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A 3-Component Mixture of Exponential Distribution Assuming Doubly Censored Data: Properties and Bayesian Estimation

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ABSTRACT

The output of an engineering process is the result of several inputs, which may be homogeneous or heterogeneous and to study them, we need a model which should be flexible enough to summarize efficiently the nature of such processes. As compared to simple models, mixture models of underlying lifetime distributions are intuitively more appropriate and appealing to model the heterogeneous nature of a process in survival analysis and reliability studies. Moreover, due to time and cost constraints, in the most lifetime testing experiments, censoring is an unavoidable feature. This article focuses on studying a mixture of exponential distributions, and we considered this particular distribution for three reasons. The first reason is its application in reliability modeling of electronic components and the second important reason is its skewed behavior. Similarly, the third and the most important reason is that exponential distribution has the memory-less property. In particular, we deal with the problem of estimating the parameters of a 3-component mixture of exponential distributions using type-II doubly censoring sampling scheme. The elegant closed-form expressions for the Bayes estimators and their posterior risks are derived under squared error loss function, precautionary loss function and DeGroot loss function assuming the noninformative (uniform and Jeffreys') and the informative priors. A detailed Monte Carlo simulation and real data studies are carried out to investigate the performance (in terms of posterior risks) of the Bayes estimators. From results, it is observed that the Bayes estimates assuming the informative prior perform better than the noninformative priors.

1. INTRODUCTION

Exponential distribution has been successfully used to model lifetimes of industrial objects. It is a very flexible distribution that can express a wide range of distribution shapes and can be fitted to a wide range of empirical data. Because of its memory-less property, it is commonly used in testing of the objects whose lifetimes do not depend on their age.

Mixture models play a significant role in many real-life applications since the last few decades. Finite mixtures of lifetime distributions have been proved to be of considerable interest both in terms of their methodological development and practical applications. As defined in Mendenhall and Hader [1], for practical purposes, an engineer may divide causes of failure of a system, or a device, into two or more different types of causes. For example, to know the proportion of failure due to a certain cause and to improve the manufacturing process, Acheson and McElwee [2] divided causes of electronic tube failures into gaseous defects, mechanical defects and normal deterioration of the cathode. Similarly, an engineering system may be composed of different homogeneous and/or heterogeneous subsystems. Instead of single probability models, heterogeneity in the nature of such systems can be captured through mixture models. Another important feature of mixture models is that when a population is supposed to comprise a number of subpopulations mixed in an unknown proportion, common available distributions do not exhibit the situation at hand. Some applications of mixture of exponential distributions include McCullagh [3], Hebert and Scariano [4], Raqab and Ahsanullah [5], Ali *et al.* [6] and Abu-Taleb *et al.* [7]. Direct applications of mixture models can be seen mostly in industrial engineering (Ali *et al.* [8]), medicine (Chivers [9] and Burekhardt [10]), biology (Bhattacharya [11] and Gregor [12]), social sciences (Harris [13]), economics (Jedidi *et al.* [14]), life testing (Shawky and Bakoban [15]) and reliability analysis (Sultan *et al.* [16]).

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In many applications, data can be considered as coming from a mixture of two or more distributions. This motivates the researchers to mix common statistical distributions to get a new distribution. Several authors have extensively applied 2-component mixture modeling in different practical problems using the Bayesian approach. For instance, we refer to Liu [17], Saleem and Irfan [18], Saleem *et al.* [19], Santos [20], Al-Hussaini and Hussein [21], Mohammadi and Salehi-Rad [22], Kazmi *et al.* [23], Ahmad and Al-Zaydi [24], Ali *et al.* [25], Mohammadi *et al.* [26], Ali [27], Ateya [28], Feroze and Aslam [29], Mohamed *et al.* [30] and Zhang and Huang [31] for the Bayesian estimation of mixture models. However, limited work is available in the literature on the Bayesian analysis of the 3-component mixture distribution.

Due to time and cost limitations, it is impossible to continue the testing until the last observation. Therefore, the values which are greater than pre-fixed life-test termination time are taken as censored observations. It is worth mentioning that censoring is a property of data sets and not of parameters and commonly used in lifetime experiments. A valuable account on censoring is given in Romeu [32], Gijbels [33] and Kalbfleisch and Prentice [34] and the referenced cited therein.

Motivated by the applications of the exponential distribution and mixture models, the focus of the present article is to develop a 3-component mixture of exponential distributions (3-CMEDs) from Bayesian perspective. We assume that all the parameters of a 3-CMED are unknown and estimate them by considering different priors and loss functions. In addition to this, a type-II doubly censoring sampling scheme is also considered in this article.

The rest of the article is arranged as follows: Development of a 3-CMEDs is given in Section 2. Sampling scheme and likelihood function of the mixture model are defined in Section 3. The joint posterior distributions assuming the non-informative and the informative priors are derived in Sections 4 and 5, respectively. The marginal posterior distributions are derived in Sections 6. In Section 7, the Bayesian estimation under squared error loss function (SELF), precautionary loss function (PLF) and DeGroot loss function (DLF) are presented. The posterior predictive distribution and the Bayesian predictive intervals are described in Section 8. The elicitation of hyperparameters is discussed in Section 9. The simulation study and the real-life data application are explained in Sections 10 and 11, respectively. Finally, some concluding remarks are given in Section 12.

2. A 3-COMPONENT MIXTURE OF THE EXPONENTIAL DISTRIBUTIONS

As defined by Barger [35] and Střelec and Stehlík [36], the probability density function of a finite 3-CMED with mixing proportions p_1 and p_2, is given by

$$f(y;\Phi) = p_1 f_1(y;\Phi_1) + p_2 f_2(y;\Phi_2) + (1-p_1-p_2) f_3(y;\Phi_3), p_1, p_2 \geq 0, p_1 + p_2 \leq 1, \tag{1}$$

where $\Phi = (\theta_1, \theta_2, \theta_3, p_1, p_2)$, $\Phi_m = \theta_m$, $m = 1,2,3$ and $f_m(y;\Phi_m)$ is the pdf of the mth component defined as

$$f_m(y;\Phi_m) = \theta_m \exp(-\theta_m y), 0 < y < \infty, \theta_m > 0, m = 1,2,3.$$

The cdf of a 3-component mixture of the exponential distributions is defined as

$$F(y;\Phi) = p_1 F_1(y;\Phi_1) + p_2 F_2(y;\Phi_2) + (1-p_1-p_2) F_3(y;\Phi_3), \tag{2}$$

where $F_m(y;\Phi_m)$, the cdf of the mth component, is

$$F_m(y;\Phi_m) = 1 - \exp(\theta_m y), 0 < y < \infty, \theta_m > 0, m = 1,2,3.$$

Following two theorems give the characterizations of von Mises distribution by truncated first moment.

Theorem 1 *Suppose that the random variable Y satisfies the conditions given in Assumption $\mathcal{A}$, with pdf $f(y;\Phi)$, cdf $F(y;\Phi)$ with $\alpha = 0$ and $\beta = \infty$. Then $E(Y|Y<y) = g(y;\Phi)\,\tau(y;\Phi)$, where $\tau(y;\Phi) = \dfrac{f(y;\Phi)}{F(y;\Phi)}$ and $g(y;\Phi) = \dfrac{\frac{p_1}{\theta_1}\Gamma_{\theta_1 y}(2) + \frac{p_2}{\theta_2}\Gamma_{\theta_2 y}(2) + \frac{(1-p_1-p_2)}{\theta_3}\Gamma_{\theta_3 y}(2)}{p_1\theta_1\exp(-\theta_1 y) + p_2\theta_2\exp(-\theta_2 y) + (1-p_1-p_2)\theta_3\exp(-\theta_3 y)}$ if and only if $f(y;\Phi) = p_1\theta_1\exp(-\theta_1 y) + p_2\theta_2\exp(-\theta_2 y) + (1-p_1-p_2)\theta_3\exp(-\theta_3 y)$.*

Proof: We have $f(y;\Phi)\,g(y;\Phi) = \int_0^y u f(u;\Phi)\,du$.

If $f(y;\Phi) = p_1\theta_1\exp(-\theta_1 y) + p_2\theta_2\exp(-\theta_2 y) + (1-p_1-p_2)\theta_3\exp(-\theta_3 y)$,

$$\text{then } f(y;\Phi)\,g(y;\Phi) = \int_0^y u\left(p_1\theta_1\exp(-\theta_1 u) + p_2\theta_2\exp(-\theta_2 u) + (1-p_1-p_2)\theta_3\exp(-\theta_3 u)\right)du$$
$$= \frac{p_1}{\theta_1}\Gamma_{\theta_1 y}(2) + \frac{p_2}{\theta_2}\Gamma_{\theta_2 y}(2) + \frac{(1-p_1-p_2)}{\theta_3}\Gamma_{\theta_3 y}(2),$$

where $\Gamma_y(2) = \int_0^y u \exp(-u)\, du$.

Thus $g(y;\Phi) = \frac{\frac{p_1}{\theta_1}\Gamma_{\theta_1 y}(2)+\frac{p_2}{\theta_2}\Gamma_{\theta_2 y}(2)+\frac{(1-p_1-p_2)}{\theta_3}\Gamma_{\theta_3 y}(2)}{p_1\theta_1\exp(-\theta_1 y)+p_2\theta_2\exp(-\theta_2 y)+(1-p_1-p_2)\theta_3\exp(-\theta_3 y)}$

then

$$g'(y;\Phi) = y - \frac{\frac{p_1}{\theta_1}\Gamma_{\theta_1 y}(2)+\frac{p_2}{\theta_2}\Gamma_{\theta_2 y}(2)+\frac{(1-p_1-p_2)}{\theta_3}\Gamma_{\theta_3 y}(2)}{p_1\theta_1\exp(-\theta_1 y)+p_2\theta_2\exp(-\theta_2 y)+(1-p_1-p_2)\theta_3\exp(-\theta_3 y)}p(y;\Phi),$$

where

$$p(y;\Phi) = -\frac{p_1\theta_1^2\exp(-\theta_1 y)+p_2\theta_2^2\exp(-\theta_2 y)+(1-p_1-p_2)\theta_3^2\exp(-\theta_3 y)}{p_1\theta_1\exp(-\theta_1 y)+p_2\theta_2\exp(-\theta_2 y)+(1-p_1-p_2)\theta_3\exp(-\theta_3 y)}.$$

Thus $g'(y;\Phi) = y - g(y;\Phi)\,p(y;\Phi)$

and

$$\frac{y-g'(y;\Phi)}{g(y;\Phi)} = -\frac{p_1\theta_1^2\exp(-\theta_1 y)+p_2\theta_2^2\exp(-\theta_2 y)+(1-p_1-p_2)\theta_3^2\exp(-\theta_3 y)}{p_1\theta_1\exp(-\theta_1 y)+p_2\theta_2\exp(-\theta_2 y)+(1-p_1-p_2)\theta_3\exp(-\theta_3 y)}.$$

By Lemma 1, we obtain

$$\frac{f'(y;\Phi)}{f(y;\Phi)} = -\frac{p_1\theta_1^2\exp(-\theta_1 y)+p_2\theta_2^2\exp(-\theta_2 y)+(1-p_1-p_2)\theta_3^2\exp(-\theta_3 y)}{p_1\theta_1\exp(-\theta_1 y)+p_2\theta_2\exp(-\theta_2 y)+(1-p_1-p_2)\theta_3\exp(-\theta_3 y)}.$$

Integrating both sides of the equation, we obtain

$f(y;\Phi) = c\left(p_1\theta_1\exp(-\theta_1 y)+p_2\theta_2\exp(-\theta_2 y)+(1-p_1-p_2)\theta_3\exp(-\theta_3 y)\right)$, where c is a constant.

Using the boundary conditions $F(0)=0$ and $F(\infty)=1$, we obtain

$$f(y;\Phi) = p_1\theta_1\exp(-\theta_1 y)+p_2\theta_2\exp(-\theta_2 y)+(1-p_1-p_2)\theta_3\exp(-\theta_3 y).$$

Theorem 2 *Suppose that the random variable Y satisfies the conditions given in Assumption $\mathcal{A}$, with pdf $f(y;\Phi)$, cdf $F(y;\Phi)$ with $\alpha = 0$ and $\beta = \infty$. Then $E(Y|Y>y) = h(y;\Phi)\,r(y;\Phi)$, where $r(y;\Phi) = \frac{f(y;\Phi)}{1-F(y;\Phi)}$ and $h(y;\Phi) = \frac{E(Y)-\left(\frac{p_1}{\theta_1}\Gamma_{\theta_1 y}(2)+\frac{p_2}{\theta_2}\Gamma_{\theta_2 y}(2)+\frac{(1-p_1-p_2)}{\theta_3}\Gamma_{\theta_3 y}(2)\right)}{p_1\theta_1\exp(-\theta_1 y)+p_2\theta_2\exp(-\theta_2 y)+(1-p_1-p_2)\theta_3\exp(-\theta_3 y)}$ if and only if $f(y;\Phi) = p_1\theta_1\exp(-\theta_1 y)+p_2\theta_2\exp(-\theta_2 y)+(1-p_1-p_2)\theta_3\exp(-\theta_3 y)$.*

Proof: We have $f(y;\Phi)\,h(y;\Phi) = \int_y^\infty u f(u;\Phi)\,du = E(Y) - \int_0^y u f(u;\Phi)\,du$

If $f(y;\Phi) = p_1\theta_1\exp(-\theta_1 y)+p_2\theta_2\exp(-\theta_2 y)+(1-p_1-p_2)\theta_3\exp(-\theta_3 y)$,

$$\begin{aligned}\text{then } f(y;\Phi)\,h(y;\Phi) &= E(Y) - \int_0^y u\left(p_1\theta_1\exp(-\theta_1 u)+p_2\theta_2\exp(-\theta_2 u)+(1-p_1-p_2)\theta_3\exp(-\theta_3 u)\right)du \\ &= E(Y) - \left(\frac{p_1}{\theta_1}\Gamma_{\theta_1 y}(2)+\frac{p_2}{\theta_2}\Gamma_{\theta_2 y}(2)+\frac{(1-p_1-p_2)}{\theta_3}\Gamma_{\theta_3 y}(2)\right).\end{aligned}$$

Thus

$$h(y;\Phi) = \frac{E(Y)-\left(\frac{p_1}{\theta_1}\Gamma_{\theta_1 y}(2)+\frac{p_2}{\theta_2}\Gamma_{\theta_2 y}(2)+\frac{(1-p_1-p_2)}{\theta_3}\Gamma_{\theta_3 y}(2)\right)}{p_1\theta_1\exp(-\theta_1 y)+p_2\theta_2\exp(-\theta_2 y)+(1-p_1-p_2)\theta_3\exp(-\theta_3 y)}.$$

Now

$$h'\left(y;\Phi\right)=-y-\frac{E\left(Y\right)-\left(\frac{p_1}{\theta_1}\Gamma_{\theta_1 y}\left(2\right)+\frac{p_2}{\theta_2}\Gamma_{\theta_2 y}\left(2\right)+\frac{\left(1-p_1-p_2\right)}{\theta_3}\Gamma_{\theta_3 y}\left(2\right)\right)}{p_1\theta_1\exp\left(-\theta_1 y\right)+p_2\theta_2\exp\left(-\theta_2 y\right)+\left(1-p_1-p_2\right)\theta_3\exp\left(-\theta_3 y\right)}p\left(y\right),$$

$$h'\left(y;\Phi\right)=-y-h\left(y;\Phi\right)p\left(y;\Phi\right)$$

and $\dfrac{y+h'\left(y;\Phi\right)}{h\left(y;\Phi\right)}=-p\left(y;\Phi\right)$

where $p\left(y;\Phi\right)=-\dfrac{p_1\theta_1^2\exp\left(-\theta_1 y\right)+p_2\theta_2^2\exp\left(-\theta_2 y\right)+\left(1-p_1-p_2\right)\theta_3^2\exp\left(-\theta_3 y\right)}{p_1\theta_1\exp\left(-\theta_1 y\right)+p_2\theta_2\exp\left(-\theta_2 y\right)+\left(1-p_1-p_2\right)\theta_3\exp\left(-\theta_3 y\right)}$.

By Lemma 2, we obtain

$$\frac{f'\left(y;\Phi\right)}{f\left(y;\Phi\right)}=-p\left(y\right)=-\frac{p_1\theta_1^2\exp\left(-\theta_1 y\right)+p_2\theta_2^2\exp\left(-\theta_2 y\right)+\left(1-p_1-p_2\right)\theta_3^2\exp\left(-\theta_3 y\right)}{p_1\theta_1\exp\left(-\theta_1 y\right)+p_2\theta_2\exp\left(-\theta_2 y\right)+\left(1-p_1-p_2\right)\theta_3\exp\left(-\theta_3 y\right)}.$$

Integrating both sides of the equation, we obtain

$f\left(y;\Phi\right)=c\left(p_1\theta_1\exp\left(-\theta_1 y\right)+p_2\theta_2\exp\left(-\theta_2 y\right)+\left(1-p_1-p_2\right)\theta_3\exp\left(-\theta_3 y\right)\right)$, where c is a constant.

Using the boundary conditions $F(0)=0$ and $F(\infty)=1$, we obtain

$$f\left(y;\Phi\right)=p_1\theta_1\exp\left(-\theta_1 y\right)+p_2\theta_2\exp\left(-\theta_2 y\right)+\left(1-p_1-p_2\right)\theta_3\exp\left(-\theta_3 y\right).$$

3. LIKELIHOOD FUNCTION FOR A 3-CMED UNDER DOUBLY CENSORED DATA

To explain the construction of likelihood function, suppose n units are placed in a life testing experiment. Let $y_r, y_{r+1}, \dots, y_w$ be the ordered observations that can only be observed. The remaining $r-1$ smallest observations and the $n-w$ largest observations are censored from the study. So, $y_{1r_1}, \dots, y_{1w_1}, y_{2r_2}, \dots, y_{2w_2}$ and $y_{3r_3}, \dots, y_{3w_3}$ are failed observations belonging to subpopulation-I, subpopulation-II and subpopulation-III, respectively. Therefore, the rest of the observations which are either less than y_r or greater than y_w assumed to be censored from each component, such that $y_r=\min\left(y_{1r_1}, y_{2r_2}, y_{3r_3}\right)$ and $y_w=\max\left(y_{1w_1}, y_{2w_2}, y_{3w_3}\right)$, whereas the numbers $s_1=w_1-r_1+1$, $s_2=w_2-r_2+1$ and $s_3=w_3-r_3+1$ of failed observations can be obtained from subpopulation-I, subpopulation-II and subpopulation-III, respectively. The remaining $n-(w-r+3)$ observations are assumed to be censored observations where $r=r_1+r_2+r_3$, $w=w_1+w_2+w_3$ and $s=s_1+s_2+s_3$. The likelihood function of type-II doubly censored sample, $\mathbf{y}=\left\{\left(y_{1r_1},\dots,y_{1w_1}\right),\left(y_{2r_2},\dots,y_{2w_2}\right),\left(y_{3r_3},\dots,y_{3w_3}\right)\right\}$, from a 3-component mixture distribution is

$$L\left(\Phi|\mathbf{y}\right)\propto\left\{\prod_{i=r_1}^{w_1}p_1 f_1\left(y_{1i}\right)\right\}\left\{\prod_{i=r_2}^{w_2}p_2 f_2\left(y_{2i}\right)\right\}\left\{\prod_{i=r_3}^{w_3}\left(1-p_1-p_2\right)f_3\left(y_{3i}\right)\right\}$$
$$\times\left\{F_1\left(y_{1r_1}\right)\right\}^{r_1-1}\left\{F_2\left(y_{2r_2}\right)\right\}^{r_2-1}\left\{F_3\left(y_{3r_3}\right)\right\}^{r_3-1}\left\{1-F\left(y_w\right)\right\}^{n-w}.$$

On simplification, the likelihood function of the 3-CMED becomes

$$L\left(\Phi|\mathbf{y}\right)\propto\sum_{v_1=0}^{r_1-1}\sum_{v_2=0}^{r_2-1}\sum_{v_3=0}^{r_3-1}\sum_{v_4=0}^{n-w}\sum_{v_5=0}^{v_4}\prod_{k=1}^{3}(-1)^{v_k}\binom{r_k-1}{v_k}\binom{n-w}{v_4}\binom{v_4}{v_5}\exp\left[-\theta_1\left\{\sum_{i=r_1}^{w_1}y_{1i}+v_1 y_{1r_1}+(n-w-v_4)y_w\right\}\right]$$
$$\exp\left[-\theta_2\left\{\sum_{i=r_2}^{w_2}y_{2i}+v_2 y_{2r_2}+(v_4-v_5)y_w\right\}\right]\exp\left[-\theta_3\left\{\sum_{i=r_3}^{w_3}y_{3i}+v_3 y_{3r_3}+v_5 y_w\right\}\right]$$
$$\theta_1^{s_1}\theta_2^{s_2}\theta_3^{s_3}p_1^{s_1+n-w-v_4}p_2^{s_2+v_4-v_5}\left(1-p_1-p_2\right)^{s_3+v_5}. \tag{3}$$

In the next section, we discuss the joint posterior distribution for Bayesian analysis.

4. THE JOINT POSTERIOR DISTRIBUTIONS ASSUMING THE NIPS

There are some situations where no prior information about the parameter(s) of interest is available or a researcher is uncomfortable with the subjective knowledge. Thus, in such situations, one can use the noninformative priors (NIPs). Box and Taio [37] argued that NIP as a prior which gives little information relative to the experiment. Similarly, Bernardo [38] also argued that NIP should be regarded as a reference prior, that is, a prior convenient to use as a standard when analyzing statistical data. Later, Bernardo and Smith [39] defined NIP as the priors having the minimal effect relative to data.

In the existing literature, the most commonly used NIPs are the uniform prior (UP) and the Jeffreys' prior (JP). Both priors are used only when no formal prior information is available. To obtain JP, Jeffreys [40,41] suggested a method based on the square-root of the Fisher information. Later on, Geisser [42] also proposed some techniques to determine NIP.

In this section, the joint posterior distributions of parameters given data y are derived assuming the UP and the JP.

4.1. The Joint Posterior Distribution Assuming the UP

To derive the joint posterior distribution, we assume the improper UP for the unknown component parameter θ_m, that is, $\theta_m \sim Uniform\,(0, \infty)$, $m = 1, 2, 3$ and the UP over the interval $(0, 1)$ for the unknown proportion parameter p_u, that is, $p_u \sim Uniform\,(0, 1)$, $u = 1, 2$. Assuming independence of parameters, the joint prior distribution of the parameters $\theta_1, \theta_2, \theta_3, p_1$ and p_2 is given by $\pi_1(\Phi) \propto 1$. Therefore, the joint posterior distribution of parameters $\theta_1, \theta_2, \theta_3, p_1$ and p_2 given data y is given by

$$g_1(\Phi|\mathbf{y}) = \frac{L(\Phi|y)\,\pi_1(\Phi)}{\int_\Phi L(\Phi|\mathbf{y})\,\pi_1(\Phi)\,d\Phi}$$

$$g_1(\Phi|\mathbf{y}) = \frac{1}{\Omega_1}\left\{\begin{array}{l}\sum_{v_1=0}^{r_1-1}\sum_{v_2=0}^{r_2-1}\sum_{v_3=0}^{r_3-1}\sum_{v_4=0}^{n-w}\sum_{v_5=0}^{v_4}\prod_{k=1}^{3}(-1)^{v_k}\binom{r_k-1}{v_k}\binom{n-w}{v_4}\binom{v_4}{v_5}\theta_1^{A_{11}-1}\theta_2^{A_{21}-1}\theta_3^{A_{31}-1}\\ \exp(-B_{11}\theta_1)\exp(-B_{21}\theta_2)\exp(-B_{31}\theta_3)\,p_1^{A_{01}-1}p_2^{B_{01}-1}\left(1-p_1-p_2\right)^{C_{01}-1}\end{array}\right\}, \tag{4}$$

where

$A_{11} = s_1 + 1, A_{21} = s_2 + 1, A_{31} = s_3 + 1, A_{01} = s_1 + n - w - v_4 + 1, B_{01} = s_2 + v_4 - v_5 + 1, C_{01} = s_3 + v_5 + 1,$

$B_{11} = \sum_{i=r_1}^{w_1} y_{1i} + v_1 y_{1r_1} + (n - w - v_4)\,y_w, B_{21} = \sum_{i=r_2}^{w_2} y_{2i} + v_2 y_{2r_2} + (v_4 - v_5)\,y_w, B_{31} = \sum_{i=r_3}^{w_3} y_{3i} + v_3 y_{3r_3} + v_5 y_w,$

$\Omega_1 = \sum_{v_1=0}^{r_1-1}\sum_{v_2=0}^{r_2-1}\sum_{v_3=0}^{r_3-1}\sum_{v_4=0}^{n-w}\sum_{v_5=0}^{v_4}\prod_{l=1}^{3}(-1)^{v_l}\binom{r_l-1}{v_l}\binom{n-w}{v_4}\binom{v_4}{v_5}\Gamma(A_{11})\Gamma(A_{21})\Gamma(A_{31})B_{11}^{-A_{11}}B_{21}^{-A_{21}}B_{31}^{-A_{31}}B(A_{01}, B_{01}, C_{01}).$

4.2. The Joint Posterior Distribution Assuming the JP

The JP is defined as $p(\theta_m) \propto \sqrt{|I(\theta_m)|}$, $m = 1, 2, 3$, where $I(\theta_m) = -E\left[\frac{\partial^2 f(y;\theta_m)}{\partial\theta_m^2}\right]$ is the Fisher's information. The prior distributions of the proportion parameters p_1 and p_2 are again assumed as $p_u \sim Uniform\,(0, 1)$, $u = 1, 2$. Assuming independence of parameters, the joint prior distribution of parameters $\theta_1, \theta_2, \theta_3, p_1$ and p_2 is $\pi_2(\Phi) \propto \frac{1}{\theta_1\theta_2\theta_3}$. By combining the likelihood function and the joint prior distribution, we obtain the joint posterior distribution of parameters $\theta_1, \theta_2, \theta_3, p_1$ and p_2 given data $\mathbf{y}$ as

$$g_2(\Phi|\mathbf{y}) = \frac{L(\Phi|\mathbf{y})\,\pi_2(\Phi)}{\int_\Phi L(\Phi|\mathbf{y})\,\pi_2(\Phi)\,d\Phi}$$

$$g_2(\Phi|\mathbf{y}) = \frac{1}{\Omega_2}\left\{\begin{array}{l}\sum_{v_1=0}^{r_1-1}\sum_{v_2=0}^{r_2-1}\sum_{v_3=0}^{r_3-1}\sum_{v_4=0}^{n-w}\sum_{v_5=0}^{v_4}\prod_{k=1}^{3}(-1)^{v_k}\binom{r_k-1}{v_k}\binom{n-w}{v_4}\binom{v_4}{v_5}\theta_1^{A_{12}-1}\theta_2^{A_{22}-1}\theta_3^{A_{32}-1}\\ \exp(-B_{12}\theta_1)\exp(-B_{22}\theta_2)\exp(-B_{32}\theta_3)\,p_1^{A_{02}-1}p_2^{B_{02}-1}\left(1-p_1-p_2\right)^{C_{02}-1}\end{array}\right\}, \tag{5}$$

where

$A_{12}=s_1, A_{22}=s_2, A_{32}=s_3, A_{02}=s_1+n-w-v_4+1, B_{02}=s_2+v_4-v_5+1, C_{02}=s_3+v_5+1,$

$B_{12}=\sum_{i=r_1}^{w_1} y_{1i}+v_1y_{1r_1}+(n-w-v_4)\,y_w, B_{22}=\sum_{i=r_2}^{w_2} y_{2i}+v_2y_{2r_2}+(v_4-v_5)\,y_w, B_{32}=\sum_{i=r_3}^{w_3} y_{3i}+v_3y_{3r_3}+v_5y_w,$

$$\Omega_2=\sum_{v_1=0}^{r_1-1}\sum_{v_2=0}^{r_2-1}\sum_{v_3=0}^{r_3-1}\sum_{v_4=0}^{n-w}\sum_{v_5=0}^{v_4}\prod_{k=1}^{3}(-1)^{v_k}\binom{r_k-1}{v_k}\binom{n-w}{v_4}\binom{v_4}{v_5}\Gamma(A_{12})\,\Gamma(A_{22})\,\Gamma(A_{32})\,B_{12}^{-A_{12}}B_{22}^{-A_{22}}B_{32}^{-A_{32}}B(A_{02},B_{02},C_{02}).$$

5. THE JOINT POSTERIOR DISTRIBUTION ASSUMING THE INFORMATIVE PRIOR

The information available on the parameter(s) of interest is quantified as an informative prior. In this article, we consider the gamma distributions as the prior distributions for component parameters θ_m, that is, $\theta_m \sim Gamma\,(a_m,b_m)$, $m=1,2,3$ and bivariate beta distribution is assumed as the prior distribution for the proportion parameters p_1 and p_2, that is, $p_1,p_2 \sim BivariateBeta\,(a,b,c)$. So, the joint prior distribution of parameters $\theta_1,\theta_2,\theta_3,p_1$ and p_2 is written as

$$\pi_3(\Phi)=\frac{b_1^{a_1}}{\Gamma(a_1)}\theta_1^{a_1-1}\exp(-b_1\theta_1)\frac{b_2^{a_2}}{\Gamma(a_2)}\theta_2^{a_2-1}\exp(-b_2\theta_2)\frac{b_3^{a_3}}{\Gamma(a_3)}\theta_3^{a_3-1}\exp(-b_3\theta_3)\frac{p_1^{a-1}p_2^{b-1}\left(1-p_1-p_2\right)^{c-1}}{B(a,b,c)}.$$

Thus, the joint posterior distribution of parameters $\theta_1,\theta_2,\theta_3,p_1$ and p_2 given data $\mathbf{y}$ is

$$g_3\left(\Phi|\mathbf{y}\right)=\frac{L\left(\Phi|\mathbf{y}\right)\pi_3(\Phi)}{\int_\Phi L\left(\Phi|\mathbf{y}\right)\pi_3(\Phi)\,d\Phi}$$

$$g_3\left(\Phi|\mathbf{y}\right)=\frac{1}{\Omega_3}\left\{\begin{array}{l}\sum_{v_1=0}^{r_1-1}\sum_{v_2=0}^{r_2-1}\sum_{v_3=0}^{r_3-1}\sum_{v_4=0}^{n-w}\sum_{v_5=0}^{v_4}\prod_{k=1}^{3}(-1)^{v_k}\binom{r_k-1}{v_k}\binom{n-w}{v_4}\binom{v_4}{v_5}\theta_1^{A_{13}-1}\theta_2^{A_{23}-1}\theta_3^{A_{33}-1}\\ \exp(-B_{13}\theta_1)\exp(-B_{23}\theta_2)\exp(-B_{33}\theta_3)\,p_1^{A_{03}-1}p_2^{B_{03}-1}\left(1-p_1-p_2\right)^{C_{03}-1}\end{array}\right\}, \quad (6)$$

where

$A_{13}=s_1+a_1, A_{23}=s_2+a_2, A_{33}=s_3+a_3, B_{13}=\sum_{i=r_1}^{w_1} y_{1i}+v_1y_{1r_1}+(n-w-v_4)\,y_w+b_1, B_{23}=\sum_{i=r_2}^{w_2} y_{2i}+v_2y_{2r_2}+(v_4-v_5)\,y_w+b_2,$

$B_{33}=\sum_{i=r_3}^{w_3} y_{3i}+v_3y_{3r_3}+v_5y_w+b_3,$

$A_{03}=s_1+n-w-v_4+a, B_{03}=s_2+v_4-v_5+b, C_{03}=s_3+v_5+c,$

$$\Omega_3=\sum_{v_1=0}^{r_1-1}\sum_{v_2=0}^{r_2-1}\sum_{v_3=0}^{r_3-1}\sum_{v_4=0}^{n-w}\sum_{v_5=0}^{v_4}\prod_{k=1}^{3}(-1)^{v_k}\binom{r_k-1}{v_k}\binom{n-w}{v_4}\binom{v_4}{v_5}\Gamma(A_{13})\,\Gamma(A_{23})\,\Gamma(A_{33})\,B_{13}^{-A_{13}}B_{23}^{-A_{23}}B_{33}^{-A_{33}}B(A_{03},B_{03},C_{03}).$$

6. BAYESIAN ESTIMATION UNDER LOSS FUNCTIONS

In this section, we derived the algebraic expressions of Bayes estimators and their associated posterior risks using the UP, the JP and the IP under three different loss functions, namely SELF, PLF and DLF. If $L(\theta,\hat{\omega})$ is a loss function, the expected value of the loss function for a given decision with respect to the posterior distribution is known as the posterior risk function. The Bayes estimator $\hat{\omega}$ is obtained by minimizing the posterior expectation with respect to parameter, that is, defined as $\hat{\omega}=E_{\theta|y}\{L(\theta,\hat{\omega})\}$, where $L(\theta,\hat{\omega})$ is the loss incurred estimating θ by $\hat{\omega}$. The SELF, is defined as $L(\theta,\hat{\omega})=(\theta-\hat{\omega})^2$, was introduced by Legendre [43] to develop the least square theory. Later, Norstrom [44] discussed an asymmetric PLF and also introduced a special case of general class of PLFs defined as $L(\theta,\hat{\omega})=(\hat{\omega})^{-1}(\theta-\hat{\omega})^2$. The DLF is presented by DeGroot [45] and is defined as $L(\theta,\hat{\omega})=(\hat{\omega})^{-2}(\theta-\hat{\omega})^2$. For a given prior, the general form of the Bayes estimators and their posterior risks under SELF, PLF and DLF are given in Table 1.

Table 1 | Bayes estimators and posterior risks under SELF, PLF and DLF.

Loss Function	Bayes Estimators	Posterior Risks
$SELF = L\left(\theta,\hat{\omega}\right) = \left(\theta-\hat{\omega}\right)^2$	$\hat{\omega} = E_{\theta\|y}\left(\theta\right)$	$\rho\left(\hat{\omega}\right) = E_{\theta\|y}\left(\theta^2\right) - \left\{E_{\theta\|y}\left(\theta\right)\right\}^2$
$PLF = L\left(\theta,\hat{\omega}\right) = \left(\hat{\omega}\right)^{-1}\left(\theta-\hat{\omega}\right)^2$	$\hat{\omega} = \sqrt{E_{\theta\|y}\left(\theta^2\right)}$	$\rho\left(\hat{\omega}\right) = 2\sqrt{E_{\theta\|y}\left(\theta^2\right)} - 2E_{\theta\|y}\left(\theta\right)$
$DLF = L\left(\theta,\hat{\omega}\right) = \left(\hat{\omega}\right)^{-2}\left(\theta-\hat{\omega}\right)^2$	$\hat{\omega} = E_{\theta\|y}\left(\theta^2\right)\left\{E_{\theta\|y}\left(\theta\right)\right\}^{-1}$	$\rho\left(\hat{\omega}\right) = 1 - \left\{E_{\theta\|y}\left(\theta\right)\right\}^2\left\{E_{\theta\|y}\left(\theta^2\right)\right\}^{-1}$

SELF, squared error loss function; PLF, precautionary loss function; DLF, DeGroot loss function.

6.1. Expressions of the Bayes Estimators and Posterior Risks Under SELF

The algebraic expressions for Bayes estimators and posterior risks assuming the UP, the JP and the IP of parameters $\theta_1, \theta_2, \theta_3, p_1$ and p_2 under SELF are obtained with respective marginal distribution as

$$\hat{\theta}_{\varpi(SELF)} = \frac{1}{\Omega_\xi}\left\{\begin{array}{l}\sum_{v_1=0}^{r_1-1}\sum_{v_2=0}^{r_2-1}\sum_{v_3=0}^{r_3-1}\sum_{v_4=0}^{n-w}\sum_{v_5=0}^{v_4}\prod_{k=1}^{3}(-1)^{v_k}\binom{r_k-1}{v_k}\binom{n-w}{v_4}\binom{v_4}{v_5}\Gamma\left(A_{\varpi\xi}+1\right)\Gamma\left(A_{\pi\xi}\right)\Gamma\left(A_{\eta\xi}\right)\\ B_{\varpi\xi}^{-\left(A_{\varpi\xi}+1\right)}B_{\pi\xi}^{-A_{\pi\xi}}B_{\eta\xi}^{-A_{\eta\xi}}B\left(A_{0\xi},C_{0\xi}\right)B\left(B_{0\xi},A_{0\xi}+C_{0\xi}\right)\end{array}\right\} \quad (7)$$

$$\hat{p}_{\alpha(SELF)} = \frac{1}{\Omega_\xi}\left\{\begin{array}{l}\sum_{v_1=0}^{r_1-1}\sum_{v_2=0}^{r_2-1}\sum_{v_3=0}^{r_3-1}\sum_{v_4=0}^{n-w}\sum_{v_5=0}^{v_4}\prod_{k=1}^{3}(-1)^{v_k}\binom{r_k-1}{v_k}\binom{n-w}{v_4}\binom{v_4}{v_5}\Gamma\left(A_{1\xi}\right)\Gamma\left(A_{2\xi}\right)\Gamma\left(A_{3\xi}\right)\\ B_{1\xi}^{-A_{1\xi}}B_{2\xi}^{-A_{2\xi}}B_{3\xi}^{-A_{3\xi}}B\left(\Gamma_{0\xi},C_{0\xi}\right)B\left(\Delta_{0\xi}+1,\Gamma_{0\xi}+C_{0\xi}\right)\end{array}\right\} \quad (8)$$

$$\rho\left(\hat{\theta}_{\varpi(SELF)}\right) = \frac{1}{\Omega_\xi}\left\{\begin{array}{l}\sum_{v_1=0}^{r_1-1}\sum_{v_2=0}^{r_2-1}\sum_{v_3=0}^{r_3-1}\sum_{v_4=0}^{n-w}\sum_{v_5=0}^{v_4}\prod_{k=1}^{3}(-1)^{v_k}\binom{r_k-1}{v_k}\binom{n-w}{v_4}\binom{v_4}{v_5}\Gamma\left(A_{\varpi\xi}+2\right)\Gamma\left(A_{\pi\xi}\right)\Gamma\left(A_{\eta\xi}\right)\\ B_{\varpi\xi}^{-\left(A_{\varpi\xi}+2\right)}B_{\pi\xi}^{-A_{\pi\xi}}B_{\eta\xi}^{-A_{\eta\xi}}B\left(A_{0\xi},C_{0\xi}\right)B\left(B_{0\xi},A_{0\xi}+C_{0\xi}\right)\end{array}\right\} - \left\{\hat{\theta}_{\varpi(SELF)}\right\}^2 \quad (9)$$

$$\rho\left(\hat{p}_{\alpha(SELF)}\right) = \frac{1}{\Omega_\xi}\left\{\begin{array}{l}\sum_{v_1=0}^{r_1-1}\sum_{v_2=0}^{r_2-1}\sum_{v_3=0}^{r_3-1}\sum_{v_4=0}^{n-w}\sum_{v_5=0}^{v_4}\prod_{k=1}^{3}(-1)^{v_k}\binom{r_k-1}{v_k}\binom{n-w}{v_4}\binom{v_4}{v_5}\Gamma\left(A_{1\xi}\right)\Gamma\left(A_{2\xi}\right)\Gamma\left(A_{3\xi}\right)\\ B_{1\xi}^{-A_{1\xi}}B_{2\xi}^{-A_{2\xi}}B_{3\xi}^{-A_{3\xi}}B\left(\gamma_{0\xi},C_{0\xi}\right)B\left(\Delta_{0\xi}+2,\gamma_{0\xi}+C_{0\xi}\right)\end{array}\right\} - \left\{\hat{p}_{\alpha(SELF)}\right\}^2, \quad (10)$$

where α, β, γ and Δ take the values as (i) $\alpha = 1, \beta = 2, \gamma = B, \Delta = A$ and (ii) $\alpha = 2, \beta = 1, \alpha = 2, \beta = 1, \gamma = A, \Delta = B$. Also, $\xi = 1$ for the UP, $\xi = 2$ for the JP and $\xi = 3$ for the IP.

6.2. Expressions of the Bayes Estimators and Posterior Risks Under PLF

The respective marginal distribution yields the algebraic expressions for Bayes estimators and posterior risks assuming the UP, the JP and the IP of parameters $\theta_1, \theta_2, \theta_3, p_1$ and p_2 under PLF as

$$\hat{\theta}_{\varpi(PLF)} = \sqrt{\frac{1}{\Omega_\xi}\left\{\begin{array}{l}\sum_{v_1=0}^{r_1-1}\sum_{v_2=0}^{r_2-1}\sum_{v_3=0}^{r_3-1}\sum_{v_4=0}^{n-w}\sum_{v_5=0}^{v_4}\prod_{k=1}^{3}(-1)^{v_k}\binom{r_k-1}{v_k}\binom{n-w}{v_4}\binom{v_4}{v_5}\Gamma\left(A_{\varpi\xi}+2\right)\Gamma\left(A_{\pi\xi}\right)\Gamma\left(A_{\eta\xi}\right)\\ B_{\varpi\xi}^{-\left(A_{\varpi\xi}+2\right)}B_{\pi\xi}^{-A_{\pi\xi}}B_{\eta\xi}^{-A_{\eta\xi}}B\left(A_{0\xi},C_{0\xi}\right)B\left(B_{0\xi},A_{0\xi}+C_{0\xi}\right)\end{array}\right\}}. \quad (11)$$

$$\hat{p}_{\alpha(PLF)} = \sqrt{\frac{1}{\Omega_\xi}\left\{\begin{array}{l}\sum_{v_1=0}^{r_1-1}\sum_{v_2=0}^{r_2-1}\sum_{v_3=0}^{r_3-1}\sum_{v_4=0}^{n-w}\sum_{v_5=0}^{v_4}\prod_{k=1}^{3}(-1)^{v_k}\binom{r_k-1}{v_k}\binom{n-w}{v_4}\binom{v_4}{v_5}\Gamma\left(A_{1\xi}\right)\Gamma\left(A_{2\xi}\right)\Gamma\left(A_{3\xi}\right)\\ B_{1\xi}^{-A_{1\xi}}B_{2\xi}^{-A_{2\xi}}B_{3\xi}^{-A_{3\xi}}B\left(\Gamma_{0\xi},C_{0\xi}\right)B\left(\Delta_{0\xi}+2,\Gamma_{0\xi}+C_{0\xi}\right)\end{array}\right\}}. \quad (12)$$

$$\rho\left(\hat{\theta}_{\varpi(PLF)}\right)=2\sqrt{\frac{1}{\Omega_\xi}\left\{\begin{array}{l}\sum_{v_1=0}^{r_1-1}\sum_{v_2=0}^{r_2-1}\sum_{v_3=0}^{r_3-1}\sum_{v_4=0}^{n-w}\sum_{v_5=0}^{v_4}\prod_{k=1}^{3}(-1)^{v_k}\binom{r_k-1}{v_k}\binom{n-w}{v_4}\binom{v_4}{v_5}\Gamma\left(A_{\varpi\xi}+2\right)\Gamma\left(A_{\pi\xi}\right)\Gamma\left(A_{\eta\xi}\right)\\ B_{\varpi\xi}^{-\left(A_{\varpi\xi}+2\right)}B_{\pi\xi}^{-A_{\pi\xi}}B_{\eta\xi}^{-A_{\eta\xi}}B\left(A_{0\xi},C_{0\xi}\right)B\left(B_{0\xi},A_{0\xi}+C_{0\xi}\right)\end{array}\right\}}$$

$$-\frac{2}{\Omega_\xi}\left\{\begin{array}{l}\sum_{v_1=0}^{r_1-1}\sum_{v_2=0}^{r_2-1}\sum_{v_3=0}^{r_3-1}\sum_{v_4=0}^{n-w}\sum_{v_5=0}^{v_4}\prod_{k=1}^{3}(-1)^{v_k}\binom{r_k-1}{v_k}\binom{n-w}{v_4}\binom{v_4}{v_5}\Gamma\left(A_{\varpi\xi}+1\right)\Gamma\left(A_{\pi\xi}\right)\Gamma\left(A_{\eta\xi}\right)\\ B_{\varpi\xi}^{-\left(A_{\varpi\xi}+1\right)}B_{\pi\xi}^{-A_{\pi\xi}}B_{\eta\xi}^{-A_{\eta\xi}}B\left(A_{0\xi},C_{0\xi}\right)B\left(B_{0\xi},A_{0\xi}+C_{0\xi}\right)\end{array}\right\}. \tag{13}$$

$$\rho\left(\hat{p}_{\alpha(PLF)}\right)=2\sqrt{\frac{1}{\Omega_\xi}\left\{\begin{array}{l}\sum_{v_1=0}^{r_1-1}\sum_{v_2=0}^{r_2-1}\sum_{v_3=0}^{r_3-1}\sum_{v_4=0}^{n-w}\sum_{v_5=0}^{v_4}\prod_{k=1}^{3}(-1)^{v_k}\binom{r_k-1}{v_k}\binom{n-w}{v_4}\binom{v_4}{v_5}\Gamma\left(A_{1\xi}\right)\Gamma\left(A_{2\xi}\right)\Gamma\left(A_{3\xi}\right)\\ B_{1\xi}^{-A_{1\xi}}B_{2\xi}^{-A_{2\xi}}B_{3\xi}^{-A_{3\xi}}B\left(\gamma_{0\xi},C_{0\xi}\right)B\left(\Delta_{0\xi}+2,\gamma_{0\xi}+C_{0\xi}\right)\end{array}\right\}}$$

$$-\frac{2}{\Omega_\xi}\left\{\begin{array}{l}\sum_{v_1=0}^{r_1-1}\sum_{v_2=0}^{r_2-1}\sum_{v_3=0}^{r_3-1}\sum_{v_4=0}^{n-w}\sum_{v_5=0}^{v_4}\prod_{k=1}^{3}(-1)^{v_k}\binom{r_k-1}{v_k}\binom{n-w}{v_4}\binom{v_4}{v_5}\Gamma\left(A_{1\xi}\right)\Gamma\left(A_{2\xi}\right)\Gamma\left(A_{3\xi}\right)\\ B_{1\xi}^{-A_{1\xi}}B_{2\xi}^{-A_{2\xi}}B_{3\xi}^{-A_{3\xi}}B\left(\gamma_{0\xi},C_{0\xi}\right)B\left(\Delta_{0\xi}+1,\gamma_{0\xi}+C_{0\xi}\right)\end{array}\right\}. \tag{14}$$

6.3. Expressions of the Bayes Estimators and Posterior Risks Under DLF

The algebraic expressions for Bayes estimators and posterior risks assuming the UP, the JP and the IP of parameters $\theta_1, \theta_2, \theta_3, p_1$ and p_2 under DLF are derived with respective marginal distribution as

$$\hat{\theta}_{\varpi(DLF)}=\frac{\left\{\begin{array}{l}\sum_{v_1=0}^{r_1-1}\sum_{v_2=0}^{r_2-1}\sum_{v_3=0}^{r_3-1}\sum_{v_4=0}^{n-w}\sum_{v_5=0}^{v_4}\prod_{k=1}^{3}(-1)^{v_k}\binom{r_k-1}{v_k}\binom{n-w}{v_4}\binom{v_4}{v_5}\Gamma\left(A_{\varpi\xi}+2\right)\Gamma\left(A_{\pi\xi}\right)\Gamma\left(A_{\eta\xi}\right)\\ B_{\varpi\xi}^{-\left(A_{\varpi\xi}+2\right)}B_{\pi\xi}^{-A_{\pi\xi}}B_{\eta\xi}^{-A_{\eta\xi}}B\left(A_{0\xi},C_{0\xi}\right)B\left(B_{0\xi},A_{0\xi}+C_{0\xi}\right)\end{array}\right\}}{\left\{\begin{array}{l}\sum_{v_1=0}^{r_1-1}\sum_{v_2=0}^{r_2-1}\sum_{v_3=0}^{r_3-1}\sum_{v_4=0}^{n-w}\sum_{v_5=0}^{v_4}\prod_{k=1}^{3}(-1)^{v_k}\binom{r_k-1}{v_k}\binom{n-w}{v_4}\binom{v_4}{v_5}\Gamma\left(A_{\varpi\xi}+1\right)\Gamma\left(A_{\pi\xi}\right)\Gamma\left(A_{\eta\xi}\right)\\ B_{\varpi\xi}^{-\left(A_{\varpi\xi}+1\right)}B_{\pi\xi}^{-A_{\pi\xi}}B_{\eta\xi}^{-A_{\eta\xi}}B\left(A_{0\xi},C_{0\xi}\right)B\left(B_{0\xi},A_{0\xi}+C_{0\xi}\right)\end{array}\right\}}. \tag{15}$$

$$\hat{p}_{\alpha(DLF)}=\frac{\left\{\begin{array}{l}\sum_{v_1=0}^{r_1-1}\sum_{v_2=0}^{r_2-1}\sum_{v_3=0}^{r_3-1}\sum_{v_4=0}^{n-w}\sum_{v_5=0}^{v_4}\prod_{k=1}^{3}(-1)^{v_k}\binom{r_k-1}{v_k}\binom{n-w}{v_4}\binom{v_4}{v_5}\Gamma\left(A_{1\xi}\right)\Gamma\left(A_{2\xi}\right)\Gamma\left(A_{3\xi}\right)\\ B_{1\xi}^{-A_{1\xi}}B_{2\xi}^{-A_{2\xi}}B_{3\xi}^{-A_{3\xi}}B\left(\gamma_{0\xi},C_{0\xi}\right)B\left(\Delta_{0\xi}+2,\gamma_{0\xi}+C_{0\xi}\right)\end{array}\right\}}{\left\{\begin{array}{l}\sum_{v_1=0}^{r_1-1}\sum_{v_2=0}^{r_2-1}\sum_{v_3=0}^{r_3-1}\sum_{v_4=0}^{n-w}\sum_{v_5=0}^{v_4}\prod_{k=1}^{3}(-1)^{v_k}\binom{r_k-1}{v_k}\binom{n-w}{v_4}\binom{v_4}{v_5}\Gamma\left(A_{1\xi}\right)\Gamma\left(A_{2\xi}\right)\Gamma\left(A_{3\xi}\right)\\ B_{1\xi}^{-A_{1\xi}}B_{2\xi}^{-A_{2\xi}}B_{3\xi}^{-A_{3\xi}}B\left(\gamma_{0\xi},C_{0\xi}\right)B\left(\Delta_{0\xi}+1,\gamma_{0\xi}+C_{0\xi}\right)\end{array}\right\}}. \tag{16}$$

$$\rho\left(\hat{\theta}_{\varpi(DLF)}\right)=1-\frac{\left\{\begin{array}{l}\sum_{v_1=0}^{r_1-1}\sum_{v_2=0}^{r_2-1}\sum_{v_3=0}^{r_3-1}\sum_{v_4=0}^{n-w}\sum_{v_5=0}^{v_4}\prod_{k=1}^{3}(-1)^{v_k}\binom{r_k-1}{v_k}\binom{n-w}{v_4}\binom{v_4}{v_5}\Gamma\left(A_{\varpi\xi}+1\right)\Gamma\left(A_{\pi\xi}\right)\Gamma\left(A_{\eta\xi}\right)\\ B_{\varpi\xi}^{-\left(A_{\varpi\xi}+1\right)}B_{\pi\xi}^{-A_{\pi\xi}}B_{\eta\xi}^{-A_{\eta\xi}}B\left(A_{0\xi},C_{0\xi}\right)B\left(B_{0\xi},A_{0\xi}+C_{0\xi}\right)\end{array}\right\}^2}{\Omega_\xi\left\{\begin{array}{l}\sum_{v_1=0}^{r_1-1}\sum_{v_2=0}^{r_2-1}\sum_{v_3=0}^{r_3-1}\sum_{v_4=0}^{n-w}\sum_{v_5=0}^{v_4}\prod_{k=1}^{3}(-1)^{v_k}\binom{r_k-1}{v_k}\binom{n-w}{v_4}\binom{v_4}{v_5}\Gamma\left(A_{\varpi\xi}+2\right)\Gamma\left(A_{\pi\xi}\right)\Gamma\left(A_{\eta\xi}\right)\\ B_{\varpi\xi}^{-\left(A_{\varpi\xi}+2\right)}B_{\pi\xi}^{-A_{\pi\xi}}B_{\eta\xi}^{-A_{\eta\xi}}B\left(A_{0\xi},C_{0\xi}\right)B\left(B_{0\xi},A_{0\xi}+C_{0\xi}\right)\end{array}\right\}}. \tag{17}$$

$$\rho\left(\hat{p}_{\alpha(DLF)}\right)=1-\frac{\left\{\begin{array}{l}\sum_{v_1=0}^{r_1-1}\sum_{v_2=0}^{r_2-1}\sum_{v_3=0}^{r_3-1}\sum_{v_4=0}^{n-w}\sum_{v_5=0}^{v_4}\prod_{k=1}^{3}(-1)^{v_k}\binom{r_k-1}{v_k}\binom{n-w}{v_4}\binom{v_4}{v_5}\Gamma\left(A_{1\xi}\right)\Gamma\left(A_{2\xi}\right)\Gamma\left(A_{3\xi}\right)\\ B_{1\xi}^{-A_{1\xi}}B_{2\xi}^{-A_{2\xi}}B_{3\xi}^{-A_{3\xi}}B\left(\gamma_{0\xi},C_{0\xi}\right)B\left(\Delta_{0\xi}+1,\gamma_{0\xi}+C_{0\xi}\right)\end{array}\right\}^2}{\Omega_\xi\left\{\begin{array}{l}\sum_{v_1=0}^{r_1-1}\sum_{v_2=0}^{r_2-1}\sum_{v_3=0}^{r_3-1}\sum_{v_4=0}^{n-w}\sum_{v_5=0}^{v_4}\prod_{k=1}^{3}(-1)^{v_k}\binom{r_k-1}{v_k}\binom{n-w}{v_4}\binom{v_4}{v_5}\Gamma\left(A_{1\xi}\right)\Gamma\left(A_{2\xi}\right)\Gamma\left(A_{3\xi}\right)\\ B_{1\xi}^{-A_{1\xi}}B_{2\xi}^{-A_{2\xi}}B_{3\xi}^{-A_{3\xi}}B\left(\gamma_{0\xi},C_{0\xi}\right)B\left(\Delta_{0\xi}+2,\gamma_{0\xi}+C_{0\xi}\right)\end{array}\right\}}. \tag{18}$$

7. THE POSTERIOR PREDICTIVE DISTRIBUTION AND BAYESIAN PREDICTIVE INTERVAL

A significant feature of the Bayesian methodology is the predictive distribution used to predict the future observation $X = Y_{n+1}$ of a random variable given the data $\mathbf{y}$, already observed. Al-Hussaini *et al.* [46], Bolstad [47] and Bansal [48] have given a detailed discussion on prediction and predictive distribution in the Bayesian paradigm. We, now, present the derivation of posterior predictive distribution and Bayesian predictive interval.

The posterior predictive distribution of a future observation $X = Y_{n+1}$ given data $\mathbf{y}$ assuming the UP, the JP and the IP is defined as

$$f(x|\mathbf{y})=\int_{\Phi}p\left(x|\Phi\right)g_\xi\left(\Phi|\mathbf{y}\right)d\Phi. \tag{19}$$

So, the posterior predictive distribution given in (19) after substituting and simplifying is

$$\begin{aligned}f(x|\mathbf{y})=&\frac{1}{\Omega_\xi}\left\{\begin{array}{l}\sum_{v_1=0}^{r_1-1}\sum_{v_2=0}^{r_2-1}\sum_{v_3=0}^{r_3-1}\sum_{v_4=0}^{n-w}\sum_{v_5=0}^{v_4}\prod_{k=1}^{3}(-1)^{v_k}\binom{r_k-1}{v_k}\binom{n-w}{v_4}\binom{v_4}{v_5}\Gamma\left(A_{1\xi}+1\right)\Gamma\left(A_{2\xi}\right)\Gamma\left(A_{3\xi}\right)\\ \left(B_{1\xi}+x\right)^{-\left(A_{1\xi}+1\right)}B_{2\xi}^{-A_{2\xi}}B_{3\xi}^{-A_{3\xi}}B\left(A_{0\xi}+1,C_{0\xi}\right)B\left(B_{0\xi},A_{0\xi}+C_{0\xi}+1\right)\end{array}\right\}\\ &+\frac{1}{\Omega_\xi}\left\{\begin{array}{l}\sum_{v_1=0}^{r_1-1}\sum_{v_2=0}^{r_2-1}\sum_{v_3=0}^{r_3-1}\sum_{v_4=0}^{n-w}\sum_{v_5=0}^{v_4}\prod_{k=1}^{3}(-1)^{v_k}\binom{r_k-1}{v_k}\binom{n-w}{v_4}\binom{v_4}{v_5}\Gamma\left(A_{1\xi}\right)\Gamma\left(A_{2\xi}+1\right)\Gamma\left(A_{3\xi}\right)\\ B_{1\xi}^{-A_{1\xi}}\left(B_{2\xi}+x\right)^{-\left(A_{2\xi}+1\right)}B_{3\xi}^{-A_{3\xi}}B\left(A_{0\xi},C_{0\xi}\right)B\left(B_{0\xi}+1,A_{0\xi}+C_{0\xi}\right)\end{array}\right\}\\ &+\frac{1}{\Omega_\xi}\left\{\begin{array}{l}\sum_{v_1=0}^{r_1-1}\sum_{v_2=0}^{r_2-1}\sum_{v_3=0}^{r_3-1}\sum_{v_4=0}^{n-w}\sum_{v_5=0}^{v_4}\prod_{k=1}^{3}(-1)^{v_k}\binom{r_k-1}{v_k}\binom{n-w}{v_4}\binom{v_4}{v_5}\Gamma\left(A_{1\xi}\right)\Gamma\left(A_{2\xi}\right)\Gamma\left(A_{3\xi}+1\right)\\ B_{1\xi}^{-A_{1\xi}}B_{2\xi}^{-A_{2\xi}}\left(B_{3\xi}+x\right)^{-\left(A_{3\xi}+1\right)}B\left(A_{0\xi},C_{0\xi}+1\right)B\left(B_{0\xi},A_{0\xi}+C_{0\xi}+1\right)\end{array}\right\}.\end{aligned} \tag{20}$$

In order to construct a Bayesian predictive interval, suppose L and U be the two endpoints of the Bayesian predictive interval. These two endpoints can be obtained using the posterior predictive distribution defined in (20). A $100(1-\gamma)\%$ Bayesian predictive interval (L, U) can be obtained by solving the following equations:

$$\int_0^L f(x|\mathbf{y})\,dx = \frac{\gamma}{2} = \int_U^\infty f(x|\mathbf{y})\,dx.$$

On simplifying the above equations, the Bayesian predictive interval (L, U) can be written as

$$\frac{1}{\Omega_\xi}\left\{\begin{array}{l}\sum_{v_1=0}^{r_1-1}\sum_{v_2=0}^{r_2-1}\sum_{v_3=0}^{r_3-1}\sum_{v_4=0}^{n-w}\sum_{v_5=0}^{v_4}\prod_{k=1}^{3}(-1)^{v_k}\binom{r_k-1}{v_k}\binom{n-w}{v_4}\binom{v_4}{v_5}\frac{\Gamma(A_{1\xi}+1)\Gamma(A_{2\xi})\Gamma(A_{3\xi})}{A_{1\xi}}\\ \left(B_{1\xi}^{-A_{1\xi}}-(B_{1\xi}+L)^{-A_{1\xi}}\right)B_{2\xi}^{-A_{2\xi}}B_{3\xi}^{-A_{3\xi}}B(A_{0\xi}+1,C_{0\xi})B(B_{0\xi},A_{0\xi}+C_{0\xi}+1)\end{array}\right\}$$
$$+\frac{1}{\Omega_\xi}\left\{\begin{array}{l}\sum_{v_1=0}^{r_1-1}\sum_{v_2=0}^{r_2-1}\sum_{v_3=0}^{r_3-1}\sum_{v_4=0}^{n-w}\sum_{v_5=0}^{v_4}\prod_{k=1}^{3}(-1)^{v_k}\binom{r_k-1}{v_k}\binom{n-w}{v_4}\binom{v_4}{v_5}\frac{\Gamma(A_{1\xi})\Gamma(A_{2\xi}+1)\Gamma(A_{3\xi})}{A_{2\xi}}\\ B_{1\xi}^{-A_{1\xi}}\left(B_{2\xi}^{-A_{2\xi}}-(B_{2\xi}+L)^{-A_{2\xi}}\right)B_{3\xi}^{-A_{3\xi}}B(A_{0\xi},C_{0\xi})B(B_{0\xi}+1,A_{0\xi}+C_{0\xi})\end{array}\right\}$$
$$+\frac{1}{\Omega_\xi}\left\{\begin{array}{l}\sum_{v_1=0}^{r_1-1}\sum_{v_2=0}^{r_2-1}\sum_{v_3=0}^{r_3-1}\sum_{v_4=0}^{n-w}\sum_{v_5=0}^{v_4}\prod_{k=1}^{3}(-1)^{v_k}\binom{r_k-1}{v_k}\binom{n-w}{v_4}\binom{v_4}{v_5}\frac{\Gamma(A_{1\xi})\Gamma(A_{2\xi})\Gamma(A_{3\xi}+1)}{A_{3\xi}}\\ B_{1\xi}^{-A_{1\xi}}B_{2\xi}^{-A_{2\xi}}\left(B_{3\xi}^{-A_{3\xi}}-(B_{3\xi}+L)^{-A_{3\xi}}\right)B(A_{0\xi},C_{0\xi}+1)B(B_{0\xi},A_{0\xi}+C_{0\xi}+1)\end{array}\right\}=\frac{\gamma}{2}$$

and

$$\frac{1}{\Omega_\xi}\left\{\begin{array}{l}\sum_{v_1=0}^{r_1-1}\sum_{v_2=0}^{r_2-1}\sum_{v_3=0}^{r_3-1}\sum_{v_4=0}^{n-w}\sum_{v_5=0}^{v_4}\prod_{k=1}^{3}(-1)^{v_k}\binom{r_k-1}{v_k}\binom{n-w}{v_4}\binom{v_4}{v_5}\frac{\Gamma(A_{1\xi}+1)\Gamma(A_{2\xi})\Gamma(A_{3\xi})}{A_{1\xi}}\\ (B_{1\xi}+U)^{-A_{1\xi}}B_{2\xi}^{-A_{2\xi}}B_{3\xi}^{-A_{3\xi}}B(A_{0\xi}+1,C_{0\xi})B(B_{0\xi},A_{0\xi}+C_{0\xi}+1)\end{array}\right\}$$
$$+\frac{1}{\Omega_\xi}\left\{\begin{array}{l}\sum_{v_1=0}^{r_1-1}\sum_{v_2=0}^{r_2-1}\sum_{v_3=0}^{r_3-1}\sum_{v_4=0}^{n-w}\sum_{v_5=0}^{v_4}\prod_{k=1}^{3}(-1)^{v_k}\binom{r_k-1}{v_k}\binom{n-w}{v_4}\binom{v_4}{v_5}\frac{\Gamma(A_{1\xi})\Gamma(A_{2\xi}+1)\Gamma(A_{3\xi})}{A_{2\xi}}\\ B_{1\xi}^{-A_{1\xi}}(B_{2\xi}+U)^{-A_{2\xi}}B_{3\xi}^{-A_{3\xi}}B(A_{0\xi},C_{0\xi})B(B_{0\xi}+1,A_{0\xi}+C_{0\xi})\end{array}\right\}$$
$$+\frac{1}{\Omega_\xi}\left\{\begin{array}{l}\sum_{v_1=0}^{r_1-1}\sum_{v_2=0}^{r_2-1}\sum_{v_3=0}^{r_3-1}\sum_{v_4=0}^{n-w}\sum_{v_5=0}^{v_4}\prod_{k=1}^{3}(-1)^{v_k}\binom{r_k-1}{v_k}\binom{n-w}{v_4}\binom{v_4}{v_5}\frac{\Gamma(A_{1\xi})\Gamma(A_{2\xi})\Gamma(A_{3\xi}+1)}{A_{3\xi}}\\ B_{1\xi}^{-A_{1\xi}}B_{2\xi}^{-A_{2\xi}}(B_{3\xi}+U)^{-A_{3\xi}}B(A_{0\xi},C_{0\xi}+1)B(B_{0\xi},A_{0\xi}+C_{0\xi}+1)\end{array}\right\}=\frac{\gamma}{2}.$$

8. ELICITATION OF HYPERPARAMETERS

Elicitation is a tool used to quantify a person's belief and knowledge about the parameter(s) of interest and in the Bayesian perspective, elicitation, most often, arises as a method for specifying the hyperparameter of the prior distribution for the random parameter(s). Elicitation has remained a challenging problem for the Bayesian statistician. However, in this study, we adopt a method based on predictive probabilities, given by Aslam [49].

For eliciting the hyperparameters, we use the prior predictive distribution (PPD). The PPD for a random variable Y is

$$p(y) = \int_\Phi p(y|\Phi)\,\pi_3(\Phi)\,d\Phi$$

$$p(y) = \frac{1}{(a+b+c)}\left\{\frac{aa_1b_1^{a_1}}{(b_1+y)^{a_1+1}}+\frac{ba_2b_2^{a_2}}{(b_2+y)^{a_2+1}}+\frac{ca_3b_3^{a_3}}{(b_3+y)^{a_3+1}}\right\}. \tag{21}$$

To elicit the nine hyperparameters involved in the PPD in (21), we considered the following nine intervals (0, 1), (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7), (7, 8) and (8, 9) and assumed the following probabilities 0.57, 0.20, 0.10, 0.05, 0.02, 0.015, 0.01, 0.005 and 0.003, respectively. It is worth mentioning that these probabilities might have been obtained from the expert(s) as their opinion about the likelihood of these intervals. Moreover, different intervals could also be considered. Then, nine equations are solved simultaneously by using Mathematica software for eliciting the hyperparameters $a_1, b_1, a_2, b_2, a_3, b_3, a, b$ and c. Using the above defined procedure, we obtain the following hyperparameter values: 3.8330, 3.7310, 3.3570, 3.1360, 2.9030, 2.7330, 3.0280, 0.6995 and 2.7350.

9. MONTE CARLO SIMULATION STUDY

From Equations (7–8), (11–12) and (15–16), it is clear that comparing Bayes estimators (under different priors and loss functions) analytically is almost impossible. Therefore, a Monte Carlo simulation study is conducted to assess the performance of the Bayes estimators under different priors, loss functions, parametric values, sample sizes and left and right test termination times. For different values of each of the five parameters $\theta_1, \theta_2, \theta_3, p_1$ and p_2 of a 3-CMED, we calculate the Bayes estimates and their posterior risks by using the following steps:

1. A sample from the mixtures may be generated through the Mathematica package as follows:
   i. Generate $p_1 n$ observations randomly from first component density $f_1\left(y;\Phi_1\right)$.
   ii. Generate $p_2 n$ observations randomly from second component density $f_2\left(y;\Phi_2\right)$.
   iii. Generate remaining $\left(1-p_1-p_2\right)n$ observations randomly from third component density $f_3\left(y;\Phi_3\right)$.
2. Select a sample censored at fixed test termination times on left and right, that is, y_r and y_w.
3. Take observations which are less than y_r and greater than y_w as censored ones.
4. Using the Steps 1–3 for the fixed values of parameters, test termination time and sample size, generate 1000 samples.
5. Calculate the Bayes estimates and posterior risks of parameters $\theta_1, \theta_2, \theta_3, p_1$ and p_2 based on 1000 repetitions by solving (7)–(18).

The above steps 1–5 are used for each of the sample sizes $n = 40, 80,$ and 140. The choice of the vector of the parameters is $\left(\theta_1, \theta_2, \theta_3, p_1, p_2\right) \in (2, 3, 4, 0.2, 0.4)$ by taking the following left and right test termination times $\left(y_r, y_w\right) \in (0.01, 0.8)$. It is worth mentioning that the choices of left and right test termination times are made in such a way that the censoring rate in the resulting sample remains in between 7% to 20%. The resulting simulated results have been presented in Tables 2–4. The simulated results for $\left(\theta_1, \theta_2, \theta_3, p_1, p_2\right) \in (2, 3, 4, 0.2, 0.4)$ with $\left(y_r, y_w\right) \in (0.005, 1.2)$ are available with the first author and can be obtained on demand.

From the Tables 2–4, it was observed that the degree of over-estimation (and/or under-estimation) of Bayes estimate of component and proportion parameters using the NIP (UP and JP) and the IP under SELF, PLF and DLF was greater for smaller sample size as compared to a larger sample size for the fixed left and right test termination times $\left(y_r, y_w\right)$. Also, the degree of under-estimation (and/or over-estimation) of

Table 2 Bayes estimates (BE) and posterior risks (PR) of 3-CMED using the UP, the JP and the IP under SELF with parameters $\theta_1 = 2, \theta_2 = 3, \theta_3 = 4, p_1 = 0.2, p_2 = 0.4$.

y_r, y_w	n	Prior Distribution		$\hat{\theta}_{1(SELF)}$	$\hat{\theta}_{2(SELF)}$	$\hat{\theta}_{3(SELF)}$	$\hat{p}_{1(SELF)}$	$\hat{p}_{2(SELF)}$
0.01, 0.8	40	UP	BE	3.395080	3.605950	4.564170	0.206830	0.394006
			PR	3.189950	1.380540	1.915510	0.004616	0.006416
		JP	BE	2.784800	3.430350	4.284070	0.205418	0.395564
			PR	2.936520	1.313790	1.790290	0.004524	0.006386
		IP	BE	1.556880	2.355800	2.722950	0.238333	0.351486
			PR	0.278267	0.386267	0.506775	0.004511	0.005442
	80	UP	BE	2.719220	3.291960	4.265730	0.204338	0.397148
			PR	1.146740	0.636585	0.911545	0.002552	0.003508
		JP	BE	2.456510	3.253020	4.144790	0.203833	0.397956
			PR	0.986246	0.621705	0.870667	0.002469	0.003467
		IP	BE	1.723430	2.612290	3.163760	0.220376	0.373973
			PR	0.240709	0.293481	0.410092	0.002441	0.003127
	140	UP	BE	2.452820	3.176990	4.153470	0.203616	0.398351
			PR	0.620311	0.380634	0.520377	0.001613	0.001898
		JP	BE	2.284210	3.164150	4.066220	0.203075	0.398527
			PR	0.569558	0.315852	0.434408	0.001586	0.001858
		IP	BE	1.831800	2.747620	3.432180	0.213538	0.381421
			PR	0.196568	0.227209	0.331019	0.001510	0.001718

SELF, squared error loss function; UP, uniform prior; JP, Jeffreys' prior; CMED, component mixture of exponential distribution.

Table 3 Bayes estimates (BE) and posterior risks (PR) of 3-CMED using the UP, the JP and the IP under PLF with parameters $\theta_1 = 2, \theta_2 = 3, \theta_3 = 4, p_1 = 0.2, p_2 = 0.4$.

y_r, y_w	n	Prior Distribution		$\hat{\theta}_{1(PLF)}$	$\hat{\theta}_{2(PLF)}$	$\hat{\theta}_{3(PLF)}$	$\hat{p}_{1(PLF)}$	$\hat{p}_{2(PLF)}$
0.01, 0.8	40	UP	BE	3.699000	3.849730	4.767100	0.217908	0.405841
			PR	0.717074	0.356772	0.394466	0.022028	0.016086
		JP	BE	3.053300	3.664720	4.384080	0.216180	0.404422
			PR	0.693655	0.352933	0.380654	0.022006	0.016010
		IP	BE	1.634710	2.431220	2.795510	0.245329	0.360427
			PR	0.173252	0.154085	0.178664	0.018648	0.015317
	80	UP	BE	2.888690	3.446400	4.376220	0.210699	0.403334
			PR	0.369371	0.186865	0.206311	0.012482	0.008699
		JP	BE	2.638560	3.326240	4.190550	0.210432	0.403134
			PR	0.361023	0.184805	0.204036	0.012440	0.008668
		IP	BE	1.781800	2.667790	3.237220	0.226288	0.378155
			PR	0.134257	0.108032	0.126989	0.010931	0.008318
	140	UP	BE	2.623740	3.268870	4.229650	0.205550	0.402903
			PR	0.242676	0.125873	0.129712	0.008339	0.005497
		JP	BE	2.476790	3.165040	4.142140	0.205369	0.402307
			PR	0.225903	0.121273	0.125100	0.008282	0.005367
		IP	BE	1.875440	2.777690	3.451640	0.217994	0.384240
			PR	0.107602	0.088266	0.096329	0.007372	0.005126

PLF, precautionary loss function; UP, uniform prior; JP, Jeffreys' prior; CMED, component mixture of exponential distribution.

Table 4 Bayes estimates (BE) and posterior risks (PR) of 3-CMED using the UP, the JP and the IP under DLF with parameters $\theta_1 = 2, \theta_2 = 3, \theta_3 = 4, p_1 = 0.2, p_2 = 0.4$.

y_r, y_w	n	Prior Distribution		$\hat{\theta}_{1(DLF)}$	$\hat{\theta}_{2(DLF)}$	$\hat{\theta}_{3(DLF)}$	$\hat{p}_{1(DLF)}$	$\hat{p}_{2(DLF)}$
0.01, 0.8	40	UP	BE	4.159730	4.017760	4.978260	0.227945	0.412073
			PR	0.191222	0.092280	0.083481	0.100028	0.039774
		JP	BE	3.488430	3.779970	4.726180	0.221350	0.411067
			PR	0.215770	0.095264	0.085669	0.102242	0.039905
		IP	BE	1.724710	2.487160	2.887600	0.256362	0.367605
			PR	0.104512	0.062960	0.063214	0.074873	0.038481
	80	UP	BE	3.082070	3.515470	4.415310	0.217892	0.407875
			PR	0.129499	0.055152	0.048539	0.060574	0.021708
		JP	BE	2.818450	3.420770	4.355840	0.214324	0.406578
			PR	0.134775	0.055252	0.048982	0.060620	0.021766
		IP	BE	1.870780	2.735790	3.306760	0.231815	0.381260
			PR	0.075300	0.039535	0.038694	0.048479	0.021646
	140	UP	BE	2.694600	3.341800	4.324170	0.209112	0.405395
			PR	0.104543	0.021811	0.024297	0.042675	0.019338
		JP	BE	2.511670	3.303690	4.193300	0.207597	0.404229
			PR	0.110332	0.023206	0.028588	0.043811	0.020176
		IP	BE	1.928560	2.760290	3.556690	0.221630	0.386456
			PR	0.053041	0.019671	0.023283	0.032184	0.017784

DLF, DeGroot loss function; UP, uniform prior; JP, Jeffreys' prior; CMED, component mixture of exponential distribution.

component and proportion parameters was observed lower for a smaller left test termination time y_r and larger for the right test termination time y_w as compared to a larger left test termination time y_r and a smaller right test termination time y_w for a fixed sample size. It has also been observed that the bias in the Bayes estimates reduced to zero as the sample size was increased at varying left and right test termination times. Moreover, we also observed from the simulation study that the Bayes estimates tend to converge to the true parameter values with a smaller left test termination time y_r and a larger right test termination time y_w as compared to a larger left test termination time y_r and a smaller right test termination time y_w for different sample sizes.

The posterior risk of the Bayes estimates is a suitable criterion for comparing the performance of the different loss functions. From our study, we observed that the posterior risk was directly proportional to true parametric values and was inversely proportional to the sample size. It was seen that the posterior risks of the component and the proportion parameters using the NIP (UP and JP) and the IP under SELF, PLF and DLF were inversely proportional to sample size for fixed left and right test termination times. The same observation was made for smaller left and larger right test termination times as compared to larger left and smaller right test termination times at varying sample sizes.

As far as the problem of selecting a suitable prior is concerned, it can be seen that, having the least associated amount of posterior risk of the Bayes estimates for a given loss function, the IP is more efficient prior amongst the different considered priors in this study. Also, it can be seen that the UP (JP) emerges as the best prior than the JP (UP) under DLF (SELF and PLF) due to smaller associated posterior risk. On the other hand, for estimating the component parameters, the DLF performs better than PLF and SELF, whereas the performance of SELF is superior to PLF and DLF for estimating the proportion parameters. It was also observed that the selection of the best prior or suitable loss function is independent on left and right test termination times and sample sizes. It is worth mentioning that the selection of the best prior (loss function) for a given loss function (prior) is made on the basis of the minimum posterior risks.

10. REAL DATA APPLICATION

In this section, we present the analysis of a real-life data to illustrate the methodology discussed in the previous sections. The main idea of the present section is to determine whether the results and properties of the Bayes estimators explored by a simulation study, have the same behavior under a real-life situation. Therefore, for this purpose, we use the data set reported in Gómez *et al.* [50] about the fatigue life fracture of Kevlar 373/epoxy that are subject to constant pressure at the 90% stress level until all had failed, that is, we have a complete data with the exact times of failure. To illustrate the proposed methodology, the data are randomly grouped into three sets of observations with 26 observations belonging to subpopulation-I, 25 observations belonging to subpopulation-II and remaining 25 observations belonging to subpopulation-III. To implement doubly censored samplings, we consider $y_{1r_1}, \ldots, y_{1w_1}, y_{2r_2}, \ldots, y_{2w_2}$ and $y_{3r_3}, \ldots, y_{3w_3}$ failed observations belong to subpopulation-I, subpopulation-II and subpopulation-III, respectively. The rest of the observations which are less than 0.05 and greater than 0.34 assumed to be censored observations from each component, such that $y_r = \min\left(y_{1r_1}, y_{2r_2}, y_{3r_3}\right) = 0.05$ and $y_w = \max\left(y_{1w_1}, y_{2w_2}, y_{3w_3}\right) = 0.34$. Notice that the numbers of failed observations, $s_1 = w_1 - r_1 + 1 = 19$, $s_2 = w_2 - r_2 + 1 = 20$ and $s_3 = w_3 - r_3 + 1 = 19$, can be observed from subpopulation-I, subpopulation-II and subpopulation-III, respectively. The remaining $n-(w-r+3) = 18$ observations are assumed to be censored observations and $w-r+3 = 58$ are the uncensored observations, such that $r = r_1 + r_2 + r_3$, $w = w_1 + w_2 + w_3$ and $s = s_1 + s_2 + s_3$. The data are summarized as below:

$$n_1 = 26,\ r_1 = 4,\ w_1 = 22,\ n_2 = 25,\ r_2 = 3,\ w_2 = 22,\ n_3 = 25,\ r_3 = 3,\ w_3 = 21,\ n = 76,\ r = 10,\ w = 65,\ s = 58,\ \sum_{i=r_1}^{w_1} y_{1i} = 3.05256,$$
$$\sum_{i=r_2}^{w_2} y_{2i} = 3.19514,\ \sum_{i=r_3}^{w_3} y_{3i} = 2.97166.$$

Since $n - (w - r + 3) = 18$, we have almost 23.68% doubly censored data. Bayes estimates and posterior risks using the NIP (UP and JP) and the IP under SELF, PLF and DLF are showcased in Table 5.

It is observed that results obtained through real data are compatible with the simulation results, as discussed in the previous section. Table 5 also reveals that the performance of the IP is the best as compared to the NIP (UP and JP), that is, in terms of the minimum posterior risks. Moreover, it is noticed that results are relatively more precise with the UP (JP) than the JP (UP) under DLF (SELF and PLF). In addition, it can be seen that SELF (DLF) performs better than PLF and DLF (PLF and SELF) for estimating proportion (component) parameters.

Table 5 Bayes estimates (BE) and posterior risks (PR) of 3-CMED using the UP, the JP and the IP under SELF, PLF and DLF with a real-life mixture data.

Loss Function	Prior		$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_1$
SELF	UP	BE	5.565440	5.131310	5.357700	0.324222	0.347934
		PR	2.304870	1.980750	2.285350	0.003952	0.004180
	JP	BE	5.313890	4.913600	5.101590	0.324376	0.347532
		PR	2.136780	1.840220	2.106690	0.003924	0.004138
	IP	BE	3.163980	3.375500	3.465220	0.344839	0.320345
		PR	0.450884	0.527196	0.601823	0.003577	0.003425
PLF	UP	BE	5.768790	5.320820	5.566900	0.330260	0.353890
		PR	0.406710	0.379013	0.418386	0.012076	0.011912
	JP	BE	5.511280	5.097420	5.304050	0.330369	0.353435
		PR	0.394780	0.367639	0.404913	0.011986	0.011807
	IP	BE	3.234450	3.452710	3.551000	0.349986	0.325647
		PR	0.140936	0.154417	0.171552	0.010295	0.010605

(*continued*)

Table 5 Bayes estimates (BE) and posterior risks (PR) of 3-CMED using the UP, the JP and the IP under SELF, PLF and DLF with a real-life mixture data. *(Continued)*

Loss Function	Prior		$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_1$
DLF	UP	BE	5.979580	5.517320	5.784260	0.336411	0.359948
		PR	0.069259	0.069964	0.073744	0.036231	0.033378
	JP	BE	5.716000	5.288110	5.514540	0.336472	0.359438
		PR	0.070348	0.070822	0.074883	0.035950	0.033126
	IP	BE	3.306490	3.531680	3.638900	0.355210	0.331037
		PR	0.043099	0.044223	0.047727	0.029200	0.032300

SELF, squared error loss function; PLF, precautionary loss function; DLF, DeGroot loss function; CMED, component mixture of exponential distribution; UP, uniform prior; JP, Jeffreys' prior.

Table 6 Bayesian predictive interval (*L, U*) of 3-CMED using the UP, the JP and the IP with real-life mixture data.

UP		JP		IP	
L	***U***	***L***	***U***	***L***	***U***
0.009775	1.102830	0.010232	1.183630	0.015505	1.026150

CMED, component mixture of exponential distribution; UP, uniform prior; JP, Jeffreys' prior.

The 90% Bayesian predictive intervals (*L, U*) using the NIP (UP and JP) and the IP are presented in Table 6. It can be seen that the 90% Bayesian predictive intervals using the IP are narrower than the Bayesian predictive intervals using the NIP (UP and JP).

11. CONCLUSION

In this article, a 3-CMED under doubly censoring sampling scheme is considered for modeling lifetime data. Assuming different NIP and IP, expressions of the Bayes estimators and their posterior risks under different loss functions are derived. To judge the relative performance of the Bayes estimators and also to deal with the problems of selecting a suitable priors and loss functions by assuming different sample sizes and various left and right test termination times, a comprehensive simulation and real-life study have been conducted in this article. The simulation study revealed some important and interesting properties of the Bayes estimators. From numerical results given in Tables 2–4, we observed that an increase in sample size or decrease in left and increase in right test termination times provides improved Bayes estimates. We also observed that the effect of left and right test termination times, sample size and different parameter values on the Bayes estimates is in the form of over-estimation and/or under-estimation. To be more specific, a larger (smaller) sample size results in a smaller (larger) degree of under-estimation and/or over-estimation of parameters at the fixed left and right test termination times. However, the extent of over-estimation and/or under-estimation of parameters is quite smaller (larger) with a relatively smaller left and larger right (larger left and smaller right) test termination times for a fixed sample size. It is also observed that as the sample size (left and right test termination time) increases (increases and decreases) the posterior risks of Bayes estimates of parameters decrease (increase) for a fixed left and right test termination times (sample size). Finally, we conclude that for a Bayesian analysis of mixture data under doubly censoring sampling scheme, the IP performance is good for the DLF (SELF) for estimating component (proportion) parameters. However, if only the NIPs are considered, the JP (UP) are suitable with SELF (DLF) for estimating proportion (component) parameters. Moreover, the results of real data coincide with the simulated results that confirm the correctness of our simulation scheme.

In future, this work can be extended by comparing the Bayesian estimates with the maximum likelihood estimates by assuming record values and other different types of censoring schemes. Moreover, the performance of the Bayes estimators under other different loss functions can also be assessed.

CONFLICT OF INTEREST

The authors have declared that there is no conflict of interests exist.

AUTHORS' CONTRIBUTIONS

Conceptualization: Muhammad Tahir, Muhammad Aslam; Formal analysis: Muhammad Abid, Sajid Ali; Methodology: Muhammad Tahir, Mohammad Ahsanullah; Software: Muhammad Abid, Sajid Ali; Supervision: Muhammad Aslam; Writing –original draft: Muhammad Tahir, Sajid Ali, Muhammad Abid and Writing –review & editing: Muhammad Aslam, Mohammad Ahsanullah.

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