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Research Article

New Kind of *MV*-Modules

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ABSTRACT

In this paper, by considering the notion of *MV*-modules, which is the structure that naturally correspond to *lu*-modules over *lu*-rings, we investigate some properties of a new kind of *MV*-modules, that we introduced in Borzooei and Saidi Goraghani, Free MV-modules, J. Intell. Fuzzy Syst. 31 (2016), 151–161 as A_k-modules. With the current situation, it was not easy for us to work on some concepts such as free *MV*-modules and Noetherian *MV*-modules. So we limited our scope of work by introducing a new kind of *MV*-modules. We define and study the notions of free A_k-modules, radical of A_k-modules and Noetherian A_k-modules, where A is a product *MV*-algebra and $k \in \mathbb{N}$. For example, we state a general representation for a free A_k-module, and we obtain conditions in which an A_k-module can be Noetherian.

1. INTRODUCTION

MV-algebras were defined by Chang [1] as algebras corresponding to the Lukasiewicz infinite valued propositional calculus. These algebras have appeared in the literature under different names and polynomially equivalent presentation: *CN*-algebras, Wajsberg algebras, bounded commutative *BCK*-algebras and bricks. It is discovered that *MV*-algebras are naturally related to the Murray-von Neumann order of projections in operator algebras on Hilbert spaces and that they play an interesting role as invariant of approximately finite-dimensional C^*-algebras. They are also naturally related to Ulam's searching games with lies. *MV*-algebras admit a natural lattice reduction and hence a natural order structure. Many important properties can be derived from the fact, established by Chang that nontrivial *MV*-algebras are subdirect products of *MV*-chains, that is, totally ordered *MV*-algebras. To prove this fundamental result, Chang introduced the notion of prime ideal in an *MV*-algebra. The categorical equivalence between *MV*-algebras and *lu*-groups leads to the problem of defining a product operation on *MV*-algebras, in order to obtain structures corresponding to *l*-rings. A *product MV-algebra* (or *PMV*-algebra, for short) is an *MV*-algebra which has an associative binary operation " . ". It satisfies an extra property which will be explained in Preliminaries. During the last years, *PMV*-algebras were considered and their equivalence with a certain class of *l*-rings with strong unit was proved. It seems quite natural to introduce modules over such algebras, generalizing the divisible *MV*-algebras and the *MV*-algebras obtained from Riesz spaces and to prove natural equivalence theorems. Hence, the notion of *MV*-modules was introduced as an action of a *PMV*-algebra over an *MV*-algebra by Di Nola [2]. Since then, we and some researches have worked on *MV*-modules (see, for instance, [3–12]). Most published papers are about ideals of *MV*-modules. We were surious to investigate concepts such as injective (projective) *MV*-modules, free *MV*-modules and so on. In 2018, we introduced injective *MV*-modules (see [8]), but with the current situation, it was not possible for us to achieve all our goals. So we first limited our scope of work by introducing a new kind of *MV*-modules, namely A_k-modules. Since *MV*-modules are in their infancy, stating and opening of any subject in this field can be useful. Hence, in this paper, we present definitions of free A_k-modules, radical of A_k-modules and Noetherian A_k-modules, where A is a *PMV*-algebra and $k \in \mathbb{N}$. In Section 3, we state when an A_k-module is equal to $\sum_{j=1}^{n} A$, for some $n \in \mathbb{N}$. In Section 4, we state a method to obtain the radical of an A-ideal in A_k-modules. In Section 5, we obtain conditions in which an A_k-module can be Noetherian. We hope that we have taken an effective step in this regards.

2. PRELIMINARIES

In this section, we review related lemmas and theorems that we use in the next sections.

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Definition 2.1. [13] An MV-*algebra* is a structure $M = (M, \oplus, ', 0)$ of type $(2, 1, 0)$ such that

$(MV1)$ $(M, \oplus, 0)$ is an Abelian monoid,

$(MV2)$ $(a')' = a$,

$(MV3)$ $0' \oplus a = 0'$,

$(MV4)$ $(a' \oplus b)' \oplus b = (b' \oplus a)' \oplus a$,

If we define the constant $1 = 0'$ and operations $\odot$ and $\ominus$ by $a \odot b = (a' \oplus b')'$, $a \ominus b = a \odot b'$, then

$(MV5)$ $(a \oplus b) = (a' \odot b')'$,

$(MV6)$ $x \oplus 1 = 1$,

$(MV7)$ $(a \ominus b) \oplus b = (b \ominus a) \oplus a$,

$(MV8)$ $a \oplus a' = 1$,

for every $a, b \in M$. It is clear that $(M, \odot, 1)$ is an Abelian monoid. Now, if we define auxiliary operations $\vee$ and $\wedge$ on M by $a \vee b = (a \odot b') \oplus b$ and $a \wedge b = a \odot (a' \oplus b)$, for every $a, b \in M$, then $(M, \vee, \wedge, 0)$ is a *bounded distributive lattice*.

An MV-algebra M is a *Boolean* algebra if and only if the operation "$\oplus$" is idempotent, that is, $x \oplus x = x$, for every $x \in M$. In an MV-algebra M, the following conditions are equivalent: (i) $a' \oplus b = 1$, (ii) $a \odot b' = 0$, (iii) $b = a \oplus (b \ominus a)$, (iv) $\exists c \in M$ such that $a \oplus c = b$, for every $a, b, c \in M$. For any two elements a, b of the MV-algebra M, $a \leq b$ if and only if a, b satisfy the above equivalent conditions $(i) - (iv)$. An ideal of MV-algebra M is a subset I of M, satisfying the following conditions: $(I1)$: $0 \in I$, $(I2)$: $x \leq y$ and $y \in I$ imply $x \in I$, $(I3)$: $x \oplus y \in I$, for every $x, y \in I$. In an MV-algebra M, the *distance function* $d : M \times M \to M$ is defined by $d(x, y) = (x \ominus y) \oplus (y \ominus x)$ which satisfies (i): $d(x, y) = 0$ if and only if $x = y$, (ii): $d(x, y) = d(y, x)$, (iii): $d(x, z) \leq d(x, y) \oplus d(y, z)$, (iv): $d(x, y) = d(x', y')$, (v): $d(x \oplus z, y \oplus t) \leq d(x, y) \oplus d(z, t)$, for every $x, y, z, t \in M$. Let I be an ideal of MV-algebra M. We denote $x \sim y$ $(x \equiv_I y)$ if and only if $d(x, y) \in I$, for every $x, y \in M$. So $\sim$ is a congruence relation on M. Denote the equivalence class containing x by $\frac{x}{I}$ and $\frac{M}{I} = \left\{\frac{x}{I} : x \in M\right\}$. Then $\left(\frac{M}{I}, \oplus, ', \frac{0}{I}\right)$ is an MV-algebra, where $\left(\frac{x}{I}\right)' = \frac{x'}{I}$ and $\frac{x}{I} \oplus \frac{y}{I} = \frac{x \oplus y}{I}$, for all $x, y \in M$. Let M and K be two MV-algebras. A mapping $f : M \to K$ is called an MV-*homomorphism* if $(H1)$: $f(0) = 0$, $(H2)$: $f(x \oplus y) = f(x) \oplus f(y)$ and $(H3)$: $f(x') = (f(x))'$, for every $x, y \in M$. If f is one to one (onto), then f is called an MV-monomorphism (MV-epimorphism) and if f is onto and one to one, then f is called an MV-isomorphism.

Lemma 2.2. [13] In every MV-algebra M, the natural order "$\leq$" has the following properties:

i. $x \leq y$ if and only if $y' \leq x'$,

ii. if $x \leq y$, then $x \oplus z \leq y \oplus z$, for every $z \in M$.

Lemma 2.3. [13] Let M and N be two MV-algebras and $f : M \to N$ be an MV-*homomorphism*. Then the following properties hold:

i. $Ker(f)$ is an ideal of M,

ii. if f is an MV-epimorphism, then $\frac{M}{Kerf} \cong N$.

Definition 2.4. [14,15] An l-group is an algebra $(G, +, -, 0, \vee, \wedge)$, where the following properties hold:

(a) $(G, +, -, 0)$ is a group,

(b) $(G, \vee, \wedge)$ is a lattice,

(c) $x \leq y$ implies that $x + a \leq y + a$, for any $x, y, a, b \in G$.

A strong unit $u > 0$ is a positive element with property that for any $g \in G$ there exits $n \in \omega$ such that $g \leq nu$. The Abelian l-groups with strong unit will be simply called lu-groups [16].

The category whose objects are MV-algebras and whose homomorphisms are MV-homomorphisms is denoted by MV. The category whose objects are pairs (G, u), where G is an Abelian l-group and u is a strong unit of G and whose homomorphisms are l-group homomorphisms is denoted by Ug. The functor that establishes the categorial equivalence between MV and Ug is

$$\Gamma : Ug \to MV,$$

where $\Gamma(G, u) = [0, u]_G$, for every lu-group (G, u) and $\Gamma(h) = h|_{[0,u]}$, for every lu-group homomorphism h. The above results allows us to consider an MV-algebra, when necessary, as an interval in the positive cone of an l-group. Thus, many definitions and properties can be transferred from l-groups to MV-algebras. For example, the group addition becomes a partial operation when it is restricted to an interval, so we define a *partial addition* on an MV-algebra M as follows:

$x + y$ is defined if and only if $x \leq y'$ and in this case, $x + y = x \oplus y$, for every $x, y \in M$. Moreover, if $z + x \leq z + y$, then $x \leq y$ (see [2]).

Definition 2.5. [16] An l-ring is a structure $(R, +, ., 0, \leq)$, where $(R, +, 0, \leq)$ is an L-group such that, for any $x, y \in R$,

$$x \geq 0 \text{ and } y \geq 0 \text{ implies } x.y \geq 0.$$

A *product* MV-*algebra* (or PMV-algebra, for short) is a structure $A = (A, \oplus, ., ', 0)$, where $(A, \oplus, ', 0)$ is an MV-algebra and " . " is a binary associative operation on A such that the following property is satisfied: if $x+y$ is defined, then $x.z+y.z$ and $z.x+z.y$ are defined and $(x+y).z = x.z+y.z$, $z.(x+y) = z.x+z.y$, for every $x, y, z \in A$, where " + " is the partial addition on A. A unity for the product is an element $e \in A$ such that $e.x = x.e = x$, for every $x \in A$. If A has a unity for product, then $e = 1$. A PMV-algebra A is called *commutative*, if $x.y = y.x$, for every $x, y \in A$ [4]. A PMV-homomorphism is an MV-homomorphism which also commutes with the product operation. A $\cdot$-ideal of A is an ideal I of A such that if $a \in I$ and $b \in A$ entail $a.b \in I$ and $b.a \in I$. The set of $\cdot$-ideals of A is denoted by $Id(A)$.

An lu-ring is a pair (R, u), where $(R, +, ., 0, \leq)$ is an l-ring and u is a strong unit of R (i.e., u is a strong unit of the underlying l-group) such that $u.u \leq u$. The category whose objects are pairs (R, u), where R is an l-ring and u is a strong unit of R and whose homomorphisms are l-ring homomorphisms is denoted by UR. The functor that establishes the categorial equivalence between PMV and UR is

$$\Gamma : UR \to PMV,$$

where $\Gamma(R, u) = [0, u]_R$, for every lu-ring (R, u) and $\Gamma(h) = h|_{[0,u]}$, for every lu-ring homomorphism h.

Definition 2.6. [6] Let I be a proper $\cdot$-ideal of A. I is called a $\cdot$-prime ideal of A, if $x.y \in I$ implies that $x \in I$ or $y \in I$, for any $x, y \in A$.

Proposition 2.7. *[4] Let A be a PMV-algebra. Then $\sum_{i=1}^{n} A$ is a PMV-algebra.*

Lemma 2.8. [16] Let A be a PMV-algebra. Then $a \leq b$ implies that $a.c \leq b.c$ and $c.a \leq c.b$ for every $a, b, c \in A$.

Definition 2.9. [2] Let $A = (A, \oplus, ., ', 0)$ be a PMV-algebra, $M = (M, \oplus, ', 0)$ be an MV-algebra and the operation $\Phi : A \times M \to M$ be defined by $\Phi(a, x) = ax$, which satisfies the following axioms:

$(AM1)$ If $x + y$ is defined in M, then $ax + ay$ is defined in M and $a(x + y) = ax + ay$,

$(AM2)$ If $a + b$ is defined in A, then $ax + bx$ is defined in M and $(a + b)x = ax + bx$,

$(AM3)$ $(a.b)x = a(bx)$, for every $a, b \in A$ and $x, y \in M$.

Then M is called a (left) *MV-module* over A or briefly an A-module. We say that M is a *unitary* MV-module if A has a unity for the product and

$(AM4)$ $1_A x = x$, for every $x \in M$. Moreover, M is called a *Boolean* A-*module* if $ax \oplus ay \leq a(x \oplus y)$, for every $a \in A$ and $x, y \in M$ (see [4]).

Lemma 2.10. [2] Let A be a PMV-algebra and M be an A-module. Then

(a) $0x = 0$,

(b) $a0 = 0$,

(c) $ax' \leq (ax)'$,

(d) $a'x \leq (ax)'$,

(e) $(ax)' = a'x + (1x)'$,

(f) $x \leq y$ implies $ax \leq ay$,

(g) $a \leq b$ implies $ax \leq bx$,

(h) $a(x \oplus y) \leq ax \oplus ay$,

(i) $d(ax, ay) \leq ad(x, y)$,

(j) if $x \equiv_I y$, then $ax \equiv_I ay$, where I is an ideal of A,

(k) if M is a unitary MV-module, then $(ax)' = a'x + x'$, for every $a, b \in A$ and $x, y \in M$.

Definition 2.11. [2] Let A be a PMV-algebra and M be an A-module. Then an ideal $N \subseteq M$ is called an A-*ideal* of M if $(I4)$: $ax \in N$, for every $a \in A$ and $x \in N$. Let N be a proper A-ideal of M. Then N is called a *prime* A-ideal of M, if $am \in N$ implies that $m \in N$ or $a \in (N : M)$, for any $a \in A$ and $m \in M$, where $(N : M) = \{a \in A : aM \subseteq N\}$ (see [6]).

Lemma 2.12. [4] Let M be an A-module and N be an A-ideal of M. Then

$$(N : M) = \{a \in A : aM \subseteq N\}$$

is an ideal of A.

Definition 2.13. [2] Let A be a PMV-algebra and M_1, M_2 be two A-modules. A map $f : M_1 \to M_2$ is called an A-*module* homomorphism or (A-*homomorphism*, for short) if f is an MV-homomorphism and

$(H4)$: $f(ax) = af(x)$, for every $x \in M_1$ and $a \in A$.

Definition 2.14. [4] Let M_1 and M_2 be two A-modules. Then the map $f : M_1 \to M_2$ is called an A'-*homomorphism* if and only if it satisfies in $(H1)$, $(H3)$, $(H4)$ and

$(H'2)$: if $x + y$ is defined in M_1, then $h(x + y) = h(x \oplus y) = h(x) \oplus h(y)$, for every $x, y \in M_1$, where " + " is the partial addition on M_1. If h is one to one (onto), then h is called an A'-monomorphism (epimorphism). If h is onto and one to one, then h is called an A'-isomorphism and we write $M_1 \cong {}'M_2$.

Definition 2.15. [4] Let M be a unitary A-module and there exists $k \in \mathbb{N}$ such that $\sum_{i=1}^{n} a_i' m_i \leq \left(\sum_{i=1}^{n} a_i m_i\right)'$, for every $1 \leq n \leq k$, $a_i \in A$ and $m_i \in M$. Then M is called an A_k-*module*. If $\sum_{i=1}^{n} a_i' m_i \leq \left(\sum_{i=1}^{n} a_i m_i\right)'$, for every $n \in \mathbb{N}$, then M is called an $A_{\mathbb{N}}$-module.

Proposition 2.16. *[4] M is an A_k-module if and only if $\left(\sum_{i=1}^{n} a_i m_i\right)' = \left(\sum_{i=1}^{n} m_i\right)' \oplus \sum_{i=1}^{n} a_i' m_i$, for every $a_i \in A$, $m_i \in M$ and $n \leq k$.*

Definition 2.17. [4] Let M be an A-module, $\varnothing \neq T \subseteq M$ and

$$M = \left\{ \sum_{i=1}^{n} x_i t_i : x_i \in A, t_i \in T, \text{for} 1 \leq i \leq n \,\text{and}\, x_1 t_1 + \cdots \right.$$
$$\left. + x_n t_n \text{ is defined in } M \right\},$$

where " $+$" is the partial addition (similarly, if T is infinite set, then M can be defined). Then we say that M is *generated* by T and we set $M =\prec T \succ$. If $|T| < \infty$, then M is called a *finitely generated* A-module. Specially, if $M =\prec m \succ$, where $m \in M$, then M is called a *cyclic* A-module.

Definition 2.18. [7] Let M be an A-module. An A-ideal N of M is called a maximal A-ideal of M, if there exist no A-ideal K of M containing N such that $N \subsetneq K \subsetneq M$. The set of all maximal A-ideals of M is showed by $Max(M)$. Let I be a proper A-ideal in M. The intersection of all maximal A-ideals of M which contain I is called the radical of I and is denoted by $Rad(I)$. Moreover, the intersection all maximal A-ideals of M is showed by $Rad(M)$.

Definition 2.19. [17] Let M be an A-module. Then

i. M is called *Noetherian* if M satisfies the ascending chain condition (ACC): any chain $N_1 \subseteq N_2 \subseteq \cdots$ of A-ideals of M is stationary.

ii. We say M has *maximum condition*, if every non-empty family of A-ideals of M has a maximal element.

Note. From now on, in this paper, we let A be a PMV-algebra, M be an MV-algebra and $\sum_{i=1}^{n} x_i$ means $x_1 \oplus x_2 \oplus \cdots \oplus x_n$.

3. FREE A_k-MODULES

In this section, we present definition of free A_k-modules and we obtain some results on them. For example, we state a general representation for a free A_k-module.

Definition 3.1. Let M be an A_k-*module*. If M is a free A-module, then M is called a free A_k-module.

Example 3.2.

i. By Lemma 2.10(d), every free A-module M is a free A_1-module.

ii. If A is the Boolean algebra with two elements, then every MV-algebra M is an A_N-module. Now, if M is a free A-module, then M is a free A_N-module.

iii. Consider $L_2 = \{0, 1\}$, $L_4 = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$, $a \oplus b = min\{1, a + b\}$, $a' = 1 - a$ and $+, -, .$ are ordinary operations in $\mathbb{R}$. Then it is routine to show that $(L_2, \oplus, ', ., 0)$ is a PMV-algebra and $(L_4, \oplus, ', 0)$ is an MV-algebra. Let operation $\cdot : L_2 \times L_4 \to L_4$ is defined by $a \cdot b = a.b$, for every $a \in L_2$ and $b \in L_4$. Then it is easy to show that L_4 is an L_2-module. Now, since

$$\frac{2}{3} = 1.\frac{2}{3} \ , \ \frac{1}{3} = 1.\frac{1}{3} \ , \ 1 = 1.\frac{1}{3} + 1.\frac{2}{3} = 1.\frac{1}{3} \oplus 1.\frac{2}{3}$$

we have $L_4 = \prec \left\{\frac{1}{3}, \frac{2}{3}\right\} \succ$. It is easy to see that L_4 is a free A_N-module, where $A = L_2$.

Lemma 3.3. Let M be an $A_{\mathbb{N}}$-module, I be an ideal of A and N be an A-ideal of M. Then

$$IM \oplus N = \left\{\sum_{i=1}^{k} r_i m_i \oplus n \ : \ r_i \in I, \ m_i \in M, n \in N\right\}$$

is an A-ideal of M.

Proof. (I_1) and (I_3) are clear.

(I_2): Let $m \le b$, where $m \in M$ and $b \in IM \oplus N$. Since $b \in IM \oplus N$, there exist $n \in N$, $r_1, \cdots, r_k \in I$ and $m_1, \cdots, m_k \in M$ such that $b = \sum_{i=1}^{k} r_i m_i \oplus n$ and so $m' \oplus \sum_{i=1}^{k} r_i m_i \oplus n = 1$. By Lemma 2.10(d), $r_i' m_i + r_i m_i$ is defined and it is clear that $r_i + r_i'$ is defined, too, for every $r_i \in A$, $m_i \in M$ and $1 \le i \le k$. Hence,

$$\begin{aligned}
m' \oplus \sum_{i=1}^{k} m_i \oplus n &= m' \oplus \sum_{i=1}^{k} (r_i \oplus r'_i)\, m_i \oplus n \\
&= m' \oplus \sum_{i=1}^{k} (r_i + r'_i)\, m_i \oplus n \\
&= m' \oplus \sum_{i=1}^{k} (r_i m_i + r'_i m_i) \oplus n \\
&= m' \oplus \sum_{i=1}^{k} (r_i m_i \oplus r'_i m_i) \oplus n \\
&= m' \oplus \sum_{i=1}^{k} r_i m_i \oplus \sum_{i=1}^{k} r'_i m_i \oplus n = 1 \oplus \sum_{i=1}^{k} r'_i m_i \\
&= 1
\end{aligned}$$

and so by Proposition 2.16,

$$\begin{aligned}
m &= m \wedge b = \\
m \odot (m' \oplus b) &= \left(m' \oplus \left(m' \oplus \sum_{i=1}^{k} r_i m_i \oplus n\right)'\right)' \\
&= \left(m' \oplus \left(1m' \oplus \sum_{i=1}^{k} r_i m_i \oplus 1n\right)'\right)' \\
&= \left(m' \oplus \left(\sum_{i=1}^{k} r_i' m_i \oplus \left(m' \oplus n \oplus \sum_{i=1}^{k} m_i\right)'\right)\right)' \\
&= \left(m' \oplus \sum_{i=1}^{k} r_i' m_i \oplus 1'\right)' \\
&= \left(m' \oplus \sum_{i=1}^{k} r_i' m_i\right)' \\
&= \left(1m' \oplus \sum_{i=1}^{k} r_i' m_i \oplus 0n\right)' \\
&= \sum_{i=1}^{k} r_i m_i \oplus 1n \oplus \left(m' \oplus \sum_{i=1}^{k} m_i \oplus n\right)' \\
&= \sum_{i=1}^{k} r_i m_i \oplus n \\
&\in IM \oplus N.
\end{aligned}$$

(I_4): Let $a \in A$ and $\sum_{i=1}^{k} r_i m_i \oplus n \in IM \oplus N$. Since by Lemma 2.10(h),

$$a\left(\sum_{i=1}^{k} r_i m_i \oplus n\right) \le \sum_{i=1}^{k} a(r_i m_i) \oplus an = \sum_{i=1}^{k} (a.r_i) m_i \oplus an$$

and since $a.r_i \in I$, for any $1 \le i \le k$, we have $\sum_{i=1}^{k} (a.r_i) m_i \oplus an \in IM \oplus N$ and so by (I_2), $a(\sum_{i=1}^{k} r_i m_i \oplus n) \in IM \oplus N$. Therefore, $IM \oplus N$ is an A-ideal of M.

Theorem 3.4. *Let A be commutative and M be a cyclic $A_{\mathbb{N}}$-module. Then for every A-ideal N of M, there exists an ideal I of A such that $N = IM$.*

Proof. Since M is a cyclic $A_{\mathbb{N}}$-module, there exists $m \in M$ such that $M = \prec m \succ$. By Lemma 2.12, $(N : M)$ is an ideal of A and by Lemma 3.3, $(N : M)M$ is an A-ideal of M. Now, we show that $N = (N : M)M$. It is clear that $(N : M)M \subseteq N$. Now, let $n \in N$. Then there exists $a \in A$ such that $n = am$. Since

$$\begin{aligned}
aM &= a \prec m \succ = \{a(a_i m) : a_i \in A\} = \{(a.a_i)m : a_i \in A\} \\
&= \{(a_i.a)m) : a_i \in A\} = \{a_i(am) : a_i \in A\} \\
&= \{a_i n : a_i \in A\} \\
&\subseteq N,
\end{aligned}$$

we have $a \in (N : M)$ and so $n \in (N : M)M$. Hence, $N \subseteq (N : M)M$ and so $N = (N : M)M$. Now, by considering $I = (N : M)$, we get that $N = IM$.

Theorem 3.5. *M is a free A_k-module with basis $T = \{t_1, \cdots, t_n\}$ that $\sum_{j=1}^{n} t_j = 1$ and $n \leq k$ if and only if $M \cong ' \sum_{j=1}^{n} A$.*

Proof. $(\Rightarrow)$ Let M be a free A_k-module with a basis T and $\sum_{j=1}^{n} t_j = 1$. By Proposition 2.7, $\sum_{j=1}^{n} A$ is a PMV-algebra. Now, let the operation $\cdot : A \times \sum_{j=1}^{n} A \to \sum_{j=1}^{n} A$ be defined by $a \cdot \{a_j\}_{j=1}^{n} = \{a.a_j\}_{j=1}^{n} = \{aa_j\}_{j=1}^{n}$, for every $\{a_j\}_{j=1}^{n} \in \sum_{j=1}^{n} A$ and $a \in A$. It is easy to show that $\sum_{j=1}^{n} A$ is an A-module. Let $\Phi : \sum_{j=1}^{n} A \to M$ be defined by $\Phi(\{a_j\}_{j=1}^{n}) = \sum_{j=1}^{n} a_j t_j$, for every $a_j \in A$ and $t_j \in T$. Then it is clear that ϕ is well defined and $\phi(0) = 0$. If $\{a_j\}_{j=1}^{n} + \{b_j\}_{j=1}^{n}$ is defined in $\sum_{j=1}^{n} A$, then $a_j + b_j$ is defined in A and so $a_j t_j + b_j t_j$ is defined in M, for every $1 \leq j \leq n$ and $t_j \in T$. Hence,

$$\begin{aligned}
\phi\left(\left\{a_j\right\}_{j=1}^{n} + \left\{b_j\right\}_{j=1}^{n}\right) &= \phi\left(\left\{a_j + b_j\right\}_{j=1}^{n}\right) = \sum_{j=1}^{n}(a_j + b_j)t_j \\
&= \sum_{j=1}^{n}(a_j t_j + b_j t_j) \\
&= \sum_{j=1}^{n}(a_j t_j \oplus b_j t_j) = \sum_{j=1}^{n} a_j t_j \oplus \sum_{j=1}^{n} b_j t_j \\
&= \phi\left(\left\{a_j\right\}_{j=1}^{n}\right) \oplus \phi\left(\left\{b_j\right\}_{j=1}^{n}\right).
\end{aligned}$$

Also,

$$\begin{aligned}
\phi\left(\alpha\left\{a_j\right\}_{j=1}^{n}\right) &= \phi\left\{\alpha a_j\right\}_{j=1}^{n}\Big) = \sum_{j=1}^{n}\left(\alpha.a_j\right) t_j \\
&= \sum_{j=1}^{n} \alpha(a_j t_j) = \alpha \sum_{j=1}^{n} a_j t_j = \alpha\phi\left(\left\{a_j\right\}_{j=1}^{n}\right).
\end{aligned}$$

Finally, for every $\{a_j\}_{j=1}^{n} \in \sum_{j=1}^{n} A$,

$$\begin{aligned}
\phi\left(\left(\left\{a_j\right\}_{j=1}^{n}\right)'\right) &= \phi\left(\left\{a_j'\right\}_{j=1}^{n}\right) = \sum_{j=1}^{n} a_j' t_j = \sum_{j=1}^{n} a_j' t_j \oplus \left(\sum_{j=1}^{n} t_j\right)' \\
&= \left(\sum_{j=1}^{n} a_j t_j\right)' = \left(\phi\left(\left\{a_j\right\}_{j=1}^{n}\right)\right)'.
\end{aligned}$$

Hence, ϕ is an A'-homomorphism. Let $\phi(\{a_j\}_{j=1}^{n}) = 0$, for any $\{a_j\}_{j=1}^{n} \in \sum_{j=1}^{n} A$. Since T is a linearly independent set, it is easy to show that $a_j = 0$, for every $1 \leq j \leq n$ and so $Ker(\phi) = 0$. It results that ϕ is an A'-monomorphism. It is clear that ϕ is an A'-epimorphism and so $M \cong ' \sum_{j=1}^{n} A$.

$(\Leftarrow)$ Let $M \cong ' \sum_{j=1}^{n} A$. We construct a basis for $\sum_{j=1}^{n} A$. Let $\theta_j = \{u_i\}_{i=1}^{n}$ such that

$$u_i = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

We show that $K = \{\theta_j : 1 \leq j \leq n\}$ is a basis for $\sum_{j=1}^{n} A$. Let $\{a_j\}_{j=1}^{n} \in \sum_{j=1}^{n} A$. Since $1.a = a.1 = a$, for every $a \in A$,

$$\begin{aligned}
\left\{a_j\right\}_{j=1}^{n} &= \{a_1, 0, \cdots, 0\} \oplus \{0, a_2, 0, \cdots, 0\} \oplus \cdots \oplus \{0, \cdots, 0, a_n\} \\
&= \{a_1.1, 0, \cdots, 0\} \oplus \{0, a_2.1, 0, \cdots, 0\} \oplus \cdots \oplus \{0, \cdots, 0, a_n.1\} \\
&= a_1.\{1, 0, \cdots, 0\} \oplus a_2.\{0, 1, \cdots, 0\} \oplus \cdots \oplus a_n.\{0, \cdots, 0, 1\} \\
&= a_1.\theta_1 \oplus a_2.\theta_2 \oplus \cdots \oplus a_n.\theta_n.(\text{it is easy to see that } a_1\theta_1 \\
&\quad + \cdots + a_n\,\theta_n \text{ is defined.})
\end{aligned}$$

Hence, $\sum_{j=1}^{n} A = \prec K \succ$. Now, let $\sum_{j=1}^{n} a_j.\theta_j = 0$. Then it is easy to show that $a_j = 0$, for every $1 \leq j \leq n$. Now, if $\phi : \sum_{j=1}^{n} A \to M$ is an A'-isomorphism, then $\{\phi(\theta_j) : 1 \leq j \leq n\}$ is a basis for M. Moreover, it is clear that $\sum_{j=1}^{n} \theta_j = 1$.

4. RADICAL OF A_k-MODULES

In [7], the radical of an A-ideal in MV-modules had been defined by maximal ideals. In [10], we presented definition of radical of an A-ideal in MV-modules by prime A-ideals. As fundamental result of this section, in an A_N-module, we state a method to obtain the radical of an A-ideal.

Definition 4.1. [10] Let M be an A-module and N be an A-ideal of M. The intersection of all prime A-ideals of M, including N, is called *radical* of N and it is shown by $rad_M(N)$ or $rad(N)$. If N is a prime A-ideal of M, then it is clear that $rad(N) = N$. If there exists no prime A-ideal of M including N, then we let $rad_M(N) = M$.

Example 4.2. Let $A = \{0, 1, 2, 3\}$ and the operations " $\oplus$" and " . " on A be defined as follows:

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
3	3	3	3	3

.	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

Consider $0' = 3$, $1' = 2$, $2' = 1$ and $3' = 0$. Then it is easy to show that $(A, \oplus, ', ., 0)$ is a PMV-algebra and $(A, \oplus, ', 0)$ is an MV-algebra. Now, let the operation $\cdot : A \times A \to A$ be defined by $a \cdot b = a.b$, for every $a, b \in A$. It is easy to show that A is an MV-module on A, $I = \{0, 1\}$, $J = \{0, 2\}$ are prime A-ideals of A and $\{0\}$ is not a prime A-ideal of A. Also, $rad(I) = \{0, 1\}$ and $rad(\{0\}) = \{0\}$.

Theorem 4.3. *Let M be a free $A_{\mathbb{N}}$-module and P be a $\cdot$-prime ideal of A. Then $(PM : M) = P$ and PM is a prime A-ideal of M.*

Proof. It is clear that $P \subseteq (PM : M)$. Let $X = \{x_1, \cdots, x_n\}$ be a basis of M, $a \in (PM : M)$ and $a \notin P$. Choose $x_a \in X$. Since $ax_a \in PM \subseteq M$, we have $ax_a = \sum_{i=1}^{n} a_i x_i$, where $a_1 x_1 + \cdots + a_n x_n$ is defined. We have $ax_a \oplus (a_i x_i)' = 1$, for every $1 \leq i \leq n$. Then by Proposition 2.10 (j), $ax_a \oplus a_i' x_i \oplus x_i' = 1$ and so by Proposition 2.16, $a' x_a \oplus a_i x_i \oplus (x_a \oplus x_i \oplus x_i')' = 0$. Hence, $a' x_a = 0$, $a_i x_a = 0$ and so $a' = a_i = 0$, for every $1 \leq i \leq n$. It means that $a = 1$ and so $x_a = \sum_{i=1}^{n} a_i x_i = 0$, which is a contradiction. Therefore, $(PM : M) \subseteq P$ and hence $(PM : M) = P$. Now, let

$ax \in PM$, where $a \in A$ and $x \in M$. Then there exist $s_i \in A$, for $1 \le i \le n$ such that $x = \sum_{i=1}^{n} s_i x_i$, where $s_1x_1 + \cdots + s_nx_n$ is defined. Let $a \notin P$. Since $(a.s_i)x_i \le \sum_{i=1}^{n}(a.s_i)x_i = ax \in PM$, $a.s_i \in P$, for every $1 \le i \le n$. Since $a \notin P$, we have $s_i \in P$, for every $1 \le i \le n$ and so $x = \sum_{i=1}^{n} s_i x_i \in PM$.

Theorem 4.4. *Let M be an A-module and L, N be A-ideals of M. Then*

i. $N \subseteq rad(N)$,

ii. *if* $L \subseteq N$, *then* $rad(L) \subseteq rad(N)$,

iii. $rad(rad(N) = rad(N)$,

iv. $rad(N \cap L) \subseteq rad(N) \cap rad(L)$.

Proof. The proof is routine.

Definition 4.5. Let I be an ideal of A. The intersection of all $\cdot$-prime ideals of A including I is denoted by $r_A(I)$ or $r(I)$. If there exists no $\cdot$-prime ideal of A including I, then we let $r_A(I) = A$.

Example 4.6. In Example 4.2, $I = \{0, 1\}$ is a $\cdot$-prime ideal of A and $I = \{0\}$ is not a $\cdot$-prime ideal of A. It is easy to see that $r(I) = \{0, 1\}$ and $r(\{0\}) = \{0\}$.

(ii) Let $M_2(\mathbb{R})$ be the ring of square matrixes of order 2 with real elements and let 0 be the matrix with all elements 0. If we define the order relation on components

$$A = (a_{ij})_{i,j=1,2} \ge 0 \text{ if and only if } a_{ij} \ge 0 \text{ for any } i, j,$$

then $M_2(\mathbb{R})$ is an l-ring. If $v = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, then $(M_2(\mathbb{R}), v)$ is an lu-ring and so $A = \Gamma(M_2(\mathbb{R}), v)$ is a PMV-algebra. It is easy to see that $Id(A) = \{\{0\}, A\}$ and $\{0\}$ is not a $\cdot$-prime ideal of A. Then $r(\{0\}) = A$.

Theorem 4.7. *Let A be unital, M be an $A_{\mathbb{N}}$-module, N be an A-ideal of M and I be an ideal of A. Then*

i. $rad(IM) = rad(r(I)M)$,

ii. $r(N : M) \subseteq (rad(N) : M)$.

Proof. (i) By Lemma 3.3, IM is an A-ideal of M. It is clear that $I \subseteq r(I)$. Then $IM \subseteq r(I)M$ and so by Theorem 4.4 (ii), $rad(IM) \subseteq rad(r(I)M)$. Now, let P be an arbitrary prime A-ideal of M containing IM. Since $I \subseteq (IM : M) \subseteq (P : M)$, we have $r(I) \subseteq (P : M)$ and so $r(I)M \subseteq P$. Hence, $rad(r(I)M) \subseteq P$. It results that $rad(r(I)M) \subseteq rad(IM)$.

(ii) Let P be an arbitrary prime A-ideal of M containing N. Since $N \subseteq P$, it is easy to see that $(N : M) \subseteq (P : M)$. Hence, $r(N : M) \subseteq (P : M)$ and so

$$r(N : M) \subseteq \bigcap_{N \subseteq P \in Spec(M)} (P : M) = \left(\bigcap_{N \subseteq P \in Spec(M)} P : M \right) = (rad(N) : M).$$

Theorem 4.8. *Let A be commutative, M be an $A_{\mathbb{N}}$-module, N be an A-ideal of M and P be a $\cdot$-prime ideal of A. Then $A_N(P) = \{m \in M : cm \in PM \oplus N, \exists c \in A - P\}$ is an A-ideal of M and $PM \oplus N \subseteq A_N(P)$.*

Proof. First, we show that $A_N(P)$ is an ideal of M. (I_1). The proof is clear.

(I_2): Let $n \le m$ and $m \in A_N(P)$, where $m, n \in M$. Then there exists $c \in A - P$ such that $cm \in PM \oplus N$. By Proposition 2.10(f), $cn \le cm \in PM \oplus N$ and so by Lemma 3.3, $cn \in PM \oplus N$. Hence, $n \in A_N(P)$.

(I_3): Let $m, n \in A_N(P)$. Then there exists $c_1, c_2 \in A - P$ such that $c_1m, c_2n \in PM \oplus N$. Let $c = c_1.c_2$. Since P is a $\cdot$-prime ideal of A, we have $c \in A - P$. By Propositions 2.10(h) and Lemma 3.3,

$$\begin{aligned} c(m \oplus n) \le cm \oplus cn &= (c_1.c_2)m \oplus (c_1.c_2)n \\ &= c_2(c_1m) \oplus c_1(c_2n) \in PM \oplus N. \end{aligned}$$

Hence, $c(m \oplus n) \in PM \oplus N$ and so $m \oplus n \in A_N(P)$.

(I_4): Let $m \in A_N(P)$ and $a \in A$. Then there exists $c \in A - P$ such that $cm \in PM \oplus N$. Since $c(am) = (c.a)m = (a.c)m = a(cm) \in PM \oplus N$, we have $am \in A_N(P)$. Finally, let $c = 1$, for any $m \in PM \oplus N$. Then $cm = 1m = m \in PM \oplus N$ and so $m \in A_N(P)$. Therefore, $PM \oplus N \subseteq A_N(P)$.

Theorem 4.9. *Let A be commutative, M be an $A_{\mathbb{N}}$-module, N be an A-ideal of M and P be a $\cdot$-prime ideal of A. Then $A_N(P) = M$ or $A_N(P)$ is a prime A-ideal of M such that $P = (A_N(P) : M)$.*

Proof. Let $A_N(P) \neq M$. We show that $A_N(P)$ is a prime A-ideal of M and $P = (A_N(P) : M)$. By Theorem 4.8, $A_N(P)$ is an A-ideal of M. Let $xm \in A_N(P)$, for any $x \in A$ and $m \in M$. Then there exists $c \in A - P$ such that $c(xm) \in PM \oplus N$. We show that $m \in A_N(P)$ or $x \in (A_N(P) : M)$. If $x \in P$, then by Theorem 4.8, $xM \subseteq PM \oplus N \subseteq A_N(P)$. Hence, $x \in (A_N(P) : M)$. If $x \notin P$, then $x \in A - P$. Since P is a $\cdot$-prime ideal of A, we have $c.x \in A - P$. On the other hand, $c(xm) \in PM \oplus N$ and so $(c.x)m \in PM \oplus N$. Hence, $m \in A_N(P)$. Therefore, $A_N(P)$ is a prime A-ideal of M. Now, we prove that $P = (A_N(P) : M)$. Let $p \in P$. Then for any $m \in M$, $pm \in PM \oplus N$. Let $c = 1$. Then $c(pm) \in PM \oplus N$ and so $pm \in A_N(P)$. Hence, $PM \subseteq A_N(P)$ and so $P \subseteq (A_N(P) : M)$. Now, let $q \in (A_N(P) : M)$ such that $q \notin P$. Since $qM \subseteq A_N(P)$, we have $qm \in A_N(P)$, for any $m \in M$. Hence, there exists $c \in A - P$ such that $c(qm) \in PM \oplus N$ and so $(c.q)m \in PM \oplus N$. Now, since P is a $\cdot$-prime ideal of A, we have $c.q \in A - P$ and so $m \in A_N(P)$. Hence, $M = A_N(P)$, which is a contradiction. Then $q \in P$ and so $(A_N(P) : M) \subseteq P$. Therefore, $P = (A_N(P) : M)$.

Theorem 4.10. *Let A be commutative and M be an $A_{\mathbb{N}}$-module. Then for every A-ideal N of M,*

$$rad_M(N) = \bigcap \{A_N(P) : P \text{ is a } \cdot\text{-prime ideal of } A\}.$$

Proof. Let $T = \bigcap\{A_N(P) : P \text{ is a } \cdot\text{-prime ideal of } A\}$ and $m \in T$. Let L be a prime A-ideal of M including N. Hence, by Theorem 2.7, $Q = (L : M)$ is a $\cdot$-prime ideal of A. Since $m \in A_N(P)$, for every $\cdot$-prime ideal P of A, $m \in A_N(Q)$ and so there exists $c \in A - Q$ such that $cm \in QM \oplus N = (L : M)M \oplus N \subseteq L \oplus L \subseteq L$. Since L is a prime A-ideal of M and $c \notin Q = (L : M)$, $m \in L$. Hence $T \subseteq rad_M(N)$. Now, let $m \in rad_M(N)$ and P be a $\cdot$-prime ideal of A. Hence, $m \in L$, where L is every prime A-ideal of M including N. If $A_N(P) = M$, then the proof is complete. Let $A_N(P) \neq M$. By Theorem 4.9, $A_N(P)$ is a prime A-ideal of M and $P = (A_N(P) : M)$. Now, we show that $N \subseteq A_N(P)$. By Theorem 4.8, we have $PM \oplus N \subseteq A_N(P)$ and so $N \subseteq A_N(P)$. Since $m \in rad_M(N)$, $m \in A_N(P)$. Hence, $m \in T$ and so $rad_M(N) \subseteq T$. Therefore, $rad_M(N) = T$.

5. THE NOETHERIAN A_k-MODULES

In this section, we present definition of a Noetherian A_k-modules and verify some conditions on them.

Definition 5.1. Let M be an A_k-module. If M is a Noetherian A-module, then M is called a Noetherian A_k-module.

Example 5.2.

i. If we consider A as A-module, where $ab = a.b$, for every $a, b \in A$, then every ideal in A is an A-ideal of A. Now, if A is a free A_k-module with finite basis, then A is a Noetherian A_k-module.

ii. Every finite A_k-module is a Noetherian A_k-module.

Theorem 5.3. *Let M be an A-module. Then*

i. M is Noetherian if and only if M has maximum condition.

ii. M is Noetherian if and only if N is finitely generated, for any A-ideal N of M.

Proof. The proof is routine.

Lemma 5.4. Let M be an A-module and I, J be A-ideals of M. Then

$$I \oplus J = \{a \oplus b : (a \in I \text{ and } b \in J) \text{ or } (a \in J \text{ and } b \in I)\}$$

is an A-ideal of M.

Proof. $(I1)$ It is clear that $0 \in I \oplus J$.

$(I2)$ Let $t \leq a \oplus b$, where $a \oplus b \in I \oplus J$, for any $t \in M$. W. L. O. G, let $a \in I$ and $b \in J$. Then $t' \oplus a \oplus b = 1$ and so $(t' \oplus a)' \leq b \in J$. It results that $(t' \oplus a)' \in J$ and so by $(MV4)$, $t \oplus (a' \oplus t)' = a \oplus (t' \oplus a)' \in I \oplus J$. Hence, $t \in I$ or $t \in J$ and so $t = t \oplus 0 \in I \oplus J$.

$(I3)$ Let $a_1 \oplus b_1, a_2 \oplus b_2 \in I \oplus J$. Then

$$(a_1 \oplus b_1) \oplus (a_2 \oplus b_2) = (a_1 \oplus a_2) \oplus (b_1 \oplus b_2) \in I \oplus J.$$

$(I4)$ Let $s \in A$ and $a \oplus b \in I \oplus J$. Then by Lemma 2.10 (h), $s(a \oplus b) \leq sa \oplus sb \in I \oplus J$ and so by $(I2)$, $s(a \oplus b) \in I \oplus J$.

Lemma 5.5. Let A be commutative, M be an A-module and N be an A-ideal of M. Then

$[N : aA] = \{m \in M : (aA)m \subseteq N\}$ is an A-ideal of M, for every $a \in A$, where $aA = \{a.b : b \in A\}$.

Proof. $(I1)$: It is clear that $0 \in [N : aA]$.

$(I2)$: Let $m_1 \leq m_2$ and $m_2 \in [N : aA]$, for any $m_1, m_2 \in M$. Then $(a.b)m_2 \in N$, for some $b \in A$. Since $m_1 \leq m_2$, by Lemma 2.10 (f), we have $(a.b)m_1 \leq (a.b)m_2 \in N$ and so $(a.b)m_1 \in N$, for some $b \in A$. Hence, $m_1 \in [N : aA]$.

$(I3)$: Let $m_1, m_2 \in [N : aA]$, for any $m_1, m_2 \in M$. Since $m_1 \leq m_1 \vee m_2$, $(ab)m_1 \leq (ab)(m_1 \vee m_2)$ and so $(ab)m_1 \vee (ab)m_2 \leq (ab)(m_1 \vee m_2)$. Since $((ab)m_1) \odot ((ab)m_2)' \leq ((ab)m_2)'$,

$((ab)m_1) \odot ((ab)m_2)' + (ab)m_2$ is defined. Similarly, $m_1 \odot m_2' + m_2$ is defined, too. Thus, we have

$$\begin{aligned} ((ab)m_1) \odot ((ab)m_2)' + (ab)m_2 &= (ab)(m_1) \vee (ab)m_2 \leq (ab)(m_1 \vee m_2) \\ &= (ab)(m_1 \odot m_2' + m_2) \\ &= (ab)(m_1 \odot m_2') + (ab)m_2. \end{aligned}$$

Since $+$ is cancellative, we have $((ab)m_1) \odot ((ab)m_2)' \leq (ab)(m_1 \odot m_2')$. Now, if we set $m_1 \oplus m_2$ instead of m_1, then we have

$$\begin{aligned} &(ab)(m_1 \oplus m_2) \odot ((ab)m_2)' \\ &\leq (ab)((m_1 \oplus m_2) \odot m_2') = (ab)(m_1' \wedge m_2) \leq (ab)m_2. \end{aligned}$$

It follows that

$$\begin{aligned} (ab)(m_1 \oplus m_2) = (ab)((m_1 \oplus m_2) \vee (ab)m_2 \\ \leq (ab)(m_1 \oplus m_2) \odot ((ab)m_2)' \oplus (ab)m_1 \\ \leq (ab)m_2 \oplus (ab)m_1 \in N. \end{aligned}$$

Then $(ab)(m_1 \oplus m_2) \in N$ and so $m_1 \oplus m_2 \in [N : aA]$.

$(I4)$: Let $t \in A$ and $m \in [N : aA]$. Then $(a.b)m \in N$, for some $b \in A$. Since $(a.b)m \in N$, we have $t((a.b)m) \in N$, for some $b \in A$. Then $(a.b)(tm) = ((a.b).t)m = (t.(a.b))m \in N$ and so $tm \in [N : aA]$.

Lemma 5.6. Let A be commutative, M be a Boolean A_2-module and N be an A-ideal of M. Then

i. $aM = \{am : m \in M\}$ is an A-ideal of M, for every $a \in A$,

ii. if there exists $a \in A$ such that $N \oplus aM$ and $[N : aA]$ are finitely generated, then N is finitely generated,

iii. if N is not finitely generated and it is maximal in all A-ideals of M that are not finitely generated, then N is a prime A-ideal of M.

Proof. (i) $(I1)$: It is clear that $0 \in aM$.

$(I2)$: Let $t \leq am$, where $am \in aM$ and $t \in M$. Since $am \leq m$, we get $t \leq m$ and so $t' \oplus m = 1$. Now, by Proposition 2.16, since

$$\begin{aligned} t &= t \wedge am = t \odot (t' \oplus am) \\ &= (t' \oplus (t' \oplus am)')' = (t' \oplus (1t' \oplus am)')' \\ &= (t' \oplus (a'm \oplus (t' \oplus m)'))' = (t' \oplus a'm \oplus 1')' = (t' \oplus a'm)' \\ &= am \oplus (t' \oplus m)' \\ &= am \in aM, \end{aligned}$$

we get $t \in aM$.

(I_3): Let $m_1, m_2 \in aM$, for any $m_1, m_2 \in M$. Since M is a Boolean A-module, by Lemma 2.10 (h), $am_1 \oplus am_2 = a(m_1 \oplus m_2) \in aM$.

($I4$): Let $b \in A$ and $am \in aM$. Then $b(am) = (b.a)m = (a.b)m = a(bm) \in aM$.

(ii) By Lemma 5.5 and (i), aM and $[N : aA]$ are A-ideals of M and by Lemma 5.4, $N \oplus aM$ is an A-ideal of M, too. Let $\phi : [N : aA] \to N \cap aM$ be defined by $\phi(m) = am$. It is routine to show that $\frac{[N:aA]}{ker(\phi)} \simeq N \cap aM$. Since $[N : aA]$ is finitely generated, $N \cap aM$ is finitely generated, too. On the other hand, $\frac{N \oplus aM}{aM} \simeq \frac{N}{N \cap aM}$. Now, since $N \oplus aM$ is finitely generated, $\frac{N}{N \cap aM}$ is finitely generated, too. Hence, N is finitely generated.

(iii) By (ii), the proof is routine.

Theorem 5.7. *Let A be commutative, M be a Boolean A_2-module and N be an A-ideal of M. Then M is Noetherian if and only if any prime A-ideal of M is finitely generated.*

Proof. ($\Rightarrow$) Let M be a Noetherian A-module. Then by Theorem 5.3 (ii), every prime A-ideal of M is finitely generated.

($\Leftarrow$) Let every prime A-ideal of M is finitely generated. If

$$T = \{N : N \text{is an A-ideal of} M \text{ and } N \text{is not finitely generated}\} \neq \emptyset,$$

then by Zorn' s Lemma, T has a maximal element K and so by Lemma 5.6 (iii), K is a prime A-ideal of M that is not finitely generated, which is a contradiction. Hence, $T = \emptyset$ and so every A-ideal of M is finitely generated. Therefore, M is a Noetherian A-module.

6. CONCLUSION AND FUTURE DIRECTIONS

The categorical equivalence between MV-algebras and lu-groups, leads us to the definition of product operation on MV-algebras, in order to obtain structures corresponding to l-rings. For this reason, A. Di Nola introduced the notion of MV-modules. Since by current definition, it was not easy for us to achieve all our goals, we limited our scope of work. We introduced a particular category of MV-modules as A_k-modules in [4]. Now, in this paper we presented definitions of free A_k-modules, radical of A_k-modules and Noetherian A_k-modules. The obtained results in the last sections encourage us to continue this long way. In fact, there are many questions in this field that should be verified (see advance results in classical modules). Currently, we are working on a different definition for MV-modules, and we plan to continue our studies by new definition. We believe that the fuzzy (graph) structures of this algebras are also interesting (for more guidance, see [18,19,9]).

CONFLICT OF INTEREST

The authors declare that there is no conflict of interest.

AUTHORS' CONTRIBUTIONS

We have same contribution.

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