  be three fuzzy sets, then,

$$Con(A, B \cap C) = Con(A, B) \vee Con(A, C).$$

**Proof.** By the analytical expression given in Equation (4) we have that if  $x = 0$  then

$$Con(A, B \cap C)(0) = Con(A, B)(0) \vee Con(A, C)(0) = 1.$$

Let us assume  $x \neq 0$ , then by Equation (4) we have

$$Con(A, B \cap C)(x) = \sup_{u \in \mathcal{U}} \{A(u) \mid x \leq (B \cap C)(u)\}$$

which is equivalent to

$$\sup_{u \in \mathcal{U}} \{A(u) \mid x \leq B(u)\} \vee \sup_{u \in \mathcal{U}} \{A(u) \mid x \leq C(u)\}.$$

As a result,  $Con(A, B \cap C)(x) = Con(A, B)(x) \vee Con(A, C)(x)$ .

**Theorem 8.** Let  $A, B$  and  $C$  be three fuzzy sets, then,

$$Con(A \cup B, C) = Con(A, C) \vee Con(B, C).$$

**Proof.** By the analytical expression given in Equation (4) we have that if  $x = 0$  then

$$Con(A \cup B, C)(0) = Con(A, C)(0) \vee Con(B, C)(0) = 1.$$

Let us assume  $x \neq 0$ , then again by Equation (4) we have

$$Con(A \cup B, C)(x) = \sup_{u \in \mathcal{U}} \{(A \cup B)(u) \mid x \leq C(u)\}$$

which is equivalent to

$$\sup_{u \in \mathcal{U}} \{A(u) \mid x \leq C(u)\} \vee \sup_{u \in \mathcal{U}} \{B(u) \mid x \leq C(u)\}.$$

As a result,  $Con(A \cup B, C)(x) = Con(A, C)(x) \vee Con(B, C)(x)$ .

The  $f$ -degree of contradiction is invariant under transformations, similarly to the  $f$ -degree of inclusion. Let us recall that a transformation  $T: \mathcal{U} \rightarrow \mathcal{U}$  (i.e., a bijective mapping) in the universe  $\mathcal{U}$  can be extended to the set of fuzzy sets  $[0, 1]^{\mathcal{U}}$  (i.e., the fuzzy powerset of  $\mathcal{U}$ ) as  $\hat{T}: [0, 1]^{\mathcal{U}} \rightarrow [0, 1]^{\mathcal{U}}$ , which is defined by  $\hat{T}(A)(u) = A(T(u))$  for all fuzzy set  $A \in [0, 1]^{\mathcal{U}}$ .

**Proposition 9.** Let  $A$  and  $B$  be two  $L$ -fuzzy sets and let  $T: \mathcal{U} \rightarrow \mathcal{U}$  be a transformation on  $\mathcal{U}$ , then  $Con(A, B) = Con(\hat{T}(A), \hat{T}(B))$ .

**Proof.** We know that  $Con(A, B)$  and  $Con(\hat{T}(A), \hat{T}(B))$  are the infimum of the sets:

$$\{f \in \overline{\Omega} \mid A \text{ } f\text{-weak contradictory w.r.t. } B\}$$

and

$$\{f \in \overline{\Omega} \mid \hat{T}(A) \text{ } f\text{-weak contradictory w.r.t. } \hat{T}(B)\}$$

respectively. Then, if we prove that both sets are the same, necessarily  $Con(A, B) = Con(\hat{T}(A), \hat{T}(B))$ . Let  $f \in \overline{\Omega}$  such that  $A$  is  $f$ -weak contradictory w.r.t.  $B$ . Then, for all  $u \in \mathcal{U}$  we have  $A(u) \leq f(B(u))$  which, by the bijectivity of  $T$ , is equivalent to saying that for all  $u \in \mathcal{U}$  we have  $A(T(u)) \leq f(B(T(u)))$ , which is equivalent to  $\hat{T}(A)$  is  $f$ -weak contradictory w.r.t.  $\hat{T}(B)$ .



#### 4. RELATING $f$ -DEGREES OF INCLUSION AND CONTRADICTION VIA COMPLEMENTS OF FUZZY SETS

Despite the evident relationships between  $f$ -degrees of inclusion and contradiction, they measure different features. Actually, they are independent degrees and, a priori, none can be determined from the other, as the following example shows.

**Example 1.** Let us consider three fuzzy sets  $A$ ,  $B_1$  and  $B_2$  on the universe  $\mathcal{U} = \{u_1, u_2, u_3\}$  given by the table below

	$A$	$B_1$	$B_2$
$u_1$	0.2	1	0.5
$u_2$	0.5	0.8	1
$u_3$	1	0.2	0.2

Let us show firstly that  $\text{Inc}(A, B_1) = \text{Inc}(A, B_2)$  but  $\text{Con}(A, B_1) \neq \text{Con}(A, B_2)$ . By Equation (3), it is easy to check that

$$\text{Inc}(A, B_1)(x) = \text{Inc}(A, B_2)(x) = \begin{cases} x & \text{if } x \leq 0.2 \\ 0.2 & \text{otherwise.} \end{cases}$$

However, the  $f$ -degrees of contradiction differ, since by Equation (4) we have:

$$\text{Con}(A, B_1)(x) = \begin{cases} 1 & \text{if } x \leq 0.2 \\ 0.5 & \text{if } 0.2 < x \leq 0.8 \\ 0.2 & \text{otherwise.} \end{cases}$$

and

$$\text{Con}(A, B_2)(x) = \begin{cases} 1 & \text{if } x \leq 0.2 \\ 0.5 & \text{otherwise.} \end{cases}$$

Let us show now that the converse does not hold either. That is, that the  $f$ -degree of contradiction does not determine the  $f$ -degree of inclusion. For that reason, let us consider the fuzzy set  $A_2$  on  $\mathcal{U}$  given by  $A(u_1) = 1$ ,  $A(u_2) = 1$ , and  $A(u_3) = 0.2$ . Then, we have

$$\text{Con}(A_2, B_1)(x) = \text{Con}(A_2, B_2)(x) = f_{\top},$$

since  $A_2(u_1) = B_1(u_1) = 1$  and  $A_2(u_2) = B_2(u_2) = 1$  (Corollary 5). On the other hand,

$$\text{Inc}(A_2, B_1)(x) = \begin{cases} x & \text{if } x \leq 0.8 \\ 0.8 & \text{otherwise.} \end{cases}$$

and

$$\text{Inc}(A_2, B_2)(x) = \begin{cases} x & \text{if } x \leq 0.5 \\ 0.5 & \text{otherwise.} \end{cases}$$

Then,  $\text{Con}(A_2, B_1)(x) = \text{Con}(A_2, B_2)(x)$  but  $\text{Inc}(A_2, B_1) \neq \text{Inc}(A_2, B_2)$ .

In the rest of the section, we will introduce different relationships between  $f$ -degrees of inclusion and contradiction concerning complements of fuzzy sets. For this reason, hereafter, we consider a fixed involutive negation operator  $n$  so that the complement  $A^c$  of a fuzzy set  $A$  always denotes the complement w.r.t.  $n$ .

#### 4.1. From $f$ -Inclusion to $f$ -Weak-Contradiction

In this section we assume an  $f$ -inclusion between two fuzzy sets and infer a corresponding  $f$ -weak-contradiction between fuzzy sets involved with the complement. Note that in this section we are considering the  $f$ -inclusion and the  $f$ -weak-contradiction relations, not the respective  $f$ -degrees.

**Proposition 10.** Let  $f \in \Omega$  and let  $A$  and  $B$  be two fuzzy sets such that  $A$  is  $f$ -included in  $B$ . Then,  $B^c$  is  $(n \circ f)$ -weak-contradictory w.r.t.  $A$ .

**Proof.** Firstly note that, since  $f \in \Omega$ , then  $n \circ f$  is in  $\overline{\Omega}$ . Finally, since  $A$  is  $f$ -included in  $B$  we have for all  $u \in \mathcal{U}$ :

$$f(A(u)) \leq B(u) \Leftrightarrow (n \circ B)(u) \leq (n \circ f)(A(u))$$

or equivalently,  $B^c$  is  $(n \circ f)$ -weak-contradictory with  $A$ .

The following result relates the  $f$ -inclusion and  $f$ -weak-contradiction by means of Galois connections. As we show at the end of this section, the consideration of Galois connection in this case is not a hard assumption.

**Proposition 11.** Let  $f \in \Omega$  such that  $(f, g)$  forms an isotone Galois connection and let  $A$  and  $B$  be two fuzzy sets such that  $A$  is  $f$ -included in  $B$ . Then,  $A$  is  $(g \circ n)$ -weak-contradictory w.r.t.  $B^c$ .

**Proof.** Since  $(f, g)$  forms an isotone Galois connection, then  $g(1) = 1$  and  $g$  is monotonic; as a result,  $g \circ n(0) = 1$  and  $g \circ n$  is antitonic. Thus  $g \circ n \in \overline{\Omega}$ . Finally, since  $A$  is  $f$ -included in  $B$  and  $(f, g)$  forms an isotone Galois connection, we have for all  $u \in \mathcal{U}$ :

$$\begin{aligned} f(A(u)) \leq B(u) &\Leftrightarrow A(u) \leq g(B(u)) \\ &\Leftrightarrow A(u) \leq g((n \circ n)(B(u))) \end{aligned}$$

or equivalently,  $A$  is  $(g \circ n)$ -weak-contradictory with  $B^c$ .

#### 4.2. From $f$ -Weak-Contradiction to $f$ -Inclusion

Let us study now the converse relation to that given in the previous section, that is, given an  $f$ -weak-contradiction between two fuzzy sets, determine a corresponding  $f$ -inclusion relation. The main obstacle here is to relate degrees  $f \in \overline{\Omega}$  of  $f$ -contradiction with degrees in  $\Omega$  of  $f$ -inclusion. In the first result we avoid such an obstacle by requiring that  $n(x) \leq f(x)$ .

**Proposition 12.** Let  $f \in \overline{\Omega}$  such that  $n(x) \leq f(x)$  and let  $A$  and  $B$  be two fuzzy sets such that  $A$  is  $f$ -weak-contradictory w.r.t.  $B$ . Then,  $B$  is  $(n \circ f)$ -included in  $A^c$ .

**Proof.** Firstly note that, since  $f \in \overline{\Omega}$ , then  $n \circ f$  is monotonic. Moreover, since  $n(x) \leq f(x)$ , then  $n \circ f(x) \leq n \circ n(x) = x$ ; so  $f \in \Omega$ . Finally, since  $A$  is  $f$ -weak-contradictory with  $B$  we have for all  $u \in \mathcal{U}$ :

$$A(u) \leq f(B(u)) \Leftrightarrow (n \circ f)(B(u)) \leq n(A(u))$$

or equivalently,  $B$  is  $(n \circ f)$ -included in  $A^c$ .

Our second result relating  $f$ -weak-contradiction with  $f$ -inclusion considers again the use of Galois connections.

**Proposition 13.** Let  $f \in \overline{\Omega}$  such that  $n(x) \leq f(x)$  and  $(n \circ f, g \circ n)$  forms an isotone Galois connection. Let  $A$  and  $B$  be two fuzzy sets such that  $A$  is  $f$ -weak-contradictory with  $B$ . Then,  $A$  is  $(n \circ g)$ -included in  $B^c$ .

**Proof.** Let us show firstly that  $n \circ g \leq id$ . Since  $(n \circ f, g \circ n)$  forms an isotone Galois connection and  $n(x) \leq f(x)$  (in particular  $x \leq f(n(x))$ ), we have for all  $x \in [0, 1]$ :

$$\begin{aligned} x \leq f(n(x)) &\Leftrightarrow n(f(n(x))) \leq n(x) \\ &\Leftrightarrow n(x) \leq g(n(n(x))) = g(x) \\ &\Leftrightarrow n(g(x)) \leq x \end{aligned}$$

that is,  $n \circ g \leq id$ . Moreover, since  $(n \circ f, g \circ n)$  forms an isotone Galois connection,  $g \circ n$  is monotonic and then,  $n \circ g$  is monotonic as well; i.e.,  $n \circ g \in \Omega$ . Finally, since  $A$  is  $f$ -weak-contradictory with  $B$  and  $(n \circ f, g \circ n)$  forms an antitone Galois connection, we have for all  $u \in \mathcal{U}$ :

$$\begin{aligned} A(u) \leq f(B(u)) &\Leftrightarrow n(f(B(u))) \leq n(A(u)) \\ &\Leftrightarrow B(u) \leq g(A(u)) \\ &\Leftrightarrow n(g(A(u))) \leq n(B(u)) \end{aligned}$$

or equivalently,  $A$  is  $(n \circ g)$ -included in  $B^c$ .

The previous propositions are compiled together in the following theorem which relates the  $f$ -inclusion and the  $f$ -weak-contradiction in terms of negations, complements and a Galois connection.

**Theorem 14.** Let  $f \in \Omega$  such that  $(f, g)$  forms an isotone Galois connection,  $n$  an involutive negation, and let  $A$  and  $B$  be two fuzzy sets. Then the following items are equivalent

1.  $A$  is  $f$ -included in  $B$ .
2.  $A$  is  $(g \circ n)$ -weak-contradictory with  $B^c$ .
3.  $B^c$  is  $(n \circ f)$ -weak-contradictory with  $A$ .
4.  $B^c$  is  $(n \circ g \circ n)$ -included in  $A^c$ .

**Proof.** The equivalence between items 1 and 4 comes from [12]. The rest of equivalences come from Propositions 10 and 11 by noting the equivalences in the respective proofs and that if  $f \in \Omega$  and  $(f, g)$  forms an isotone Galois connection then  $n(x) \leq f(x)$  for all  $x \in [0, 1]$ , since  $f(x) \leq x \Leftrightarrow n(x) \leq f(x)$ .

This result is the cornerstone of the next section, where the  $f$ -degree of inclusion and the  $f$ -degree of contradiction are related.

### 4.3. Relating the $f$ -Degree of Inclusion and the $f$ -Degree of Contradiction

The strongest results presented up to now require us to have a Galois connection as a hypothesis. The following theorem shows that such a requirement is not a strong one when the universe considered is finite.

**Theorem 15.** Let  $\mathcal{U}$  be finite and let  $A$  and  $B$  be two fuzzy sets on  $\mathcal{U}$ . Then, there exists a mapping  $g: [0, 1] \rightarrow [0, 1]$  such that  $(Inc(A, B), g)$  forms an isotone Galois connection.

**Proof.** Let us show that  $Inc(A, B)$  satisfies the hypothesis of Lemma 1 and then, the result is straightforward. Note that the boundary condition  $Inc(A, B)(0) = 0$  holds because  $Inc(A, B) \in \Omega$ .

Now, let us assume that  $X \subset [0, 1]$  and let us prove firstly that  $f_{A,B}(\sup(X)) = \sup_{x \in X} \{f_{A,B}(x)\}$ , where  $f_{A,B}$  is the function defined in Equation (3).

$$\begin{aligned} f_{A,B}(\sup(X)) &= \inf_{u \in \mathcal{U}} \{B(u) \mid \sup(X) \leq A(u)\} \\ &= \inf_{u \in \mathcal{U}} \left\{ \bigcap_{x \in X} \{B(u) \mid x \leq A(u)\} \right\} \\ &= \sup_{x \in X} \left\{ \inf_{u \in \mathcal{U}} \{B(u) \mid x \leq A(u)\} \right\} \\ &= \sup_{x \in X} \{f_{AB}(x)\}, \end{aligned}$$

where  $\cap$  denotes the intersection between sets. Note that the first and the second equalities are true because  $\mathcal{U}$  is finite and  $[0, 1]$  is totally ordered, respectively. Finally, we have that for all  $X \subset [0, 1]$

$$\begin{aligned} Inc(A, B)(\sup(X)) &= f_{A,B}(\sup(X)) \wedge \sup(X) \\ &= \sup_{x \in X} \{f_{AB}(x)\} \wedge \sup(X) \\ &= \sup_{x \in X} \{f_{AB}(x) \wedge x\} \\ &= \sup_{x \in X} \{Inc(A, B)(x)\} \end{aligned}$$

A similar construction can be made in the case that we start from the  $f$ -weak-contradiction degree but, in this case, the construction is parametric on an involutive negation.

**Theorem 16.** Let  $\mathcal{U}$  be finite and let  $A$  and  $B$  be two fuzzy sets on  $\mathcal{U}$ . Then, there exists a mapping  $f: [0, 1] \rightarrow [0, 1]$  such that  $(f, Con(A, B) \circ n)$  forms an isotone Galois connection.

**Proof.** Let us show that  $Con(A, B) \circ n$  satisfies the hypothesis of Lemma 1 and then, the result is straightforward. Note that the boundary condition  $(Con(A, B) \circ n)(1) = Con(A, B)(0) = 1$  holds because  $Con(A, B) \in \overline{\Omega}$  and then,  $Con(A, B)(0) = 1$ .

Let us assume that  $X \subset [0, 1]$  and let us prove that  $(Con(A, B) \circ n)(\inf(X)) = \inf_{x \in X} \{(Con(A, B) \circ n)(x)\}$ . Note that, since  $n$  is an involutive negation, proving the equality  $(Con(A, B) \circ n)(\inf(X)) = \inf_{x \in X} \{(Con(A, B) \circ n)(x)\}$  is equivalent to prove  $Con(A, B)(\sup(X)) = \inf_{x \in X} \{Con(A, B)(x)\}$ , which is easier. If  $\sup(X) = 0$ , then  $X = \{0\}$  and then:

$$\begin{aligned} Con(A, B)(\sup(X)) &= Con(A, B)(0) \\ &= \inf_{x \in X} \{Con(A, B)(x)\} \end{aligned}$$

If  $\sup(X) \neq 0$ , then we have that

$$\begin{aligned} Con(A, B)(\sup(X)) &= \sup_{u \in \mathcal{U}} \{A(u) \mid \sup(X) \leq B(u)\} \\ &= \sup_{u \in \mathcal{U}} \left\{ \bigcap_{x \in X} \{A(u) \mid x \leq B(u)\} \right\} \\ &= \inf_{x \in X} \left\{ \sup_{u \in \mathcal{U}} \{A(u) \mid x \leq B(u)\} \right\} \\ &= \inf_{x \in X} \{Con(A, B)(x)\}, \end{aligned}$$

where  $\cap$  denotes the intersection between sets.

Combining Theorems 14, 15 and 16 we can relate directly the  $f$ -degrees of inclusion and contradiction. Specifically, from the  $f$ -degree of inclusion we can determine bounds for the  $f$ -degree of contradiction.

**Corollary 17.** *Let  $\mathcal{U}$  be finite, let  $A$  and  $B$  be two fuzzy sets on  $\mathcal{U}$  and let  $g$  be the mapping such that  $(\text{Inc}(A, B), g)$  forms an isotone Galois connection. Then:*

- $\text{Con}(A, B^c) \leq g \circ n$ ,
- $\text{Con}(B^c, A) \leq n \circ \text{Inc}(A, B)$ .

**Proof.** Let us prove that  $\text{Con}(A, B^c) \leq g \circ n$ . By Theorem 14, we have that  $A$  is  $(g \circ n)$ -weak-contradictory with  $B^c$ . Then, necessarily  $\text{Con}(A, B^c) \leq g \circ n$  by definition of  $\text{Con}(A, B^c)$ . Similarly, we can prove  $\text{Con}(B^c, A) \leq n \circ \text{Inc}(A, B)$ .

The final result determines a stronger relationship between both degrees. Specifically, from the  $f$ -degree of contradiction of  $A$  w.r.t.  $B$  we can determine the exact  $f$ -degree of inclusion of  $A$  in the complement of  $B$  and the exact  $f$ -degree of inclusion of  $B$  in the complement of  $A$ .

**Theorem 18.** *Let  $\mathcal{U}$  be finite, let  $A$  and  $B$  be two fuzzy sets on  $\mathcal{U}$  and let  $f$  be the mapping such that  $(f, \text{Con}(A, B) \circ n)$  forms an isotone Galois connection. Then:*

- $\text{Inc}(B, A^c) = (n \circ \text{Con}(A, B)) \wedge \text{id}$ .
- $\text{Inc}(A, B^c) = f \wedge \text{id}$ ,

**Proof.** Let us prove the first item. For  $x = 0$  we have  $0 = \text{Inc}(B, A^c)(0) = (n \circ \text{Con}(A, B)(0)) \wedge 0 = 0$ . Let us assume  $x \neq 0$  and let us show that  $n \circ \text{Con}(A, B)(x) = f_{B, A^c}(x)$ .

$$\begin{aligned} n \circ \text{Con}(A, B)(x) &= n \left( \sup_{u \in \mathcal{U}} \{A(u) \mid x \leq B(u)\} \right) \\ &= \inf_{u \in \mathcal{U}} \{n(A(u)) \mid x \leq B(u)\} \\ &= f_{B, A^c}(x). \end{aligned}$$

As a consequence,

$$\text{Inc}(B, A^c)(x) = f_{B, A^c}(x) \wedge x = (n \circ \text{Con}(A, B)(x)) \wedge (x)$$

as we wanted to prove.

The second equality can be obtained as a consequence of Theorem 14 and the previous item by noting that  $(f, \text{Con}(A, B) \circ n)$  forms an isotone Galois connection if and only if  $(n \circ f, \text{Con}(A, B))$  forms an antitone Galois connection, if and only if  $(n \circ f \circ n, n \circ \text{Con}(A, B))$  forms an isotone Galois connection, if and only if  $((n \circ f \circ n) \vee \text{id}, n \circ \text{Con}(A, B) \wedge \text{id})$  forms an isotone Galois connection. By the previous item we have  $\text{Inc}(B, A^c) = (n \circ \text{Con}(A, B)) \wedge \text{id}$  and finally, by Theorem 14, we have that

$$\text{Inc}(A^c, B) = n \circ ((n \circ f \vee n(x)) = f \wedge \text{id}.$$

## 5. CONCLUSIONS AND FUTURE WORKS

Based on the fact that the definitions of the  $f$ -degree of contradiction and  $f$ -degree of inclusion share the same roots, the relationship

between both notions has been studied. We have shown that the crisp relationships of  $f$ -inclusion and  $f$ -weak-contradiction can be related to each other by using negation operators and Galois connections [15]. Moreover, in the case of considering a finite universe, the  $f$ -degrees of inclusion and  $f$ -degrees of contradiction form part of two Galois connections, which can be related via the complement of fuzzy sets.

The consideration of negations in this study naturally suggests to further analyze the properties of the functional degrees in terms of the opposition square, which has recently been studied in the fuzzy framework [17,18], and in the framework of positive idempotent semifields where negation is carried out by inversion, see [19]. On the other hand, the results of Theorems 16 and 18 on the existence of isotone Galois connections (also called adjunctions) suggest to further investigate the possibility of defining extension of the proposed  $f$ -degrees in more general contexts, taking into account the characterisation of existence of adjoint mappings in different fuzzy settings [20,21].

## CONFLICT OF INTEREST

The authors state that they do not have any conflict of interests.

## AUTHORS' CONTRIBUTIONS

Both authors have contributed to the main conceptual ideas, technical details and writing the manuscript.

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