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Construction of Some Circular Regular Graph Designs in Blocks of Size Four Using Cyclic Shifts

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ABSTRACT

Circular regular graph designs play an important role in the design of experiments where most of the balanced incomplete block designs require a large number of blocks. In this article, circular regular graph designs are constructed in blocks of size four through cyclic shifts. Without studying the complete design, some standard properties of the designs can be observed only through the sets of shifts. Therefore, method of cyclic shifts has an edge over existing methods.

1. INTRODUCTION

Block designs are used in experimental planning with the purpose of maximizing the information extracted from a given number of experiments. If homogeneous blocks of size k are available to accommodate all the k treatments, a randomized complete block design is preferred. Incomplete block designs are used in situations where all the treatment combinations could not be run in each block. The most popular incomplete block designs are balanced incomplete block designs (BIBDs) introduced by Yates [1]. BIBDs compare all treatments pairs with equal precision. As the class of BIBDs do not fit for many experimental situations because often these designs require a large number of replications, to overcome this Bose and Nair [2] introduced a class of binary, equireplicate and proper designs called partially balanced incomplete block designs (PBIBDs). Bose [3] established the relation between PBIBDs and strongly regular graphs. Bose and Shimamoto [4] are first to introduce the concept of association scheme in PBIBDs. Bose [5] used the graph theoretic method for the study of association schemes of PBIBDs and also shown that strongly regular graph emerges from PBIBD with two associate class.

A PBIBD is obtained by identifying the v treatments with the v objects of an association scheme arranging into b blocks satisfying the following conditions:

- Each block contains k treatments.
- Each treatment occurs in r blocks.
- If two treatments are ith associates, they occur together in λ_i blocks.
- Each treatment has exactly n_i ith associates.
- Given any two treatments which are ith associates, the number of treatments common to the jth associates of the first, and the kth associates of the second is and is independent of the pair of treatments.

An associate class is a set of treatment pairs where each pair from the set occur together the same number of times, λ_i. Regular graph design (RGD) is an important class of PBIB designs with two association scheme. A RGD (v, k, r) is a collection of blocks of size k on a v-set (with no restriction on repeated blocks) such that every element occurs in r blocks and any pair of objects occur together in either λ_1 or λ_2 blocks,

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where λ_1 is some constant and $\lambda_2 = \lambda_1 + 1$ RGDs were introduced by John and Mitchell [6]. Kreher *et al.* [7] discussed the existence of resolvable RGDs with block size 4, 8, 12 and 16 points. A design is resolvable if its set of blocks can be partitioned into r parallel classes or resolutions such that each element occurs in exactly one block in each class. Barandag *et al.* [8] considered the association scheme which is related to the flag algebra of a BIBD, with $\lambda = 1$. By finding a suitable equivalence of this scheme, they constructed a 2-class association scheme. Moreover, each 2-class association scheme is equivalent to a strongly regular graph. Cakiroglu [9] constructed optimal RGDs by P^i_{jk}adding the blocks of a BIBD repeatedly to the original design and presented the best RGDs for $v \leq 20$, $k \leq 10$ and replication $r \leq 10$.

Nair and Rao [10] have developed a set of sufficient combinatorial conditions which lead to construction of confounded designs. A catalogue of different PBIB on two associate class designs can be found in Clatworthy [11]. Cheng and Wu [12] constructed nearly BIBDs. Wallis [13] discussed measures of optimality of RGDs from a combinatorial viewpoint. Various properties of these designs were also discussed. Kumar [14] has given the construction of PBIBDs through unreduced BIBDs. Waliker *et al.* [15] have established the relation between minimum dominating sets of a graph with the blocks of PBIBDs. Using method of cyclic shifts, Yasmin *et al.* [16] constructed some classes of BIBDs. In this study, RGDs are constructed in blocks of size four through method of cyclic shifts.

2. METHOD OF CYCLIC SHIFTS

Method of cyclic shifts introduced by Iqbal [17] is simplified here only for BIBDs, PBIBDs and RGDs. In this construction, v treatments are labeled as $0, 1, 2, \ldots, v-1$. For further detail, see Yasmin *et al.* [16].

Let $S_j = \left[q_{j1}, q_{j2}, \ldots, q_{j(k-1)}\right]$ be set(s) of shifts where $1 \leq q_{ji} \leq v-1$. A design is BIBD if each element of S^*_j contains all elements $1, 2, \ldots, v-1$, equal number of times, say, λ. Where $S_j* = \left[q_{j1}, q_{j2}, \ldots, q_{j(k-1)}, \left(q_{j1}+q_{j2}\right), \left(q_{j2}+q_{j3}\right), \ldots, \left(q_{j_{(k-2)}}+q_{j_{(k-1)}}\right), \left(q_{j1}+q_{j2}+q_{j3}\right), \left(q_{j2}+q_{j3}+q_{j4}\right), \ldots, \left(q_{j(k-3)}+q_{j(k-2)}+q_{j(k-1)}\right), \ldots, \left(q_{j1}+q_{j2}+\ldots+q_{j(k-1)}\right), v-q_{j1}, v-q_{j2}, \ldots, v-q_{j(k-1)}, v-\left(q_{j1}+q_{j2}\right), v-\left(q_{j2}+q_{j3}\right), \ldots, v-\left(q_{j(k-2)}+q_{j(k-1)}\right), v-\left(q_{j1}+q_{j2}+q_{j3}\right), v-\left(q_{j2}+q_{j3}+q_{j4}\right), \ldots, v-\left(q_{j(k-3)}+q_{j(k-2)}+q_{j(k-1)}\right), \ldots, v-(q_{j1}+q_{j2}+\ldots+q_{j(k-1)})\right]$. If λ has two values as λ_1 and $\lambda_2 = \lambda_1 + 1$ then it is RGD (PBIBD with two associate class).

Example 2.1

RGD is constructed from the set of shifts [2, 1, 4] for $v = 9$ and $k = 4$ with $\lambda_1 = 1$ and $\lambda_2 = 2$.

Here S = [2, 1, 4], $v = 9$ and $k = 4$ then $S^* = [2, 1, 4, 3, 5, 7, 7, 8, 5, 6, 4, 2]$ contains each of 1, 2, …, 8 either once or twice, according to the method of cyclic shifts, it is a RGD. Now we explain the procedure to complete the design from the given set of shifts [2, 1, 4].

Consider 0, 1, …, and 8 as the elements of first unit for all blocks. To get the elements of second units for all blocks, add 2 (mod 9) to the each element of first unit for all blocks. To get the elements of third units for all blocks, add 1 (mod 9) to the each element of second unit for all blocks. Similarly add 4 (mod 9) to get the elements of fourth units for all blocks.

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9
0	1	2	3	4	5	6	7	8
2	3	4	5	6	7	8	0	1
3	4	5	6	7	8	0	1	2
7	8	0	1	2	3	4	5	6

3. CONSTRUCTION OF RGDS IN BLOCKS OF SIZE FOUR

In this section, RGDs are constructed in blocks of size four through following i sets of cyclic shifts.

$$\mathbf{S}_j = \left[s_{j1}, s_{j2}, s_{j3}\right]; \quad j = 1, 2, \ldots, i.$$

Such that

i. $1 \leq s_{j1}, s_{j2}, s_{j3} \leq v-1$,

ii. $\left(s_{j1}+s_{j2}+s_{j3}\right) \bmod v \neq 0$,

iii. For $\lambda_1 = 0$ and $\lambda_2 = 1$, each of $1, 2, \ldots, v-1$ appears once or no time in S^*.

iv. For $\lambda_1 = 1$ and $\lambda_2 = 2$, each of $1, 2, \ldots, v-1$ appears once or twice in S*.

v. $S^* = \{s_{j1}, s_{j2}, s_{j3}, s_{j1}+s_{j2}, s_{j2}+s_{j3}, s_{j1}+s_{j2}+s_{j3}, v-s_{j1}, v-s_{j2}, v-s_{j3},\ v-(s_{j1}+s_{j2}), v-(s_{j2}+s_{j3}), v-(s_{j1}+s_{j2}+s_{j3}) \bmod v$

In these designs, any pair of treatments which are first associates occur in exactly λ_1 blocks and second associates occur together in exactly λ_2 blocks, together. RGDs have eight parameters (v, b, r, k, λ_1, λ_2, n_1, n_2) of kind I and six parameters ($p^i_{jk}, i, j, k = 1, 2$) of kind II. The parameters of kind II may be arranged in the form of two symmetric matrices (P-matrices).

$$P_1 = \begin{pmatrix} p^1_{11} & p^1_{12} \\ p^1_{21} & p^1_{22} \end{pmatrix} \text{ and } P_2 = \begin{pmatrix} p^2_{11} & p^2_{12} \\ p^2_{21} & p^2_{22} \end{pmatrix}$$

RGDs constructed in these series are cyclic, therefore, general expression of P matrices can be written as

$$P_1 = \begin{pmatrix} \alpha & n_1-\alpha-1 \\ n_1-\alpha-1 & n_2-n_1+\alpha+1 \end{pmatrix} \text{ and } P_2 = \begin{pmatrix} \beta & n_1-\beta \\ n_1-\beta & n_2-n_1+\beta-1 \end{pmatrix}$$

where $p^1_{11} = \alpha$ is the number of treatments common to first associates of two treatments. Where these two treatments are first associates of each other. Similarly, $p^2_{11} = \beta$ is the number of treatments common to first associates of two treatments. Which these two treatments are second associates of each other.

Series 3.1: RGDs can be constructed for $v = 6w+3$, $k = 4$, $b = vw$, $r = 4w$ with $n_1 = 4$, $n_2 = v-5$, $\lambda_1 = 1$ and $\lambda_2 = 2$ through w sets of shifts. P-matrices for these designs are

$$P_1 = \begin{pmatrix} \alpha & 3-\alpha \\ 3-\alpha & 10-v+\alpha \end{pmatrix} \text{ and } P_2 = \begin{pmatrix} \beta & 4-\beta \\ 4-\beta & 8-\mathrm{v}+\beta \end{pmatrix}$$

Example 3.1

Set of shifts [2, 1, 4] provides RGD for $v = 9$, $k = 4$ with $n_1 = n_2 = 4$, $\lambda_1 = 1$ and $\lambda_2 = 2$.

P-matrices for this design are $P_1 = \begin{pmatrix} 0 & 3 \\ 3 & 1 \end{pmatrix}$ and $P_2 = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$

For the convenience of readers, first and second associates of all treatments for this design are arranged in the following table:

Treatment No.	First Associates	Second Associates
0	1, 3, 6, 8	2, 4, 5, 7
1	0, 2, 4, 7	3, 5, 6, 8
2	1, 3, 5, 8	0, 4, 6, 7
3	0, 2, 4, 6	1, 5, 7, 8
4	1, 3, 5, 7	0, 2, 6, 8
5	2, 4, 6, 8	0, 1, 3, 7
6	0, 3, 5, 7	1, 2, 4, 8
7	1, 4, 6, 8	0, 2, 3, 5
8	0, 2, 5, 7	1, 3, 4, 6

Here, $n_1 = n_2 = 4$ and the parameters of kind II are

$p^1_{11} = \alpha = 0$, $\quad p^2_{11} = \beta = 3$, $\quad p^1_{12} = p^1_{21} = n_1 - \alpha - 1 = 4 - 0 - 1 = 3$,

$p^1_{22} = n_2 - n_1 + \alpha + 1 = 4 - 4 + 0 + 1 = 1$, $\quad p^2_{12} = p^2_{21} = n_1 - \beta = 4 - 3 = 1$,

$p^2_{22} = n_2 - n_1 + \beta - 1 = 4 - 4 + 3 - 1 = 2$

$P_1 = \begin{pmatrix} 0 & 3 \\ 3 & 1 \end{pmatrix}$, $\quad P_2 = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$

Consider two treatments that are first associates of each other, treatment 0 and 1 are first associates; there is no treatment common in first associates of treatment 0 and 1, hence $p^1_{11} = \alpha = 0$. Similarly, there are three treatments 3, 6, 8 (or 2, 4, 7) that are common in first associates of treatment 0 and second associates of treatment 1 (or common in second associates of treatment 0 and first associates of treatment 1),

then $p^1_{12} = p^1_{21} = 3$. And there is only one treatment that is common in second associates of treatment 0 and 1 which is 5, therefore, $p^1_{22} = 1$. Similarly, P_2-matrix can be constructed.

Catalogue of RGDs constructed under Series 3.1 is presented in Table A.1 of Appendix A.

Series 3.2: RGDs can be constructed for $v = 12w + 5, k = 4, b = vw, r = 4w$ with $n_1 = 4, n_2 = v - 5, \lambda_1 = 1$ and $\lambda_2 = 0$ through w sets of shifts. P-matrices for these designs are

$$P_1 = \begin{pmatrix} \alpha & 3-\alpha \\ 3-\alpha & 10-v+\alpha \end{pmatrix} \text{ and } P_2 = \begin{pmatrix} \beta & 4-\beta \\ 4-\beta & 8-v+\beta \end{pmatrix}$$

Example 3.2

Set of shifts [1, 2, 4] provides RGD for $v = 7, k = 4, b = 17, r = 4$ with $n_1 = 4, n_2 = 12, \lambda_1 = 1$ and $\lambda_2 = 0$.

P-matrices for this design are $P_1 = \begin{pmatrix} 0 & 3 \\ 3 & 9 \end{pmatrix}$ and $P_2 = \begin{pmatrix} 1 & 3 \\ 3 & 8 \end{pmatrix}$

Catalogue of RGDs constructed under Series 3.2 is presented in Table A.2 of Appendix A.

Series 3.3: RGDs can be constructed for $v = 12w - 1, k = 4, b = vw, r = 4w$ with $n_1 = v - 3, n_2 = 2, \lambda_1 = 1$ and $\lambda_2 = 2$ through w sets of shifts. P-matrices for these designs are

$$P_1 = \begin{pmatrix} \alpha & v-\alpha-4 \\ v-\alpha-4 & \alpha-v+6 \end{pmatrix} \text{ and } P_2 = \begin{pmatrix} \beta & v-\beta-3 \\ v-\beta-3 & \beta-v+4 \end{pmatrix}$$

Example 3.3

Set of shifts [1, 2, 4] provides RGD for $v = 11, k = 4, b = 11, r = 4$ with $n_1 = 8, n_2 = 2, \lambda_1 = 1$ and $\lambda_2 = 2$.

P-matrices for this design are $P_1 = \begin{pmatrix} 5 & 2 \\ 2 & 0 \end{pmatrix}$ and $P_2 = \begin{pmatrix} 7 & 1 \\ 1 & 0 \end{pmatrix}$

Following are RGDs for v = 11 and 23

Parameters							Sets of Shifts
v	b	r	n_1	n_2	α	β	
11	11	4	8	2	5	7	[1, 2, 4]
23	46	8	20	2	17	19	[1, 2, 7] + [5, 6, 8]

Series 3.4: RGDs can be constructed for $v = 12w + 5, k = 4, b = v(w + 1), r = 4(w + 1)$ with $n_1 = v - 9, n_2 = 8, \lambda_1 = 1$ and $\lambda_2 = 2$ through $w + 1$ sets of shifts. P-matrices for these designs are

$$P_1 = \begin{pmatrix} \alpha & v-\alpha-10 \\ v-\alpha-10 & \alpha-v+18 \end{pmatrix} \text{ and } P_2 = \begin{pmatrix} \beta & v-\beta-9 \\ v-\beta-9 & \beta-v+16 \end{pmatrix}$$

Example 3.4

Sets of shifts [1, 2, 4] and [5, 8, 3] provide RGD for $v = 17, k = 4, b = 34, r = 8$ with $n_1 = 8, n_2 = 8, \lambda_1 = 1$ and $\lambda_2 = 2$.

P-matrices for this design are $P_1 = \begin{pmatrix} 4 & 3 \\ 3 & 5 \end{pmatrix}$ and $P_2 = \begin{pmatrix} 3 & 5 \\ 5 & 2 \end{pmatrix}$

Catalogue of RGDs constructed under Series 3.4 is presented in Table A.3 of Appendix A.

Series 3.5: RGDs can be constructed for $v = 12w - 4, k = 4, b = vw, r = 4w$ with $n_1 = v - 6, n_2 = 5, \lambda_1 = 1$ and $\lambda_2 = 2$ through w sets of shifts. P-matrices for these designs are

$$P_1 = \begin{pmatrix} \alpha & v-\alpha-7 \\ v-\alpha-7 & \alpha-v+12 \end{pmatrix} \text{ and } P_2 = \begin{pmatrix} \beta & v-\beta-6 \\ v-\beta-6 & \beta-v+10 \end{pmatrix}$$

Example 3.5

Set of shifts [2, 1, 4] provides RGD for $v = 8, k = 4, b = 8, r = 4$ with $n_1 = 2, n_2 = 5, \lambda_1 = 1$ and $\lambda_2 = 2$.

P-matrices for this design are $P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 4 \end{pmatrix}$ and $P_2 = \begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix}$

Catalogue of RGDs constructed under Series 3.5 is presented in Table A.4 of Appendix A.

Series 3.6: RGDs can be constructed for $v = 12w + 2, k = 4, b = v(w+1), r = 4(w+1), n_1 = v - 12$ with $n_2 = 11, \lambda_1 = 1$ and $\lambda_2 = 2$ through $w + 1$ sets of shifts. P-matrices for these designs are

$$P_1 = \begin{pmatrix} \alpha & v-\alpha-7 \\ v-\alpha-7 & \alpha-v+12 \end{pmatrix} \text{ and } P_2 = \begin{pmatrix} \beta & v-\beta-6 \\ v-\beta-6 & \beta-v+10 \end{pmatrix}$$

Example 3.6

Sets of shifts [1, 2, 5] and [4, 1, 3] provide RGD for $v = 14, k = 4, b = 28, r = 8$ with $n_1 = 2, n_2 = 11, \lambda_1 = 1$ and $\lambda_2 = 2$.

P-matrices for this design are $P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 10 \end{pmatrix}$ and $P_2 = \begin{pmatrix} 0 & 2 \\ 2 & 8 \end{pmatrix}$

Catalogue of RGDs constructed under Series 3.6 is presented in Table A.5 of Appendix A.

Series 3.7: RGDs can be constructed for $v = 12w - 2, k = 4, b = vw, r = 4w$ with $n_1 = v - 4, n_2 = 3, \lambda_1 = 1$ and $\lambda_2 = 2$ through w sets of shifts. P-matrices for these designs are

$$P_1 = \begin{pmatrix} \alpha & v-\alpha-5 \\ v-\alpha-5 & \alpha-v+8 \end{pmatrix} \text{ and } P_2 = \begin{pmatrix} \beta & v-\beta-4 \\ v-\beta-4 & \beta-v+6 \end{pmatrix}$$

Example 3.7

Set of shifts [2, 1, 4] provides RGD for $v = 10, k = 4, b = 10, r = 4$ with $n_1 = 6, n_2 = 3, \lambda_1 = 1$ and $\lambda_2 = 2$.

P-matrices for this design are $P_1 = \begin{pmatrix} 2 & 3 \\ 3 & 0 \end{pmatrix}$ and $P_2 = \begin{pmatrix} 4 & 2 \\ 2 & 0 \end{pmatrix}$

Following are RGDs for v = 10 and 22:

Parameters							Sets of Shifts
v	b	r	n_1	n_2	α	β	
10	10	4	6	3	2	4	[2, 1, 4]
22	44	8	18	3	14	16	[4, 1, 7] and [2, 6, 3]

Series 3.8: RGDs can be constructed for $v = 12w + 4, k = 4, b = v(w+1), r = 4(w+1)$ with $n_1 = v - 10, n_2 = 9, \lambda_1 = 1$ and $\lambda_2 = 2$ through $w + 1$ sets of shifts. P-matrices for these designs are

$$P_1 = \begin{pmatrix} \alpha & v-\alpha-11 \\ v-\alpha-11 & \alpha-v+20 \end{pmatrix} \text{ and } P_2 = \begin{pmatrix} \beta & v-\beta-10 \\ v-\beta-10 & \beta-v+18 \end{pmatrix}$$

Example 3.8

Sets of shifts [1, 2, 4] and [5, 8, 9] provide RGD for $v = 16, k = 4, b = 32, r = 8$, with $n_1 = 6, n_2 = 9, \lambda_1 = 1$ and $\lambda_2 = 2$.

P-matrices for this design are $P_1 = \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}$ and $P_2 = \begin{pmatrix} 2 & 4 \\ 4 & 4 \end{pmatrix}$

Catalogue of RGDs constructed under Series 3.8 is presented in Table A.6 of Appendix A.

Series 3.9: RGDs can be constructed for $v = 12w + 6, k = 4, b = v(w+1), r = 4(w+1)$ with $n_1 = v - 8, n_2 = 7, \lambda_1 = 1$ and $\lambda_2 = 2$ through $w + 1$ sets of shifts. P-matrices for these designs are

$$P_1 = \begin{pmatrix} \alpha & v-\alpha-9 \\ v-\alpha-9 & \alpha-v+16 \end{pmatrix} \text{ and } P_2 = \begin{pmatrix} \beta & v-\beta-8 \\ v-\beta-8 & \beta-v+14 \end{pmatrix}$$

Example 3.9

Sets of shifts [1, 2, 4] and [5, 8, 9] provide RGD for $v = 18, k = 4, b = 36, r = 8$ with $n_1 = 10, n_2 = 7, \lambda_1 = 1$ and $\lambda_2 = 2$.

P-matrices for this design are $P_1 = \begin{pmatrix} 3 & 6 \\ 6 & 1 \end{pmatrix}$ and $P_2 = \begin{pmatrix} 6 & 4 \\ 4 & 2 \end{pmatrix}$

Catalogue of RGDs constructed under Series 3.9 is presented in Table A.7 of Appendix A

Following are some more RGDs for $k = 4$.

Design 1. Set of shifts [1, 2, 4] provides RGD for $v = 12, k = 4, b = 12, r = 4$ with $n_1 = 10, n_2 = 1, \lambda_1 = 1$ and $\lambda_2 = 2$.

P-matrices for this design are $P_1 = \begin{pmatrix} 8 & 1 \\ 1 & 0 \end{pmatrix}$ and $P_2 = \begin{pmatrix} 10 & 0 \\ 0 & 0 \end{pmatrix}$

Design 2. Set of shifts [2, 8, 1] provides RGD for $v = 14, k = 4, b = 14, r = 4$ with $n_1 = 1, n_2 = 12, \lambda_1 = 1$ and $\lambda_2 = 0$.

P-matrices for this design are $P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 12 \end{pmatrix}$ and $P_1 = \begin{pmatrix} 0 & 1 \\ 0 & 10 \end{pmatrix}$

Design 3. Set of shifts [2, 3, 4] provides RGD for $v = 15, k = 4, b = 15, r = 4$ with $n_1 = 2, n_2 = 12, \lambda_1 = 1$ and $\lambda_2 = 0$.

P-matrices for this design are $P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 11 \end{pmatrix}$ and $P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 10 \end{pmatrix}$

Design 4. Sets of shifts [3, 4, 5] and [21, 19, 25] provide RGD for $v = 27, k = 4, b = 54, r = 4$ with $n_1 = 24, n_2 = 2, \lambda_1 = 1$ and $\lambda_2 = 0$.

P-matrices for this design are $P_1 = \begin{pmatrix} 23 & 1 \\ 1 & 0 \end{pmatrix}$ and $P_1 = \begin{pmatrix} 22 & 1 \\ 1 & 0 \end{pmatrix}$

In Appendix B, designs constructed in this article are compared with the RGDs already available in literature.

4. CONCLUSION

Because of the importance of RGDs, it is much needed to have a comprehensive list/catalogues of this class of designs. Therefore, RGDs have been constructed in this article for blocks of size four through method of cyclic shifts. Nine series have been proposed along with some individual designs. Proposed designs have also been compared with the designs constructed by Bose *et al.* [18], John *et al.* [19] and Clatworthy [11]. Our proposed designs are new and have the efficiency greater than or equal to that of existing designs.

CONFLICT OF INTEREST

Certified that there is no conflict of interest.

AUTHORS' CONTRIBUTIONS

Certified that all the four authors contributed almost equally. Rashid Ahmed and Muhammad Jamil constructed RGDs for this article. Farrukh Shehzad and H. M. Kashif Rasheed calculated efficiencies for the comparison of these designs with those of existing designs.

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APPENDIX A

Table A.1 Catalog of regular graph designs (RGDs) under Series 3.1 with $\lambda_1 = 1$ and $\lambda_2 = 2$.

v	r	n_1	n_2	α	β	Sets of Shifts
9	4	4	4	0	3	[2, 1, 4]
15	8	4	10	0	1	[1, 2, 4] + [2, 3, 4]
21	12	4	16	0	0	[1, 5, 3] + [1, 10, 4] + [2, 3, 4]
27	16	4	22	0	0	[1, 5, 6] + [2, 8, 9] + [4, 3, 13] + [1, 3, 9]
33	20	4	28	0	0	[2, 8, 3] + [1, 5, 12] + [4, 7, 9] + [4, 14, 5] + [1, 6, 2]
39	20	4	34	0	0	[2, 8, 3] + [1, 5, 12] + [4, 7, 9] + [4, 14, 5] + [3, 12, 2] + [8, 1, 6]
45	28	4	40	0	0	[1, 9, 5] + [2, 1, 19] + [2, 10, 15] + [3, 8, 16] + [4, 13, 6] + [5, 7, 9] + [6, 7, 4]
51	32	4	46	0	0	[1, 10, 12] + [3, 4, 14] + [5, 4, 16] + [6, 2, 13] + [12, 5, 19] + [1, 8, 2] + [13, 7, 17] + [19, 3, 23]

Table A.2 Catalog of regular graph designs (RGDs) under Series 3.2 with $\lambda_1 = 0$ and $\lambda_2 = 1$.

v	r	n_1	n_2	α	β	Sets of Shifts
17	4	4	12	0	1	[1, 2, 4]
29	8	4	24	0	1	[1, 2, 8] + [4, 5, 7]
41	12	4	36	0	0	[1, 5, 4] + [2, 11, 7] + [14, 3, 16]
53	16	4	48	0	0	[1, 9, 12] + [2, 15, 13] + [3, 4, 20] + [5, 6, 8]
65	20	4	60	0	1	[1, 9, 12] + [2, 15, 13] + [3, 4, 20] + [6, 5, 14] + [18, 16, 23]
77	24	4	72	0	0	[2, 10, 25] + [3, 11, 22] + [4, 5, 23] + [8, 21, 17] + [15, 1, 18] + [20, 6, 24]
89	28	4	84	0	0	[1, 2, 40] + [4, 5, 30] + [7, 8, 18] + [10, 11, 16] + [12, 13, 23] + [14, 20, 24] + [22, 6, 32]

Table A.3 Catalog of regular graph designs (RGDs) under Series 3.4 with $\lambda_1 = 1$ and $\lambda_2 = 2$.

v	r	n_1	n_2	α	β	Sets of Shifts
17	8	8	8	2	4	[1, 2, 4] + [5, 8, 1]
29	12	20	8	16	13	[1, 5, 10] + [2, 9, 3] + [8, 4, 7]
41	16	32	8	26	27	[1, 5, 4] + [2, 11, 7] + [14, 3, 16] + [1, 15, 12]
53	20	44	8	39	41	[1, 9, 12] + [2, 15, 13] + [3, 4, 20] + [5, 6, 8] + [16, 1, 18]
65	24	56	8	50	52	[1, 9, 12] + [2, 15, 13] + [3, 4, 20] + [6, 5, 14] + [18, 16, 23] + [1, 32, 29]
77	28	68	8	59	61	[1, 2, 30] + [4, 5, 12] + [6, 7, 24] + [10, 8, 20] + [16, 11, 14] + [23, 22, 26] + [15, 19, 16]
89	32	80	8			[1, 2, 40] + [4, 5, 30] + [10, 11, 16] + [14, 20, 24] + [1, 8, 18] + [12, 13, 23] + [22, 6, 32] + [1, 19, 17]

Table A.4 Catalog of regular graph designs (RGDs) under Series 3.5 with $\lambda_1 = 1$ and $\lambda_2 = 2$.

v	r	n_1	n_2	α	β	Sets of Shifts
8	4	2	5	0	0	[2, 1, 4]
20	8	14	5	8	12	[1, 6, 9] + [3, 2, 8]
44	16	38	5	32	34	[1, 14, 2] + [3, 7, 12] + [6, 5, 13] + [4, 8, 9]
56	20	50	5	46	46	[2, 16, 10] + [3, 17, 4] + [5, 9, 13] + [8, 25, 12] + [1, 6, 15]
68	24	62	5	56	58	[2, 22, 10] + [13, 4, 14] + [5, 15, 25] + [7, 1, 11] + [26, 3, 30] + [6, 21, 16]

Table A.5 Catalog of regular graph designs (RGDs) under Series 3.6 with $\lambda_1 = 1$ and $\lambda_2 = 2$.

v	r	n_1	n_2	α	β	Sets of Shifts
14	8	2	11	0	0	[1, 2, 5] + [4, 1, 3]
26	12	14	11	8	6	[1, 3, 8] + [2, 5, 10]
38	16	26	11	17	16	[1, 3, 8] + [2, 5, 10] + [6, 3, 13] + [14, 18, 19]
50	20	38	11	28	31	[1, 8, 3] + [2, 5, 10] + [6, 3, 13] + [14, 4, 19] + [21, 24, 25]
62	24	50	11	40	40	[1, 8, 10] + [2, 15, 11] + [3, 20, 4] + [5, 16, 6] + [7, 13, 12] + [14, 29, 31]
74	28	62	11	50	54	[1, 8, 10] + [2, 15, 11] + [3, 20, 4] + [6, 16, 13] + [14, 7, 25] + [5, 31, 12] + [33, 34, 37]
86	32	74	11	68	64	[1, 8, 10] + [2, 15, 11] + [3, 20, 4] + [6, 16, 13] + [14, 7, 25] + [12, 31, 5] + [37, 41, 42] + [30, 33, 39]
98	36	86	11			[1, 10, 30] + [2, 15, 31] + [3, 18, 24] + [16, 4, 29] + [5, 34, 12] + [6, 19, 13] + [8, 14, 23] + [28, 27, 36] + [9, 26, 44]

Table A.6 Catalog of regular graph designs (RGDs) under Series 3.8 with $\lambda_1 = 1$ and $\lambda_2 = 2$.

v	r	n_1	n_2	α	β	Sets of Shifts
16	8	6	9	2	2	[1, 2, 4] + [5, 8, 9]
28	12	18	9	8	16	[1, 2, 4] + [5, 8, 9] + [10, 14, 12]
40	16	30	9	20	22	[1, 2, 4] + [5, 8, 9] + [14, 16, 21] + [12, 15, 20]
52	20	42	9	32	34	[1, 10, 13] + [2, 6, 12] + [3, 14, 16] + [4, 5, 21] + [7, 25, 15]
64	24	54	9	46	46	[1, 10, 13] + [2, 6, 12] + [3, 16, 14] + [4, 5, 21] + [15, 7, 25] + [27, 28, 29]
76	28	66	9	56	60	[1, 10, 13] + [2, 6, 12] + [16, 3, 14] + [4, 5, 21] + [7, 15, 25] + [31, 35, 38] + [32, 37, 34]
88	32	78	9	74	70	[1, 20, 19] + [2, 16, 26] + [4, 13, 30] + [3, 5, 28] + [6, 23, 9] + [14, 10, 25] + [11, 31, 37] + [7, 15, 12]
100	36	90	9			[1, 20, 19] + [2, 16, 26] + [4, 13, 30] + [5, 28, 3] + [6, 23, 9] + [14, 10, 25] + [8, 37, 11] + [7, 15, 12] + [41, 46, 50]

Table A.7 Catalog of regular graph designs (RGDs) under Series 3.9 with $\lambda_1 = 1$ and $\lambda_2 = 2$.

v	r	n_1	n_2	α	β	Sets of Shifts
18	8	10	7	3	6	[1, 2, 4] + [5, 8, 9]
30	12	22	7	16	19	[1, 2, 15] + [4, 5, 14] + [6, 8, 10]
42	16	34	7	26	30	[1, 5, 7] + [3, 11, 9] + [8, 2, 16] + [4, 17, 15]
54	20	46	7	38	40	[1, 5, 7] + [3, 11, 9] + [2, 8, 16] + [4, 17, 15] + [19, 25, 27]
66	24	58	7	52	50	[1, 12, 5] + [2, 27, 6] + [3, 7, 14] + [4, 19, 11] + [8, 20, 22] + [16, 9, 26]
78	28	70	7	62	66	[12, 1, 5] + [4, 19, 11] + [3, 7, 14] + [16, 9, 26] + [2, 29, 8] + [15, 17, 28] + [22, 20, 38]
90	32	82	7	78	76	[1, 15, 17] + [2, 20, 23] + [3, 4, 37] + [13, 5, 21] + [10, 14, 11] + [8, 19, 9] + [6, 34, 38] + [12, 30, 31]

APPENDIX B

v	Reference		r	λ_1, λ_2	E1	E2	Overall E
8	[b]	SR7	4	0, 2	0.75	0.86	0.84
	[b]	SR9	6	2, 3	0.83	0.87	0.85
	[d]	B6	4				0.85
9	[a]Ser3.5	(0, 2, 3, 7)	4	1, 2	0.8083	0.8661	0.8487
	[a]Ser3.1	[1, 2, 4]	4	1, 2	0.8041	0.8616	0.8319
	[b]	R8	4	3, 1	0.94	0.77	0.80
	[b]	LS1	4	1, 2	0.80	0.87	0.83
	[d]	B12	4				0.83
10	[b]	S17	4	4, 1	1	0.77	0.79
	[b]	T1	2	1, 0	0.83	0.71	0.79
	[b]	T2	4	2, 0	0.83	0.71	0.79
	[b]	T12	4	1, 2	0.80	0.87	0.82

(*continued*)

Table A.1 | *(Continued)*

v	Reference		r	λ_1, λ_2	E1	E2	Overall E
	c	T33	4				0.82
	[a]Ser3.7		4	1, 2	0.7981	0.8646	0.82
	[a]Ser3.3		4	1, 2	0.8029	0.86979	0.816
	[d]	A25	4				0.817
11	[b]	SR20	3	0, 1	0.75	0.82	0.80
	[b]	R15	4	2, 1	0.88	0.81	0.81
	c'[d]	R109/B36	4				0.813
	[a]Design 1		4	1, 2	0.8077	0.875	0.8134
	[b]	R24	4	0, 1	0.75	0.81	0.80
14	[a]Ser3.6		8	1, 2	0.7801	0.8101	0.8053
	[a]Design 2		4	0, 1	0.75	0.8077	0.8029
	[a]Ser3.1		8	1, 2	0.7776	0.8087	0.7996
15	[a]Design 3		4	0, 1	0.7452	0.8073	0.7978
	[b]	R27	4	0, 1	0.75	0.80	0.80
	[b]	R31	8	1, 2	0.78	0.81	0.80
	[b]	SR40	4	0, 1	0.75	0.80	0.79
	[b]	R36	6	2, 1	0.83	0.78	0.79
16	[b]	LS12	7	2, 1	0.82	0.78	0.79
	[b]	LS15	8	2, 1	0.81	0.78	0.80
	[a]Ser3.8		8	1, 2	0.7778	0.8077	0.7954
	[b]	C5	8	1, 2	0.78	0.81	0.79
17	[a]Ser3.2		4	0, 1	0.7356	0.7969	0.7806
	[a]Ser3.5		8	1, 2	0.775	0.8073	0.7908
18	[b]	S62	8	8, 1	1	0.72	0.73
	[a]Ser3.9		8	1, 2	0.7734	0.8071	0.7869
20	[b]	SR52	5	0, 1	0.75	0.79	0.78
	[a]Ser3.5		8	1, 2	0.7744	0.8096	0.7834
21	[a]Ser3.1		12	1, 2	0.7692	0.7894	0.7853
22	[b]	S82	10	10, 1	1	0.71	0.72
	[a]Ser3.7		8	1, 2	0.7775	.8099	0.7819
23	[a]Ser3.3		8	1, 2	0.7787	0.8112	0.7816
24	[b]	R46	7	0, 1	0.75	0.78	0.78
	[b]	R49	10	2, 1	0.80	0.78	0.78
26	[a]Ser3.5		12	1, 2	0.7683	0.7875	0.7766
	[b]	R53	8	0, 1	0.75	0.78	0.78
	[a]Design 4		8	0, 1	0.7487	0.7799	0.7774
27	[b]	R54	8	0, 1	0.75	0.78	0.78
	[a]Ser3.1		12	1, 2	0.7688	0.788	0.786
	[a]Ser3.8		12	1, 2	0.7648	0.7903	0.7731
28	[b]	SR68	7	0, 1	0.75	0.78	0.77
	[b]	R55	8	0, 1	0.75	0.78	0.77
	[b]	R56	10	2, 1	0.80	0.77	0.78
29	[a]Ser3.2		8	0, 1	0.7462	0.7773	0.7727
	[a]Ser3.4		12	1, 2	0.7693	0.7881	0.7746
30	[a]Ser3.9		12	1, 2	0.7679	0.7899	0.7731
	[b]	R57	10	2, 1	0.80	0.77	0.78
32	[b]	SR74	8	0, 1	0.75	0.77	0.77
33	[a]Ser3.1		16	1, 2	0.7641	0.7791	0.7715
38	[a]Ser3.6		16	1, 2	0.7632	0.7782	0.7676

(*continued*)

Table A.2 *(Continued)*

v	Reference		r	λ_1, λ_2	E1	E2	Overall E
39	[a]Ser3.1		16	1, 2	0.7623	0.7789	0.7666
	[a]Ser3.8		16	1, 2	0.7627	0.7786	0.7664
40	[b]	S1.12	4	1, 0	0.77	0.71	0.73
	[b]	S1.13	8	2, 0	0.77	0.71	0.73
41	[a]Ser3.2		12	0, 1	0.7483	0.7686	0.7665
	[a]Ser3.4		16	1, 2	0.7641	0.7796	0.7671
42	[a]Ser3.9		16	1, 2	0.7633	0.7799	0.7661
44	[a]Ser3.5		16	1, 2	0.7640	0.7799	0.7658
45	[a]Ser3.1		28	1, 2	0.7586	0.7674	0.7666
50	[a]Ser3.6		20	1, 2	0.7606	0.7735	0.7635
51	[a]Ser3.1		20	1, 2	0.7613	0.7736	0.7637
52	[a]Ser3.8		20	1, 2	0.7606	0.7733	0.7629
53	[a]Ser3.2		16	0, 1	0.7491	0.7643	0.7631
	[a]Ser3.4		20	1, 2	0.7617	0.7744	0.7636
54	[a]Ser3.9		20	1, 2	0.7611	0.7737	0.7627
56	[a]Ser3.5		20	1, 2	0.7619	0.7742	0.7630
62	[a]Ser3.6		24	1, 2	0.7591	0.7694	0.7610
63	[a]Ser3.1		24	1, 2	0.7594	0.7699	0.7611
64	[a]Ser3.8		24	1, 2	0.7594	0.7697	0.7609
65	[a]Ser3.2		20	0, 1	0.7494	0.7618	0.7611
	[a]Ser3.4		24	1, 2	0.7597	0.7702	0.761
66	[a]Ser3.9		24				
68	[a]Ser3.5		24	1, 2	0.7597	0.7702	0.7605
74	[a]Ser3.6		28	1, 2	0.7577	0.767	0.7591
76	[a]Ser3.8		28	1, 2	0.7579	0.7672	0.7591
77	[a]Ser3.2		24	0, 1	0.7496	0.7598	0.7593
	[a]Ser3.4		28	1, 2	0.7581	0.7671	0.759
78	[a]Ser3.9		28	1, 2	0.7582	0.7674	0.759

(a) Proposed designs of this article. (b) Designs in Bose *et al.* [18]. (c) Designs in Clatworthy [11]. (d) Designs in John *et al.* [19].