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Tests for Comparing Several Two-Parameter Exponential Distributions Based on Uncensored/Censored Samples

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ABSTRACT

The problems of comparing several exponential distributions based on type II censored samples are considered. Likelihood ratio tests (LRTs) for comparing several location parameters, comparing several scale parameters and for homogeneity of distributions are derived. The LRTs for all three problems are exact as their null distributions do not depend on any unknown parameters. Algorithms are provided to estimate the exact p-values or the percentiles of null distributions. Approximations to the null distributions that are accurate even for small sample sizes are provided. For testing the equality of scale parameters, the proposed LRT is compared with the tests based on union-intersection method and an iterative procedure. Comparison studies indicate that the LRT is more powerful than the existing ones for most parameter values. The methods are illustrated using an example involving elapsed times from randomization to diagnosis of a serious infection of chronic granulomatous disease that were collected from three different hospitals.

1. INTRODUCTION

The problem of testing equality of several distributions within a family of distributions arises in many practical situations. If independent samples are available from different sources or collected from different life testing experiments, a homogeneity test may be used to check if the samples can be modeled by a single distribution. For example, in a placebo controlled randomized clinical trial of gamma interferon in chronic granulomatous disease (CGD), the data were collected on elapsed time (in days) from randomization to diagnosis of a serious infection [1]. The samples were taken from three hospitals, and the Kolmogorov–Smirnov test indicates that all three samples satisfy the model assumption of two-parameter exponential distribution. If a homogeneity test indicates that all three samples can be modeled by a single exponential distribution, then better inferences can be made on the basis of the pooled sample. For two-parameter exponential distributions, the problem of testing the equality of parameters of several distributions was considered by [2]. These authors have proposed an exact iterated test procedures for testing the equality of the location parameters under some conditions and for testing the equality of the scale parameters. Abebe [3] have addressed the problem of comparing several location parameters with more than one control. Li *et al.* [4] have proposed parametric bootstrap simultaneous confidence intervals for pairs of means. Recently, Krishna and Goel [5] have proposed some Bayesian solutions for estimating location parameters when the data are randomly censored.

To describe the problems that will be addressed in this paper and other related problems, we first note that the two-parameter exponential distribution has the probability density function (pdf) given by

$$f(x|a,b) = \frac{1}{b}\exp\left(-\frac{x-a}{b}\right), \;\; x > a > 0, \;\; b > 0, \tag{1}$$

where a is the location parameter (also called threshold parameter) and b is the scale parameter. If it is assumed that $a = 0$, then the resulting distribution is referred to as the one-parameter exponential distribution with mean b. Let us denote the two-parameter exponential distribution by $\exp(a, b)$.

In life testing, the experiments are often discontinued as soon as a fixed number of items fail, yielding a type II censored sample. Let $X_{i1}, \ldots, X_{ir_i}$ be a type II censored sample from a life test of n_i items whose lifetimes follow an exp (a_i, b_i) distribution, $i = 1, \ldots, k$. The problems that we consider here are (i) testing the equality of the scale parameters b_i's, (ii) testing the equality of the location parameters a_i's and (iii) the homogeneity test, that is, testing $(a_1, b_1) = \ldots = (a_k, b_k)$. The LRTs that are proposed in the sequel are applicable to the case where $r_i = n_i$ for some or all $i = 1, \ldots, k$. In the following section we provide some preliminary results that are needed to derive the

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likelihood ratio tests (LRTs) for all three problems. In Section 3, we address the problem of testing the equality of scale parameters, derive the LRT and describe another two tests available in the literature. The problem of testing equality of the location parameters is addressed in Section 4, and the homogeneity test is described in Section 5. We also show that the null distributions of all three LRT statistics do not depend on any unknown parameters, and the percentiles or p-values of the LRTs can be estimated using Monte Carlo simulation. So the proposed LRTs are exact except for the simulation errors. For all three problems, we also propose convenient closed-form approximations to the null distributions, and they are very satisfactory even for small samples. Power studies and comparison of tests are given in Section 6. The proposed LRTs and other methods are illustrated using an example with real data in Section 7, and some concluding remarks are given in Section 8.

2. PRELIMINARIES

Let $X_{(1)} < \ldots < X_{(r)}$ be a type II censored sample obtained from a life test on n items whose lifetime have an exp (a, b) distribution, where a is the threshold parameter and b is the scale parameter. Let $\overline{X}_r$ denote the mean of the uncensored lifetimes in the sample.

The MLEs (see [6]) based on the censored sample are given by

$$\hat{a} = X_{(1)} \quad \text{and} \quad \hat{b} = \left[r\left(\overline{X}_r - X_{(1)}\right) + (n-r)\left(X_{(r)} - X_{(1)}\right)\right] \Big/ r. \tag{2}$$

Furthermore, the MLEs are independent with

$$\hat{a} \sim \frac{b}{2n}\chi^2_2 + a \quad \text{and} \quad \hat{b} \sim \frac{b}{2r}\chi^2_{2r-2}. \tag{3}$$

For the uncensored case, that is, when $r = n$, the MLEs are given by

$$\hat{a} = X_{(1)} \quad \text{and} \quad \hat{b} = \overline{X}_n - X_{(1)}, \tag{4}$$

and

$$\hat{a} \sim \frac{b}{2n}\chi^2_2 + a \quad \text{and} \quad \hat{b} \sim \frac{b}{2n}\chi^2_{2n-2}.$$

Log-likelihood function

Let $a = (a_1, \ldots, a_k)$ and $b = (b_1, \ldots, b_k)$. The log-likelihood function based on the k independent censored samples can be expressed as

$$l(a, b) = \sum_{i=1}^{k} \left\{ -r_i \ln b_i - \sum_{j=1}^{r_i} \frac{X_{i(j)} - a_i}{b_i} - (n_i - r_i)\frac{X_{i(r_i)} - a_i}{b_i} \right\} + C,$$

where C is a constant term. Following (2), the MLEs of a_i and b_i can be expressed as

$$\hat{a}_i = X_{i(1)} \quad \text{and} \quad \hat{b}_i = \left[r_i\left(\overline{X}_{r_i} - X_{i(1)}\right) + (n_i - r_i)\left(X_{i(r_i)} - X_{i(1)}\right)\right] \Big/ r_i, i = 1, \ldots, k, \tag{5}$$

and $l(a, b)$ at $(a, b) = (\hat{a}, \hat{b})$ can be simplified as

$$l(\hat{a}, \hat{b}) = -\sum_{i=1}^{k} r_i \ln(\hat{b}_i) - R + C, \tag{6}$$

where $R = \sum_{i=1}^{k} r_i$.

3. TESTS FOR THE EQUALITY OF SCALE PARAMETERS

To test the equality of scale parameters, consider the hypotheses

$$H_0 : b_1 = \ldots = b_k \quad \text{vs.} \quad H_a : b_i \neq b_j \text{ for some } i \neq j. \tag{7}$$

In the following, we first describe the LRT followed by an iterative test by [2], and a test based on union-intersection principle proposed in [7].

3.1. The Likelihood Ratio Test

Let b denote the common unknown parameter under H_0 in (7). Then the log-likelihood function under this null hypothesis can be expressed as

$$l(a,b) = \sum_{i=1}^{k}\left(-r_i \ln b - \sum_{j=1}^{r_i}\frac{X_{i(j)}-a_i}{b} - (n_i - r_i)\frac{X_{i(r_i)}-a_i}{b}\right) + C,$$

and is maximized at

$$\hat{a}_{ic} = X_{i(1)} \quad \text{and} \quad \hat{b}_c = \frac{1}{R}\sum_{i=1}^{k}\left\{r_i\left(\overline{X}_{r_i} - X_{i(1)}\right) + (n_i - r_i)\left(X_{i(r_i)} - X_{i(1)}\right)\right\} = \frac{\sum_{i=1}^{k} r_i \hat{b}_i}{R}, \tag{8}$$

where $R = \sum_{i=1}^{k} r_i$ and $\hat{b}_i$'s are given in (5). Thus, letting Λ_b to denote the $-2\ln$(LRT statistic), it can be shown that

$$\Lambda_b \; = \; -2\sum_{i=1}^{k} r_i \ln\left[\frac{\hat{b}_i}{\frac{\sum_{i=1}^{k} r_i \hat{b}_i}{R}}\right], \tag{9}$$

where $\hat{b}_i$ is given in (5). We note that the distribution of Λ_b under $H_0 : b_1 = \ldots = b_k$ does not depend on any parameter, and

$$\Lambda_b \sim -2\sum_{i=1}^{k} r_i \ln\left[\frac{W_i}{\frac{\sum_{i=1}^{k} r_i W_i}{R}}\right], \tag{10}$$

where W_i's are independent $\chi^2_{2r_i-2}/(2r_i)$ random variables. Notice also that the null distribution does not depend on the sample sizes and they depend only on $(r_1, \ldots, r_k)$. For a given $(r_1, \ldots, r_k)$, the exact percentiles of Λ_b can be estimated using Monte Carlo simulation as shown in the following algorithm.

Algorithm 1

For a given $(r_1, \ldots, r_k)$ and $(\hat{b}_1, \ldots, \hat{b}_k)$,

1. Compute $\Lambda_b = -2\sum_{i=1}^{k} r_i \ln\left[\frac{\hat{b}_i}{\frac{\sum_{i=1}^{k} r_i \hat{b}_i}{R}}\right]$,
2. Generate $W_i = .5\chi^2_{2r_i-2}/r_i$, $i = 1, \ldots, k$,
3. Compute $\Lambda_b^* = -2\sum_{i=1}^{k} r_i \ln\left[\frac{W_i}{\frac{\sum_{i=1}^{k} r_i W_i}{R}}\right]$,
4. Repeat steps 2 and 3, for a large number of times, say, 100,000,
5. The proportion of Λ_b^* that are greater than Λ_b is a Monte Carlo estimate of the p-value,
6. The 100 $(1-\alpha)$ percentile of these 100,000 Λ_b^*'s is a Monte Carlo estimate of the 100 $(1-\alpha)$ percentile of Λ_b.

The test statistic Λ_b is very similar to the one for testing $b_1 = \ldots = b_k$ of several one-parameter exponential distributions, except that $\hat{b}_i$ in our present problem is distributed as $\chi^2_{2r_i-2}/(2r_i)$ whereas in the one-parameter case it is distributed like $\chi^2_{2r_i}/(2r_i)$. This fact suggest that, for large samples, the null distribution of Λ_b can be approximated by χ^2_{k-1} distribution. For large samples, the null hypothesis of equal scale parameters is rejected if $\Lambda_b > \chi^2_{k-1;1-\alpha}$. A better approximation to the null distribution of Λ_b can be obtained by the moment matching method. Specifically, we determine the value of Q_b so that $E(\Lambda_b) = E(Q_b\chi^2_{k-1})$ so that

$$\frac{\Lambda_b}{Q_b} \sim \chi^2_{k-1}, \quad \text{approximately}.$$

Using the result that $E(\ln \chi_m^2) = \psi(m/2) + \ln(2)$, where ψ is the digamma function, and the expression (10), we see that

$$E(\Lambda_b) = 2E\left[\sum_{i=1}^{k} r_i \ln\left(\frac{\sum_{i=1}^{k} r_i W_i}{R}\right) - \sum_{i=1}^{k} r_i \ln(W_i)\right]$$
$$= 2\left[\sum_{i=1}^{k} r_i\left[\psi(R-k) - \psi(r_i - 1)\right] + \sum_{i=1}^{k} r_i \ln(r_i/R)\right]. \quad (11)$$

Thus, $Q_b = E(\Lambda_b)/(k-1)$, and the null hypothesis of equal shape parameters is rejected if $\Lambda_b > Q_b\chi^2_{k-1;1-\alpha}$.

To judge the accuracy of the above approximation, we estimated the upper percentiles of Λ_b in (10) using simulation with 1,000,000 runs and the percentiles of $Q_b\chi^2_{k-1}$ for $k = 3$ and 5, and some small to large values of $(r_1, \ldots, r_k)$. Recall that the distribution of Λ_b does not depend on sample sizes, and so percentiles were estimated only for some values of $(r_1, \ldots, r_k)$. These percentiles are reported in Table 1. Comparison of Monte Carlo estimates of the percentiles and those of $Q_b\chi^2_{k-1}$ shows that the latter approximate percentiles are very close to those based on simulation even when $(r_1, \ldots, r_5) = (3, 3, 3, 2, 2)$. Thus, the improved approximation can be safely used for any sample sizes with two or more uncensored data. The usual χ^2_{k-1} approximation may be used if $r_i \geq 100$ for all i; see the last row of Table 1.

3.2. An Iterative Test

We shall now describe a test for equality of scale parameters proposed by [2], which is based on an iterative procedure. Consider a sequence of nested hypotheses

$$H_2 : b_1 = b_2,\ H_3 : b_1 = b_2 = b_3, \ldots, H_k : b_1 = \ldots = b_k.$$

A test for H_i is made if and only if H_{i-1} is accepted for $i = 3, \ldots, k$. The null hypothesis (7) of equality scale parameters is accepted if and only if $H_2, \ldots, H_k$ are accepted.

The test statistic for $H_i : b_1 = \ldots = b_i$ is given by

$$F_i = \frac{r_i \hat{b}_i/(2r_i - 2)}{\sum_j^{i-1} r_j \hat{b}_j / S_{(i-1)}}, \quad (12)$$

where $S_{(i-1)} = \sum_{j=1}^{i-1}(2r_j - 2)$, $i = 2, \ldots, k$. For a given level α, choose $0 < \eta_i < \alpha$ so that $\alpha = 1 - \prod_{i=2}^{k}(1 - \eta_i)$. The iterative test rejects $H_0 : b_1 = \ldots = b_k$ if

$$F_i < F_{2r_i-2, S_{(i-1)}; \eta_i/2} \quad \text{or} \quad F_i > F_{2r_i-2, S_{(i-1)}; 1-\eta_i/2} \text{ for any } i \in \{2, \ldots, k\},$$

where $F_{m,n;q}$ denotes the 100 q percentile of an F distribution with the numerator df m and the denominator df n. If we choose $\eta_2 = \ldots = \eta_k$, then $\eta_i = 1 - (1-\alpha)^{1/(k-1)}$, $i = 2, \ldots, k$. Notice that the individual test for H_i has size η_i, but the overall test for $H_0 : b_1 = \ldots = b_k$ has size α.

Table 1 | Exact percentiles of Λ_b based on simulation and the percentiles of $Q_b\chi^2_{k-1}$.

$k = 3$	Percentiles			$k = 5$	Percentiles		
(r_1, r_2, r_3)	99	95	90	$(r_1, \ldots, r_5)$	99	95	90
(2, 2, 2)	21.53 (22.18)	14.23 (14.43)	11.03 (11.09)	(2, …, 2)	30.73 (31.46)	22.21 (22.48)	18.29 (18.43)
(3, 3, 3)	15.12 (15.31)	9.92 (9.96)	7.63 (7.66)	(3, 3, 3, 2, 2)	26.69 (26.71)	19.07 (19.08)	15.64 (15.65)
(4, 3, 4)	14.08 (14.10)	9.17 (9.17)	7.05 (7.05)	(4, 3, 4, 3, 4)	20.30 (20.30)	14.51 (14.51)	11.91 (11.90)
(3, 2, 6)	18.90 (19.13)	12.40 (12.45)	9.55 (9.57)	(3, 2, 4, 3, 2)	26.63 (26.64)	19.06 (19.04)	15.65 (15.61)
(5, 4, 3)	13.97 (13.97)	9.08 (9.09)	6.99 (6.99)	(5, 4, 3, 6, 2)	23.45 (23.07)	16.61 (16.48)	13.57 (13.52)
(8, 7, 10)	10.90 (10.90)	7.10 (7.09)	5.45 (5.45)	(8, 7, 6, 8, 20)	16.06 (16.08)	11.48 (11.49)	9.41 (9.42)
(10, 10, 10)	10.46 (10.49)	6.82 (6.82)	5.24 (5.24)	(4, 3, 2, 5, 6)	23.41 (23.07)	16.62 (16.48)	13.59 (13.52)
(20, 20, 20)	9.82 (9.81)	6.37 (6.38)	4.90 (4.90)	(20, …, 20)	14.14 (14.12)	10.09 (10.09)	8.27 (8.28)
(30, 30, 30)	9.62 (9.60)	6.25 (6.25)	4.80 (4.80)	(30, …, 30)	13.85 (13.83)	9.86 (9.88)	8.10 (8.10)
(50, 50, 50)	9.49 (9.44)	6.14 (6.14)	4.72 (4.72)	(50, …, 50)	13.62 (13.60)	9.74 (9.72)	7.97 (7.97)
(100, 100, 100)	9.33 (9.32)	6.07 (6.07)	4.66 (4.66)	(100, …, 100)	13.43 (13.44)	9.61 (9.60)	7.88 (7.87)
χ^2_{k-1} percentiles	9.21	5.99	4.61	χ^2_{k-1} percentiles	13.28	9.49	7.78

Note: Percentiles of $Q_b\chi^2_{k-1}$ are given in parentheses.

3.3. Union-Intersection Test

Shah and Rathod [7] have proposed the following test based on the union-intersection principle. The test rejects the null hypothesis of equality of scale parameters if

$$U = \frac{\max\{\hat{b}_1, \dots, \hat{b}_k\}}{\min\{\hat{b}_1, \dots, \hat{b}_k\}} > u_{1-\alpha}, \tag{13}$$

where $\hat{b}_i$ is the MLE defined in (5) and $u_{1-\alpha}$ is the 100 $(1-\alpha)$ percentile of U.

Shah and Rathod have derived analytical expressions to the distribution function of U for $k \le 3$ and $r_1 = r_2 = r_3$. Even for these special cases, calculation of the critical value $u_{1-\alpha}$ is numerically involved. As the null distribution of the test statistic U does not depend on any parameters, the critical value $u_{1-\alpha}$ can be easily estimated using Monte Carlo simulation for any $k \ge 2$ and $(r_1, \dots, r_k)$.

4. THE LRT FOR LOCATION PARAMETERS

Consider testing

$$H_0 : a_1 = \dots = a_k \quad \text{vs.} \quad H_a : a_i \ne a_j \ \text{ for some } i \ne j. \tag{14}$$

Let a denote the common unknown parameter under the H_0. Then the log-likelihood function under H_0 in (14) can be expressed as

$$l(a, b) = \sum_{i=1}^{k} \left(-r_i \ln b_i - \sum_{j=1}^{r_i} \frac{X_{i(j)} - a}{b_i} - (n_i - r_i)\frac{X_{i(r_i)} - a}{b_i} \right) + C,$$

where C is a constant term. The log-likelihood function is maximized at

$$\hat{a} = X_{(1)} = \min\{X_{1(1)}, \dots, X_{k(1)}\} \tag{15}$$

and

$$\begin{aligned} \hat{b}_{ic} &= \frac{1}{r_i}\left[\sum_{j=1}^{r_i}(X_{i(j)} - X_{(1)}) + (n_i - r_i)(X_{i(r_i)} - X_{(1)})\right] \\ &= \frac{1}{r_i}\left[r_i\hat{b}_i + n_i(X_{i(1)} - X_{(1)})\right], \quad i = 1, \dots, k, \end{aligned} \tag{16}$$

where $\hat{b}_i$'s are as defined in (5). Letting $\hat{b}_c = (\hat{b}_{1c}, \dots, \hat{b}_{kc})$, we have

$$l(\hat{a}, \hat{b}_c) = -\sum_{i=1}^{k} r_i \ln(\hat{b}_{ic}) - R + C. \tag{17}$$

Let Λ_a denote $-2\ln$(LRT statistic). Then, using $l(\hat{a}, \hat{b})$ defined in (6), it can be easily verified that

$$\begin{aligned} \Lambda_a &= -2\left\{l(\hat{a}, \hat{b}_c) - l(\hat{a}, \hat{b})\right\} \\ &= -2\sum_{i=1}^{k} r_i \ln\left[\frac{\hat{b}_i}{\hat{b}_i + n_i(X_{i(1)} - X_{(1)})/r_i}\right]. \end{aligned} \tag{18}$$

As Λ_a is invariant under the transformation $X_{ij} \to c_i X_{ij} + d$, where c_i's and d are positive constants, its null distribution does not depend on any parameters. Specifically, the null distribution of Λ_a can be evaluated empirically assuming that $a_1 = \dots = a_k = 0$ and $b_1 = \dots = b_k = 1$. Under this assumption, using the distributional results in (3), we see that

$$\Lambda_a \sim -2\sum_{i=1}^{k} r_i \ln\left[\frac{W_i}{\left(W_i + n_i(U_i - U_{(1)})\right)}\right], \tag{19}$$

where $W_i = r_i\hat{b}_i \sim \chi^2_{2r_i-2}/2$, $U_i = \chi^2_2/(2n_i)$ and $U_{(1)} = \min\{U_1, \dots, U_k\}$ and W_i's and U_i's are mutually independent. For an observed value of the LRT statistic Λ_a, the p-value and the percentiles of Λ_a can be estimated using the following algorithm.

Algorithm 2

1. For a given set of k samples, compute the LRT statistic Λ_a in (18),
2. Generate $U_i^* = \chi_2^2/n_i$ and $W_i^* = \chi^2_{2r_i-2}$, $i = 1, \dots, k$,
3. Set $U_{(1)}^* = \min\{U_1^*, \dots, U_k^*\}$,
4. Compute $\Lambda_a^* = -2\sum_{i=1}^k r_i \ln\left(\frac{W_i^*}{\left[W_i^* + n_i(U_i^* - U_{(1)}^*)\right]}\right)$,
5. Repeat steps 2 –4 for a large number of times, say, 100,000,
6. Proportion of Λ_a^*'s that are greater than Λ_a is an estimate of the p-value,
7. The 100 $(1-\alpha)$ percentile of the 100,000 Λ_a^*'s is a Monte Carlo estimate of the $100(1-\alpha)$ percentile of Λ_a.

In Table 2, we present the estimated percentiles of Λ_a (based on 1,000,000 runs) for $k = 3$ and 5, and for some values for $(r_1, \dots, r_k)$ ranging from very small to large. Notice that these are exact percentiles, except for the simulation errors.

To find a closed-form approximate distribution for Λ_a, we note that Wilks' theorem is not applicable to find an asymptotic χ^2 null distribution, because the sample space of a two-parameter exponential distribution depends on an unknown parameter. However, our extensive simulation studies indicated that, for large samples, the null distribution of Λ_a is close to χ^2_{2k-2} distribution; see the last row in Table 2. This χ^2_{2k-2} approximation is satisfactory only when all r_i's are 100 or more. This approximation can be improved along the lines of the preceding section. Specifically, we determine Q_a so that $E(\Lambda_a) = Q_a E\left(\chi^2_{2k-2}\right)$, and then approximate the distribution of Λ_a by $Q_a\chi^2_{2k-2}$. Toward this, we note that exact calculation of $E(\Lambda_a)$ seems to be difficult, and so we resort to use an approximation of $E(\Lambda_a)$. It is shown in Appendix that

$$E(\Lambda_a) \approx -2\sum_{i=1}^k r_i \left[\psi(r_i - 1) - \ln(\theta_i) + \frac{1}{2\theta_i^2}\left(r_i + \frac{n_i^2}{\left(\sum_{j=1}^k n_j\right)^2}\right)\right],$$

where $\theta_i = r_i - n_i/\sum_{j=1}^k n_j$. Finally, letting $Q_a = E(\Lambda_a)/(2k-2)$, we propose the distribution of $Q_a\chi^2_{2k-2}$ as an approximate null distribution of Λ_a.

Monte Carlo estimates (based on one million runs) of the percentiles of Λ_a based on Algorithm 2 and those based on the above approximation are given in Table 2 for $k = 3$ and 5 and for some sample sizes and $(r_1, \dots, r_k)$. Comparison of the percentiles based on Monte Carlo

Table 2 Monte Carlo estimates of the percentiles of Λ_a based on (19) and those of $Q_a\chi^2_{2k-2}$.

	$(n_1, n_2, n_3) = (9, 10, 8)$				$(n_1, \dots, n_5) = (20, \dots, 20)$		
(r_1, r_2, r_3)	99	95	90	$(r_1, \dots, r_5)$	99	95	90
(3, 2, 3)	22.43 (22.74)	15.89 (16.25)	12.96 (13.33)	(3, 4, 3, 2, 4)	31.25 (31.93)	23.93 (24.64)	20.58 (21.23)
(3, 3, 4)	19.21 (19.38)	13.69 (13.85)	11.21 (11.36)	(5, 4, 6, 5, 8)	24.87 (25.19)	19.18 (19.44)	16.55 (16.75)
(4, 5, 4)	17.38 (17.44)	12.40 (12.46)	10.18 (10.22)	(6, 3, 7, 7, 10)	24.93 (25.07)	19.16 (19.35)	16.46 (16.67)
(5, 5, 5)	16.68 (16.26)	11.88 (11.88)	9.75 (9.74)	(9, 8, 7, 10, 15)	22.57 (22.74)	17.41 (17.55)	14.99 (15.12)
(6, 7, 8)	15.59 (15.51)	11.12 (11.08)	9.11 (9.09)	(10, …, 10)	22.32 (22.47)	17.25 (17.34)	14.85 (14.94)
(9, 10, 8)	14.99 (14.94)	10.69 (10.68)	8.77 (8.75)	(5, 10, 15, 10, 15)	22.60 (22.73)	17.43 (17.55)	15.02 (15.12)
	$(n_1, n_2, n_3) = (25, 30, 22)$				$(n_1, \dots, n_5) = (60, 50, 50, 70, 60)$		
(10, 10, 10)	14.77 (14.72)	10.53 (10.52)	8.64 (8.63)	(20, …, 20)	21.19 (21.21)	16.35 (16.37)	14.08 (14.11)
(15, 10, 20)	14.33 (14.26)	10.23 (10.19)	8.39 (8.35)	(4, 5, 6, 7, 7)	24.60 (24.93)	18.98 (19.24)	16.34 (16.58)
(20, 20, 20)	14.00 (13.95)	9.98 (9.97)	8.17 (8.18)	(30, …, 30)	20.80 (20.82)	16.06 (16.07)	13.84 (13.85)
	$(n_1, n_2, n_3) = (300, 200, 100)$				$(n_1, \dots, n_5) = (200, \dots, 200)$		
(30, 40, 50)	13.61 (13.58)	9.73 (9.71)	7.97 (7.96)	(50, 40, 50, 40, 50)	20.61 (20.57)	15.87 (15.88)	13.68 (13.68)
(50, 50, 50)	13.56 (13.53)	9.68 (9.67)	7.93 (7.93)	(50, …, 50)	20.45 (20.52)	15.82 (15.84)	13.64 (13.65)
(100, 100, 100)	13.39 (13.40)	9.59 (9.58)	7.86 (7.85)	(100, …, 100)	20.25 (20.30)	15.68 (15.67)	13.50 (13.50)
(120, 150, 100)	13.37 (13.38)	9.54 (9.56)	7.81 (7.84)	(150, 130, 140, 130, 150)	20.22 (20.24)	15.61 (15.63)	13.46 (13.46)
χ^2_{2k-2} percentiles	13.28	9.49	7.78	χ^2_{2k-2} percentiles	20.09	15.51	13.36

Note: The numbers in parentheses are the approximate percentiles of Λ_a based on $Q_a\chi^2_{2k-2}$.

method and tho approximate ones are very close even for small sample such as $(r_1, r_2, r_3) = (3, 2, 3)$ and $(r_1, \dots, r_5) = (3, 4, 3, 2, 4)$. On an overall basis, we see that the approximate percentiles are very satisfactory and are safe to use in practical applications.

5. HOMOGENEITY TEST

For the homogeneity test, the hypotheses of interest are

$$H_0 : (a_1, b_1) = \dots = (a_k, b_k) \quad \text{vs.} \quad H_a : (a_i, b_i) \neq (a_j, b_j) \text{ for some } i \neq j. \tag{20}$$

Let (a, b) denote the common unknown parameter under H_0 in (20). Then the log-likelihood function under the null hypothesis (20) can be expressed as

$$l(a, b) = \sum_{i=1}^{k} \left(-r_i \ln b - \sum_{j=1}^{r_i} \frac{X_{i(j)} - a}{b} - (n_i - r_i) \frac{X_{i(r_i)} - a}{b} \right) + C,$$

where C is a constant term. The log-likelihood function is maximized at

$$\hat{a} = X_{(1)} = \min\{X_{1(1)}, \dots, X_{k(1)}\} \tag{21}$$

and

$$\begin{aligned} \hat{b} &= \frac{1}{R} \sum_{i=1}^{k} \left[\sum_{j=1}^{r_i} (X_{i(j)} - X_{(1)}) + (n_i - r_i)(X_{i(r_i)} - X_{(1)}) \right] \\ &= \frac{1}{R} \sum_{i=1}^{k} \left[r_i \hat{b}_i + n_i (X_{i(1)} - X_{(1)}) \right], \end{aligned} \tag{22}$$

where $\hat{b}_i$ is defined in (5) and $R = \sum_{i=1}^{k} r_i$. Let Λ_h denote the $-2\ln(\text{LRT statistic})$. It can be easily verified that

$$\begin{aligned} \Lambda_h &= -2 \sum_{i=1}^{k} r_i \ln \left(\frac{\hat{b}_i}{\hat{b}} \right) \\ &= -2 \sum_{i=1}^{k} r_i \ln \left[\frac{\hat{b}_i}{\frac{1}{R} \sum_{i=1}^{k} r_i \hat{b}_i + \frac{1}{R} \sum_{i=1}^{k} n_i (X_{i(1)} - X_{(1)})} \right]. \end{aligned} \tag{23}$$

Since the above statistic is location-scale invariant, its null distribution does not depend on any parameters, and so the null distribution can be evaluated empirically assuming, without loss of generality, that $a_1 = \dots = a_k = 0$ and $b_1 = \dots = b_k = 1$. Under this assumption and using the distributional results in (3), we see that

$$\Lambda_h \sim -2 \sum_{i=1}^{k} r_i \ln \left[\frac{W_i}{\frac{1}{R} \left(\sum_{i=1}^{k} r_i W_i + \sum_{i=1}^{k} n_i (U_i - U_{(1)}) \right)} \right] \tag{24}$$

where $W_i = \chi^2_{2r_i-2}/(2r_i)$, $U_i = \chi^2_2/(2n_i)$ and $U_{(1)} = \min\{U_1, \dots, U_k\}$ and the random variables W_i's and U_i's are mutually independent. Thus, the percentiles of Λ_h can be estimated using Monte Carlo simulation as in Algorithms 1 and 2.

Alternatively, a closed-form approximation to the null distribution of Λ_h can be obtained along the lines of Section 3. As noted earlier, Wilks' theorem is not applicable to find a large sample approximate χ^2 distribution. On the basis of simulation studies we found that $\Lambda_h \sim \chi^2_{3(k-1)}$ for large samples. This approximation can be improved along the lines of Section 3. That is, we determine the value of Q_h so that $E(\Lambda_h) = E(Q_h \chi^2_{3(k-1)})$ and $\Lambda_h \sim Q_h \chi^2_{3(k-1)}$, approximately. To find $E(\Lambda_h)$, we note the $\hat{b}_i$ follows a $\chi^2_{2r_i-2}/(2r_i)$ distribution and $\hat{b}$ in (22) follows a $\chi^2_{2R-2}/(2R)$. Using this distributional results in (23), it can be verified that

$$E(\Lambda_h) = 2 \sum_{i=1}^{k} r_i \left[\psi(R-1) - \psi(r_i - 1) + \ln(r_i/R) \right]. \tag{25}$$

Thus, $Q_h = E(\Lambda_h)/(3(k-1))$ and

$$\Lambda_h \sim Q_h \chi^2_{3(k-1)}, \quad \text{approximately.}$$

To judge the accuracy of the above approximation, we estimated the upper percentiles of Λ_h in (24) using simulation with 1,000,000 runs and the percentiles of $Q_h\chi^2_{3(k-1)}$ for $k = 3$ and 4, and some small to large values of sample sizes. These percentiles are reported in Table 3. Comparison of Monte Carlo estimates of the percentiles and those of $Q_h\chi^2_{2k}$ shows that the latter approximate percentiles are very close to those based on simulation even when $(r_1, r_2, r_3) = (3, 2, 2)$. Thus, the improved approximation can be safely used for all values of r_i's greater than or equal to three.

6. POWER STUDIES

6.1. Power Comparison of the Tests for Scale Parameters

To compare the LRT, iterative test (ITER) and the union-intersection test (UIT), we estimated the powers of these tests for $k = 3$ and 5, and some moderate sample sizes using simulation. As all the tests are scale invariant, the powers were estimated assuming, without loss of generality, that $b_1 = 1$ and $b_1 > \ldots > b_k$. The estimated powers at the level .05 are reported in Table 4. Examination of reported powers indicates that the UIT performs better than others only when the sample sizes are equal, one of the parameters is maximum and only one of the parameters is minimum; see the cases of $(r_1, r_2, r_3) = (10, 10, 10)$, $(b_1, b_2, b_3) = (1, 1, .4)$ and (1,1,.3), $(r_1, \ldots, r_5) = (10, \ldots, 10)$,

Table 3 Monte Carlo estimates of the percentiles of Λ_h in (9) and those of $Q_h\chi^2_{2k}$.

	Percentiles				Percentiles		
(r_1, r_2, r_3)	99	95	90	$(r_1, \ldots, r_4)$	99	95	90
(3, 2, 3)	28.10 (27.70)	20.92 (20.74)	17.62 (17.54)	(3, 2, 3, 2)	36.73 (36.19)	28.52 (28.26)	24.69 (24.53)
(3, 3, 4)	24.10 (24.04)	18.04 (18.00)	15.24 (15.22)	(4, 3, 4, 5)	28.73 (28.55)	22.38 (22.30)	19.40 (19.35)
(5, 5, 5)	20.84 (20.86)	15.62 (15.63)	13.19 (13.21)	(5, 6, 7, 9)	25.37 (25.29)	19.78 (19.75)	17.16 (17.14)
(4, 5, 4)	21.80 (21.77)	16.29 (16.31)	13.77 (13.78)	(9, 8, 10, 9)	24.12 (24.12)	18.83 (18.84)	16.35 (16.35)
(5, 4, 5)	21.35 (21.32)	15.98 (15.97)	13.50 (13.50)	(10, …, 10)	23.79 (23.83)	18.60 (18.61)	16.14 (16.15)
(7, 7, 10)	19.26 (19.22)	14.42 (14.39)	12.17 (12.17)	(15, 10, 15, 10)	23.43 (23.45)	18.30 (18.31)	15.90 (15.89)
(10, 10, 10)	18.62 (18.61)	13.94 (13.94)	11.77 (11.78)	(15, 15, 15, 15)	23.06 (23.06)	18.03 (18.01)	15.65 (15.63)
(15, 15, 15)	17.94 (17.97)	13.46 (13.46)	11.37 (11.38)	(20, 15, 20, 15)	22.89 (22.88)	17.84 (17.87)	15.49 (15.51)
(10, 15, 20)	18.11 (18.09)	13.55 (13.55)	11.46 (11.46)	(20, 20, 20, 20)	22.67 (22.69)	17.73 (17.72)	15.38 (15.38)
(50, 50, 50)	17.11 (17.14)	12.83 (12.84)	10.85 (10.85)	(50, …, 50)	22.11 (22.06)	17.24 (17.23)	14.96 (14.95)
(100, 100, 100)	16.93 (16.98)	12.70 (12.71)	10.72 (10.75)	(100, …, 100)	21.86 (21.86)	17.09 (17.07)	14.82 (14.82)
$\chi^2_{3(k-1)}$ percentiles	16.81	12.59	10.64	$\chi^2_{3(k-1)}$ percentiles	21.67	16.92	14.68

Table 4 Powers of the iterative test (ITER), UIT and the LRT for testing the equality of the scale parameters.

	Sample Sizes												
	(10, 10, 10)			(10, 10, 20)				(10, 10, 10, 10, 10)			(5, 15, 10, 5, 15)		
(b_1, b_2, b_3)	ITER	UIT	LRT	ITER	UIT	LRT	$(b_1, \ldots, b_5)$	ITER	UIT	LRT	ITER	UIT	LRT
(1, 1, 1)	.050	.050	.050	.050	.050	.050	(1, 1, 1, 1, 1)	.050	.050	.050	.050	.050	.050
(1, .8, .6)	.121	.134	.138	.182	.108	.311	(1, .9, 1, .7, 1)	.082	.089	.091	.064	.084	.079
(1, .8, .5)	.203	.228	.228	.342	.196	.377	(1, .6, 1, .9, .6)	.153	.168	.193	.189	.047	.132
(1, .5, .5)	.265	.287	.310	.311	.293	.375	(1, .6, 1, .6,1)	.173	.190	.210	.197	.111	.193
(1, .4, .6)	.373	.364	.377	.377	.411	.392	(1, .6, 1, .5, .6)	.207	.253	.292	.185	.141	.190
(1, .4, .7)	.374	.365	.367	.375	.419	.409	(1, .4, 1, 1, 1)	.367	.419	.384	.505	.100	.438
(1, .4, .8)	.385	.388	.387	.392	.457	.703	(1, .4, 1, 1, .6)	.369	.410	.431	.462	.090	.347
(1, .4, .5)	.391	.395	.422	.409	.424	.504	(1, .5, 1, .3, .5)	.547	.620	.675	.339	.360	.432
(1, 1, .4)	.443	.463	.453	.703	.473	.614	(1, .6, 1, .2, .6)	.863	.879	.862	.399	.624	.524
(1, .4, .4)	.437	.473	.508	.504	.484	.709	(1, .2, 1, 1, 1)	.874	.945	.917	.954	.633	.964
(1, .3, .7)	.611	.615	.601	.614	.677	.917	(1, .2, 1, .3, 1)	.910	.949	.972	.959	.706	.978
(1, .4, .3)	.556	.627	.655	.709	.652	.937	(1, .2, .5, .2, 1)	.934	.959	.979	.952	.745	.973
(1, 1, .3)	.706	.721	.710	.917	.775	.922	(1, .2, .4, .2, 1)	.939	.957	.980	.962	.748	.973
(1, .4, .2)	.796	.851	.859	.937	.887	.995	(1, .2, .7, .2, 1)	.937	.966	.985	.948	.768	.979
(1, .3, .2)	.819	.870	.890	.922	.932	.896	(1, .2, 1, .2, 1)	.957	.981	.994	.964	.820	.990
(1, 1, .2)	.940	.941	.936	.995	.989	.969	(1, .2, 1, .2, .2)	.958	.978	.993	.948	.970	.990

$(b_1, \ldots, b_5) = (1, .4, 1, 1, 1)$, (1,.2,1,1,1). In these cases, the UIT has larger power than those of the other tests. However, for $(r_1, \ldots, r_5) = (5, 15, 10, 5, 15)$ and $(b_1, \ldots, b_5) = (1, .4, 1, 1, 1)$, the power of the UIT is much smaller than those of other two tests. Specifically, we see under the column of $(r_1, \ldots, r_5) = (5, 15, 10, 5, 15)$ the powers of UIT are much smaller than the powers of other two tests. Between the LRT and ITER, the ITER is more powerful than the LRT in some cases; see the powers for the case $(r_1, \ldots, r_5) = (5, 15, 10, 5, 15)$. However, the LRT has appreciably larger powers than the ITER over larger parameter space, and so the LRT maybe preferred to other two tests.

6.2. Power of the LRT for Location Parameters

To judge the power properties of the LRT for the equality of location parameters, we estimated the powers by Monte Carlo simulation and reported them in Table 5. The powers were estimated for $(b_1, b_2, b_3) = (3, 4, 5)$ and (2,2,2) and sample sizes $(n_1, n_2, n_3) = (15, 15, 15)$ and (20,20,20). Given a set of sample sizes, we evaluated the powers at $(r_1, r_2, r_3) = (5, 4, 5)$ and (12,11,12) and for values of (a_1, a_2, a_3) so that $a_1 = 4 > a_2 > a_3$. We first observe from Table 5 that the powers are increasing with increasing sample sizes. For example, see the powers in the columns of $(n_1, n_2, n_3) = (15, 15, 15)$ and $(r_1, r_2, r_3) = (5, 4, 5)$, and $(n_1, n_2, n_3) = (20, 20, 20)$ and $(r_1, r_2, r_3) = (5, 4, 5)$. We also observe that the powers for larger values of (r_1, r_2, r_3) are larger than those of smaller values of (r_1, r_2, r_3) while the sample sizes and other parameters are fixed. For example, see the powers in the columns of $(n_1, n_2, n_3) = (15, 15, 15)$, $(r_1, r_2, r_3) = (5, 4, 5)$ and (12,11,12); $(n_1, n_2, n_3) = (20, 20, 20)$, $(r_1, r_2, r_3) = (5, 4, 5)$ and (12,11,12). Finally, we also notice that the powers for the case of $(b_1, b_2, b_3) = (3, 4, 5)$ are smaller than corresponding powers for $(b_1, b_2, b_3) = (2, 2, 2)$. This is expected because the variance of an $\exp(a, b)$ distribution is b^2, and so the powers for the set of k populations with larger variances are expected to be smaller than those for populations with smaller variances while all other parameters and sample sizes are fixed. Thus, the LRT for the equality of location parameters possess all natural properties of an efficient test.

6.3. Powers of the Homogeneity Test

Powers of the LRT for testing $H_0 : (a_1, b_1) = \ldots = (a_k, b_k)$ were estimated for sample sizes $(n_1, n_2, n_3) = (15, 15, 15)$ and (20,20,20), and for each set of sample sizes, $(r_1, r_2, r_3) = (7, 8, 11)$ and (14,13,10). The estimated powers are reported in Table 6. On the basis powers in Table 6, we see that the power properties of the homogeneity test is very similar to the LRT for location parameters discussed in the preceding section. In particular, the power is increasing with increasing sample sizes while other parameters are fixed and also increasing with increasing values of r_i's while other values are fixed. For example, see the columns under $(n_1, n_2, n_3) = (15, 15, 15)$ $(r_1, r_2, r_3) = (7, 8, 11)$ and $(14, 13, 10)$ in Table 6. We also notice that the power is increasing with increasing disparities among location parameters and/or scale parameters. Thus, the LRT has all natural properties of an efficient test.

7. AN EXAMPLE

The data were collected from a placebo controlled randomized clinical trial of gamma interferon in CGD. The data were subsets of large data given in Appendix D of [1]. They represent elapsed time (in days) from randomization to diagnosis of a serious infection. The samples were taken from three hospitals with ID codes 238, 204 and 332. We applied Kolmogorov–Smirnov test for the exponential distribution by [8], and the test indicates that all three samples satisfy the model assumption of two-parameter exponential distribution.

Table 5 Powers of the LRT for equality of location parameters.

$(b_1, b_2, b_3) = (3, 4, 5)$					$(b_1, b_2, b_3) = (2, 2, 2)$			
(n_1, n_2, n_3)	(15, 15, 15)		(20, 20, 20)		(15, 15, 15)		(20, 20, 20)	
	(r_1, r_2, r_3)				(r_1, r_2, r_3)			
(a_1, a_2, a_3)	(5, 4, 5)	(12, 11, 12)	(5, 4, 5)	(12, 11, 12)	(5, 4, 5)	(12, 11, 12)	(5, 4, 5)	(12, 11, 12)
(4, 4, 4)	.049	.051	.050	.049	.051	.050	.050	.050
(4, 3.8, 4)	.072	.080	.095	.104	.164	.195	.270	.341
(4, 3.5, 3.5)	.140	.151	.188	.233	.308	.453	.444	.673
(4, 3.6, 4)	.166	.195	.268	.340	.515	.683	.741	.877
(4, 4, 3.6)	.176	.198	.401	.518	.532	.677	.731	.876
(4, 3, 3)	.315	.450	.448	.681	.674	.942	.823	.994
(4, 4, 3.5)	.249	.301	.398	.513	.689	.838	.879	.952
(4, 3, 3.5)	.459	.618	.679	.839	.915	.974	.983	.996
(4, 4, 3)	.699	.836	.875	.953	.985	.996	.998	.999

Table 6 Powers of the homogeneity test.

(n_1, n_2, n_3)		(15, 15, 15)		(20, 20, 20)	
		(r_1, r_2, r_3)			
(a_1, a_2, a_3)	(b_1, b_2, b_3)	(7, 8, 11)	(14, 13, 10)	(7, 8, 11)	(14, 13, 10)
(2, 2, 2)	(3, 3, 3)	.049	.048	.049	.048
(2, 2, 2)	(3, 3, 2)	.083	.094	.084	.097
(2, 1.8, 2)	(3, 3, 3)	.094	.090	.129	.133
(2, 1.8, 2)	(3, 3, 2)	.141	.149	.195	.209
(2, 1.8, 1.7)	(3, 3, 2)	.202	.218	.285	.301
(2, 1.5, 1.7)	(3, 3, 2)	.314	.336	.506	.512
(2, 1.5, 1.7)	(3, 3, 1.5)	.498	.520	.680	.699
(2, 1.5, 1.7)	(3, 3, 1)	.792	.821	.902	.909
(2, 1.3, 1.7)	(3, 3, 1)	.920	.928	.974	.975
(2, 1.3, 1.7)	(3, 1.5, 1)	.982	.989	.998	.999

Table 7 Elapsed times.

Hospitals	Time to Infection in Days										MLEs $(\hat{a},\hat{b})$
1	253	294	19	373	334	238	118	240	99	167	(19, 194.5)
2	373	26	152	241	322	350	211	307	82		
	114	337	18	267	104						(18, 189.4)
3	146	188	304	91	121	203	264	236	207		(91, 104.6)

Notice that the samples are uncensored with $(n_1, n_2, n_3) = (10, 14, 9)$. The MLEs are $\hat{a}_i = X_{i(1)}$ and $\hat{b}_i = \overline{X}_i - X_{i(1)}$, $i = 1, 2, 3$, and the MLEs for the three samples are reported in Table 7.

For testing $H_0 : a_1 = a_2 = a_3$ vs. $H_a : a_i \neq a_j$ for some $i \neq j$, the LRT statistic $\Lambda_a = 9.63$ and the 95th percentile of the null distribution is 10.51. To find the approximate 95th percentile, we evaluated $Q_a = E(\Lambda_a)/(2k-2) = 1.105$ and $1.105 \times \chi^2_{4;.95} = 1.105 \times 9.488 = 10.48$. Notice that the exact and the approximate percentiles are practically the same. Furthermore, we estimated the p-value using Algorithm 2 with 1000000 runs as .069 and the p-value based on the $Q_a\chi^2_4$ distribution is also .069. Thus, at 5% level, the location parameters are not significantly different.

For testing $H_0 : b_1 = b_2 = b_3$, the statistic Λ_b is 2.17 and the p-value based on Algorithm 1 is .385. If we use the level of significance .05, then the exact critical value is 6.83 and $Q_b\chi^2_{2;.95} = 1.138 \times 5.99$ is 6.82, which is very close to the exact one. To apply the UIT, we estimated the 5% critical value as 3.056, and the test statistic $\max\{\hat{b}_1, \hat{b}_2, \hat{b}_3\}/\min\{\hat{b}_1, \hat{b}_2, \hat{b}_3\} = 194.5/104.6 = 1.859$. To apply the iterative test procedure, we calculated the individual test statistics $F_2 = .944$ and $F_3 = .563$. We chose the levels for the individual test as $\eta_2 = \eta_3 = \eta_4 = .01266$ so that $\alpha = 1 - (1 - 2 \times .01266)^2 = .05$. For these levels, the (lower, upper) critical values are $(.383, 2.84)$ for F_2 and $(.349, 2.36)$ for F_3. Note that both test statistics F_2 and F_3 fall in their corresponding acceptance intervals, and so the null hypothesis of equality of scale parameters is accepted. Thus, all three tests indicate that $H_0 : b_1 = b_2 = b_3$ is tenable.

For testing $H_0 : (a_1, b_1) = (a_2, b_2) = (a_3, b_3)$, the statistic Λ_h in (23) is 9.68 and the p-value based on (24) is .187. If we use the level of significance .05, then the exact critical value is 13.93; the approximate critical value $Q_h\chi^2_{6;.95} = 1.101 \times 12.59 = 13.86$. Thus, homogeneity of exponential distributions is tenable. This means that all three samples may be pooled and the pooled sample can be modeled by a single two-parameter exponential distribution.

8. CONCLUDING REMARKS

Although several tests for comparing parameters of several two-parameter exponential distributions were proposed in the literature, none of them is based on the likelihood approach. As the sample space of a two-parameter exponential distribution depends on an unknown parameter, it does not satisfy all the regularity conditions. So an LRT statistic for comparing parameters does not have the asymptotic chi-square distribution with the degrees of freedom determined by the difference between the dimensions of the parameter spaces under the alternative and null hypotheses. We have shown in this article that the null distributions of the LRT statistics for all three problems do not depend on any unknown parameters, and so the LRTs are exact. However, calculation of the percentiles of the LRT statistics involves simulation. Even though calculation of percentiles based on Monte Carlo simulation is not difficult, we provided closed-form approximate chi-square distributions for all three problems. These approximate chi-square null distributions are not only simple, but also very accurate even for small samples. These approximate null distributions maybe warranted in situations where the number of distributions to be compared is large.

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APPENDIX

Let $Z_i = W_i + n_i(U_i - U_{(1)})$, $\theta_i = E(Z_i)$, $i = 1, .., k$ and $g(x) = \ln(x)$. Recall that W_i's and U_i's are independent with

$$W_i \sim \frac{1}{2}\chi^2_{2r_i-2} \quad \text{and} \quad U_i \sim \frac{1}{2n_i}\chi^2_2, \ \ i = 1, \dots, k.$$

To find an approximation to $E\ln(Z_i)$, we shall use the result (see Section 2.9, [9]) that

$$E\ln(Z_i) \approx \ln(\theta_i) + \operatorname{Var}(Z_i)\frac{g''(\theta_i)}{2}.$$

Recall that $n_iU_i \sim \chi^2_2/2$, or n_iU_i follows a standard exponential distribution. So

$$P(U_{(1)} \le u) = 1 - \prod_i^k P(n_iU_i \ge n_iu) = 1 - \exp\left(-\sum_{i=1}^k n_iu\right),$$

which is the distribution function of an exponential distribution with mean $1/\sum_{i=1}^k n_i$. Thus,

$$E(U_{(1)}) = \frac{1}{\sum_{i=1}^k n_i} \quad \text{and} \quad \operatorname{Var}(U_{(1)}) = \frac{1}{\left(\sum_{i=1}^k n_i\right)^2} \tag{A.1}$$

Using the above expectation, we have

$$E(Z_i) = E(W_i) + n_iE(U_i - U_{(1)}) = r_i - \frac{n_i}{\sum_{j=1}^k n_j}, \ i = 1, \dots, k.$$

To find the variance,

$$\operatorname{Var}(Z_i) = \operatorname{Var}(W_i) + n_i^2\left[\operatorname{Var}(U_i) + \operatorname{Var}(U_{(1)}) - 2\operatorname{Cov}(U_i, U_{(1)})\right]. \tag{A.2}$$

To find the $\operatorname{Cov}(U_i, U_{(1)})$, we find the joint distribution of U_i and $U_{(1)}$ as

$$\begin{aligned} F_{U_i,U_{(1)}}(u, v) &= P\left(U_i \le u, U_{(1)} \le v\right) \\ &= P(U_i \le u) - P\left(U_i \le u, U_{(1)} \ge v\right) \\ &= P(U_i \le u) - P(U_i \le u, U_1 \ge v, \dots, U_k \ge v) \\ &= P(U_i \le u) - P(v \le U_i \le u)\prod_{j\ne i} P\left(n_jU_j \ge n_jv\right) \\ &= (1 - e^{-n_iu}) - (e^{-n_iv} - e^{-n_iu})\, e^{-\sum_{j\ne i} n_jv}. \end{aligned}$$

By taking the derivative with respect to (u, v), we find the joint density as

$$f_{U_i,U_{(1)}}(u, v) = n_ie^{-n_iu}\left(\sum_{j\ne i} n_j\right)e^{-\sum_{j\ne i} n_jv},$$

which shows that U_i and $U_{(1)}$ are independent. Using this fact in (A.2), we find

$$\operatorname{Var}(Z_i) = r_i + \frac{n_i^2}{\left(\sum_{j=1}^k n_j\right)^2}.$$

Thus,

$$\begin{aligned} E\ln(Z_i) &\approx \ln(\theta_i) - \frac{\operatorname{Var}(Z_i)}{2\theta_i^2} \\ &= \ln(\theta_i) - \frac{1}{2\theta_i^2}\left(r_i + \frac{n_i^2}{\left(\sum_{j=1}^k n_j\right)^2}\right), \end{aligned}$$

where $\theta_i = E(Z_i) = r_i - \frac{n_i}{\sum_{j=1}^{k} n_j}$, $i = 1, \ldots, k$. Using this approximation, we see that

$$\begin{aligned} E(\Lambda_a) &= -E\left[2\sum_{i=1}^{k} r_i \ln\left(\frac{\chi^2_{2r_i-2}/2}{Z_i}\right)\right] \\ &= -2\sum_{i=1}^{k} r_i \left[\psi(r_i - 1) - E\ln(Z_i)\right] \\ &\approx -2\sum_{i=1}^{k} r_i \left[\psi(r_i - 1) - \ln(\theta_i) + \frac{1}{2\theta_i^2}\left(r_i + \frac{n_i^2}{\left(\sum_{j=1}^{k} n_j\right)^2}\right)\right]. \end{aligned} \quad \text{(A.3)}$$