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Research Article

Group-Like Uninorms

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ABSTRACT

Uninorms play a prominent role both in the theory and the applications of aggregations and fuzzy logic. In this paper a class of uninorms, called group-like uninorms will be introduced and a complete structural description will be given for a large subclass of them. First, the four versions of a general construction—called partial lex product—will be recalled. Then two particular variants of them will be specified: the first variant constructs, starting from $\mathbb{R}$ (the additive group of the reals) and modifying it in some way by $\mathbb{Z}$'s (the additive group of the integers) what we will coin basic group-like uninorms, whereas the second variant can enlarge any group-like uninorm by a basic group-like uninorm resulting in another group-like uninorm. All group-like uninorms obtained this way are "square" and have finitely many idempotent elements. On the other hand, we prove that any square group-like uninorm which has finitely many idempotent elements can be constructed by consecutive applications of the second variant (finitely many times) using only basic group-like uninorms as building blocks. Any basic group-like uninorm can be built by the first variant using only $\mathbb{R}$ and $\mathbb{Z}$, and any square group-like uninorm which has finitely many idempotent elements can be built using the second variant using only basic group-like uninorms: ultimately, all such uninorms can be built from $\mathbb{R}$ and $\mathbb{Z}$. In this way a complete characterization for square group-like uninorms which possess finitely many idempotent elements is given. The characterization provides, for potential applications in several fields of fuzzy theory or aggregation theory, the whole spectrum of choice of those square group-like uninorms which possess finitely many idempotent elements.

1. INTRODUCTION

Aggregation operations are crucial in numerous pure and applied fields of mathematics. Fuzzy theory is another large field, involving both pure mathematics and an impressive range of applications. Mathematical fuzzy logics have been introduced in [1], and the topic is a rapidly growing field ever since. In all these fields (and the list is far from being exhaustive) a crucial role is played by t-norms, t-conorms, and uninorms [2].

Establishing the structure theory of the whole class of uninorms seems to be quite difficult and out of reach. Several authors have characterized particular subclasses of them, see, e.g., [3–11]. Uninorms are interesting not only for a structural description purpose, but also different generalizations of them play a central role in many studies, see [12,13] for example. Group-like uninorms, to be introduced below, form a subclass of involutive uninorms, and involutive uninorms play the same role among uninorms as the Łukasiewicz t-norm or in general, the class of rotation-invariant t-norms [14–18] do in the class of t-norms. Because of the wide range of theoretical and practical applications of rotation-invariant t-norms, and also only the Łukasiewicz t-norm, it can be expected that involutive uninorms will find an increasing number of applications, too. Giving an effective description for the structure of the whole class of involutive uninorms seems to be extremely complicated and far beyond the present advance of algebra. In this paper we shall give a complete characterization for a large and rich subclass, the class of square group-like uninorms which have finitely many idempotent elements.

Residuation is a crucial property in mathematical fuzzy logics and in substructural logics, in general [19,20]. Replacing the usual universe $[0,1]$ of a (residuated) involutive uninorm by an arbitrary linearly ordered set leads to the algebraic notion of involutive FL_e-chains (another standard terminology for this class of algebras is pointed involutive commutative residuated chains). Group-like uninorms will be defined as a particular subclass of involutive uninorms such that replacing their universe by an arbitrary linearly ordered set leads to the general notion of odd involutive FL_e-chains: group-like uninorms are the monoidal operations of odd involutive FL_e-chains over $[0,1]$.

The literature of FL_e-algebras (aka. pointed commutative residuated lattices) is very rich, the main reason of which is that residuated lattices are algebraic counterparts of a vast class of logics, called substructural logic, see [19] and the references therein. Involutive FL_e-algebras (aka. involutive pointed commutative residuated lattices) correspond to substructural logics which satisfy the double negation axiom $\neg\neg\varphi \leftrightarrow \varphi$. Bounded involutive FL_e-chains correspond to **IUL** (Involutive uninorm logic) [20], whereas bounded odd involutive FL_e-chains correspond to $\mathbf{IUL}^{fp}$ (Involutive uninorm logic with fixed point) [21,22]. This provides another source

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of interest for group-like uninorms, being the standard (i.e., $[0,1]$-valued) semantics for $\mathbf{IUL}^{fp}$. Prominent examples of odd involutive FL_e-algebras are lattice-ordered abelian groups and odd Sugihara monoids (see Figure 1 for an example, each), the former constitutes an algebraic semantics of Abelian logic [23, 24], and the latter constitutes an algebraic semantics of a logic at the intersection of relevance logic and fuzzy logic [25]. The above-mentioned two examples are extremal in the sense that as opposed to integral structures where the unit element is the largest one and its residual complement is the least one, the unit element (and also its residual complement) is "in the middle." These two examples are also extremal in the sense that lattice-ordered abelian groups have a single idempotent element, namely the unit element, whereas all elements of any odd Sugihara monoid are idempotent.

In order to narrow the gap between these two extremal classes, in [26,27] a deep knowledge has been gained about the structure of odd involutive FL_e-chains, including a Hahn-type embedding theorem and a representation theorem by means of linearly ordered abelian groups and there-introduced constructions, called partial lex products [27] and the more general partial *sub*lex products [26]: all odd involutive FL_e-chains which have finitely many idempotent elements have a partial *sub*lex product group-representation. Square odd involutive FL_e-algebras are those which admit a partial lex product group-representation. Representability by partial lex products has a key role in the present paper. The consequence of the main result of [26,27] to $[0,1]$-valued algebras is that all group-like uninorms which possess finitely many idempotent elements admit a partial *sub*lex product group-representation. Square group-like uninorms are defined as those group-like uninorms which admit a partial lex product group-representation. What we show in this paper is, roughly, that square group-like uninorms can be built from very specific linearly ordered abelian groups: $\mathbb{Z}$ and $\mathbb{R}$. First, we adapt the partial lex product construction to the narrower, more specific setting of group-like uninorms by introducing two particular variants of it. These variants use only $\mathbb{R}$ and $\mathbb{Z}$, the additive group of real numbers and integers, respectively. With these two variants one can construct group-like uninorms having finitely many idempotent elements. Our main theorem asserts that all square group-like uninorms having finitely many idempotent elements can be constructed by using these two variants. Ultimately, it follows that all these uninorms can be constructed by the partial lex product construction from only $\mathbb{R}$ and $\mathbb{Z}$. The price to be paid for describing such a rich class of operations by such simple and well understood building blocks is that the partial lex product construction (even in its more restricted two variants form) will be quite complex. Another interpretation of the same result is that all these uninorms can be built by the second variant of the partial lex product construction using only basic group-like uninorms. If understood this way then there is a striking similarity between this characterization and the well-known ordinal sum representation of continuous t-norms of Mostert and Shields as ordinal sums of continuous archimedean t-norms [28]: replace "t-norm" by "uninorm," "continuous" by "square group-like with finitely many idempotent elements," "continuous archimedean t-norm" by "basic group-like uninorm," and "ordinal sum construction" by "the second variant of the partial lex product construction." Besides, according to the classification of continuous archimedean t-norms, any continuous archimedean t-norm is order isomorphic to either the Łukasiewicz t-norm or the Product t-norm, so there are two prototypes. In our setting basic group-like uninorms have $\aleph_0$ prototypes, one for each natural number.

2. PRELIMINARIES

Introduced in [29], a uninorm U is a function of type $[0,1] \times [0,1] \to [0,1]$ (i.e., a binary operation over the closed real unit interval $[0,1]$) such that the following axioms are satisfied.

$U(x,y) = U(y,x)$	(Commutativity)
If $y \le z$ then $U(x,y) \le U(x,z)$	(Monotonicity)
$U(U(x,y),z) = U(x,U(y,z))$	(Associativity)
There exists $t \in]0,1[$ such that $U(x,t) = x$	(Unit Element)

A uninorm is residuated if there exists a function I_U of type $[0,1] \times [0,1] \to [0,1]$ (i.e., a binary operation on $[0,1]$) such that the following is satisfied: $U(x,y) \le z$ if and only if $I_U(x,z) \ge y$. Frequently one uses the infix notation for uninorms, too, and writes $x \circledast y$ instead of $U(x,y)$, and $x \to_{\circledast} y$ instead of $I_T(x,y)$. A generalization of residuated t-norms and uninorms is the notion of FL_e-algebras. This generalization is done by replacing $[0,1]$ by an arbitrary lattice, possibly without top and bottom elements.

Throughout the paper algebras will be denoted by bold capital letters, their underlying sets by the same regular letter.

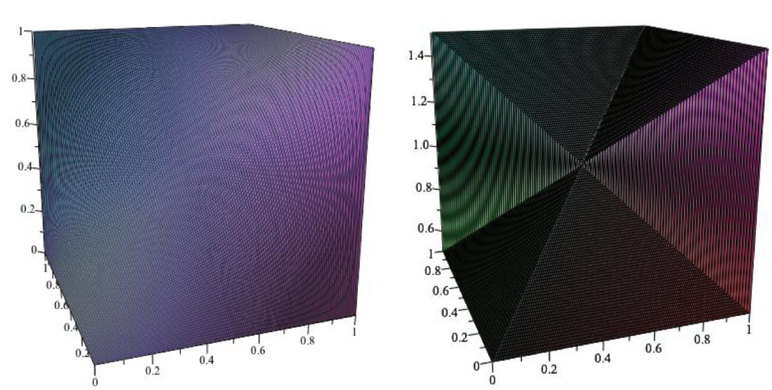

Figure 1 Visualization: the graph of the only linearly ordered abelian group over $]0;1[$ (left) and the graph of the only odd Sugihara monoid over $]0;1[$ (right).

Definition 1. An FL_e-algebra[1] is a structure $\mathbf{X} = (X, \wedge, \vee, \circledast, \rightarrow_\circledast, t, f)$ such that $(X, \wedge, \vee)$ is a lattice $(X, \leq, \circledast, t)$ is a commutative, residuated, monoid, and f is an arbitrary constant, where being residuated means that there exists a binary operation $\rightarrow_\circledast$ such that $x \circledast y \leq z$ if and only if $x \rightarrow_\circledast z \geq y$. This equivalence is often called adjointness condition $(\circledast, \rightarrow_\circledast)$ is called an adjoint pair. Equivalently, for any x, z, the set $\{v \mid x \circledast v \leq z\}$ has its greatest element, and $x \rightarrow_\circledast z$ is defined as this element: $x \rightarrow_\circledast z := \max\{v \mid x \circledast v \leq z\}$; this is often referred to as the residuation property. One defines $x' = x \rightarrow_\circledast f$ and calls an FL_e-algebra (and also its monoidal operation) involutive if $(x')' = x$ holds. The rank of an involutive FL_e-algebra is positive if $t > f$, negative if $t < f$, and 0 if $t = f$. An involutive FL_e-algebra is called odd if it is of rank 0. For an odd involutive FL_e-algebra $\mathbf{X}$, let X_{gr} be the set of invertible elements of $\mathbf{X}$. It turns out that there is a subalgebra of $\mathbf{X}$ on X_{gr}, denote it by $\mathbf{X_{gr}}$ and call it the group part of $\mathbf{X}$. We say that two consecutive elements $x \leq y$ in X form a gap in X if $x \leq z \leq y$ implies $z = x$ or $z = y$. In an FL_e-algebra call an element positive it it is greater or equal to the unit element, and strictly negative if it is smaller than the unit element.

Speaking in algebraic terms, t-norms and uninorms are the monoidal operations of commutative linearly ordered monoids over $[0, 1]$. Likewise, residuated t-norms and uninorms are just the monoidal operations of FL_e-algebras over $[0, 1]$. According to the terminology above, the class of involutive t-norms constitutes the Łukasiewicz t-norm, and all rotation-invariant t-norms (aka. monoidal operations of IMTL-algebras over $[0, 1]$) in general.

Definition 2. We call the monoidal operation $\circledast$ of an odd involutive FL_e-algebra over the real unit interval $[0, 1]$ a group-like uninorm.

We know more about the behavior of group-like uninorms (and of monoidal operations of bounded odd involutive FL_e-algebras, in general) in the boundary, as it holds true that

$$U(x,y) = \begin{cases} \in]0,1[& \text{if } x,y \in]0,1[\\ 0 & \text{if } \min(x,y) = 0 \\ 1 & \text{if } x,y > 0 \text{ and } \max(x,y) = 1 \end{cases}.$$

Therefore, values of a group-like uninorm U in the open unit square $]0, 1[^2$ fully determine U. As a consequence, one can view group-like uninorms as binary operations on $]0, 1[$, too. Because of this observation, throughout the paper we shall use the term group-like uninorm in a slightly different manner: instead of requiring the underlying universe to be $[0, 1]$, we only require that the underlying universe is order isomorphic to the open unit interval $]0, 1[$. This way, e.g., the usual addition of real numbers, that is letting $V(x, y) = x + y$, becomes a group-like uninorm in our terminology. This is witnessed by any order isomorphism from $]0, 1[$ to $\mathbb{R}$, take for instance $\varphi(x) = \tan(\pi x - \frac{\pi}{2})$. Using φ, any group-like uninorm (on $\mathbb{R}$, for example) can be carried over to $[0, 1]$ by letting, in our example,

$$U(x,y) = \begin{cases} \varphi^{-1}(V(\varphi(x), \varphi(y))) & \text{if } x,y \in]0,1[\\ 0 & \text{if } \min(x,y) = 0 \\ 1 & \text{if } x,y \neq 0 \text{ and } \max(x,y) = 1 \end{cases}.$$

[1]Other terminologies for FL_e-algebras are pointed commutative residuated lattices or pointed commutative residuated lattice-ordered monoids.

Since $]0, 1[$ and $\mathbb{R}$ are order isomorphic, technically, when proving that an operation is a group-like uninorm, we shall prove that its underlying set is order isomorphic to $\mathbb{R}$. Therefore, throughout the paper Theorem A below will play an important technical role.

Definition 3. A linearly ordered set $(X, \leq)$ is called Dedekind complete if every nonempty subset of X bounded from above by an element of X has a supremum in X. $(X, \leq)$ is called densely ordered if for any $x, y \in X$ such that $x < y$ there exists $z \in X$ such that $x < z < y$. A subset Y of X is called dense (in X) if any nonempty open interval in X contains an element from Y.

Theorem A. *([30], Theorem 2.29) A linearly ordered set $(K, \leq)$ is order isomorphic to the set of real numbers if and only if $(K, \leq)$ possesses the following four properties:*

i. $(K, \leq)$ has no least neither greatest element,

ii. $(K, \leq)$ is densely ordered,

iii. there exists a countable dense subset of $(K, \leq)$, and

iv. $(K, \leq)$ is Dedekind complete.

In the sequel we shall use this theorem in proving order isomorphism to $\mathbb{R}$ without further mention, and if the underlying sets of two odd involutive FL_e-chains $\mathbf{X}$ and $\mathbf{Y}$ are order isomorphic then we denote it by $\mathbf{X} \cong_o \mathbf{Y}$.

Definition 4. For a chain (a linearly ordered set) $(X, \leq)$ and for $x \in X$ define the predecessor $x_\downarrow$ of x to be the maximal element of the set of elements which are smaller than x, if it exists, define $x_\downarrow = x$ otherwise. Define the successor $x_\uparrow$ of x dually. We say for $Z \subseteq X$ that Z is discretely embedded into X if for $x \in Z$ it holds true that $x \notin \{x_\uparrow, x_\downarrow\} \subseteq Z$.

Nontrivial Dedekind complete linearly ordered abelian groups are widely-known to be isomorphic to either $\mathbb{Z}$ or $\mathbb{R}$. Since linearly ordered abelian groups are exactly cancellative odd involutive FL_e-chains [27], often we shall view $\mathbb{R}$ and $\mathbb{Z}$ as odd involutive FL_e-chains, and speak about an involutive FL_e-chain induced by a linearly ordered abelian group. Since an isomorphism between linearly ordered abelian groups naturally extends to the induced FL_e-algebras, the following lemma follows in a straightforward manner.

Lemma A. For any abelian group $\mathbf{G}$, if $\mathbf{G} \cong_o \mathbb{R}$ then (qua FL_e-algebras) $\mathbf{G} \cong \mathbb{R}$. Any linearly ordered abelian group $\mathbf{G}$ which is Dedekind complete and satisfies $x_\downarrow < x < x_\uparrow$ is isomorphic (qua an FL_e-algebra) to $\mathbb{Z}$.

The rest of this chapter is cited from [26,27].

Definition 5. The lexicographic product of two linearly ordered sets $\mathbf{A} = (A, \leq_1)$ and $\mathbf{B} = (B, \leq_2)$ is a linearly order set $\mathbf{A} \overrightarrow{\times} \mathbf{B} = (A \times B, \leq)$, where $A \times B$ is the Cartesian product of A and B, and $\leq$ is defined by $\langle a_1, b_1 \rangle \leq \langle a_2, b_2 \rangle$ if and only if $a_1 <_1 a_2$ or $a_1 = a_2$ and $b_1 \leq_1 b_2$. The lexicographic product $\mathbf{A} \overrightarrow{\times} \mathbf{B}$ of two FL_e-chains $\mathbf{A}$ and $\mathbf{B}$ is an FL_e-chain over the lexicographic product of their respective universes such that all operations are defined coordinatewise.

We shall view such a lexicographic product as an enlargement: each element in $\mathbf{A}$ is replaced by a whole copy of $\mathbf{B}$. Accordingly, in Definition 6 by a partial lex product of two linearly ordered sets we will mean a kind of partial enlargement: only some elements of the first algebra will be replaced by a whole copy of the second algebra.

The partial lex product construction will be crucial for our purposes.

Definition 6. Let $\mathbf{X} = (X, \wedge_X, \vee_X, *, \to_*, t_X, f_X)$ be an odd involutive FL$_e$-algebra and $\mathbf{Y} = (Y, \wedge_Y, \vee_Y, \star, \to_\star, t_Y, f_Y)$ be an involutive FL$_e$-algebra, with residual complement $'^*$ and $'^\star$, respectively.

A. Add a new element $\top$ to Y as a top element and annihilator (for $\star$), then add a new element $\bot$ to $Y \cup \{\top\}$ as a bottom element and annihilator. Extend $'^\star$ by $\bot'^\star = \top$ and $\top'^\star = \bot$. Let $\mathbf{V} \leq \mathbf{Z} \leq \mathbf{X_{gr}}$. Let

$$X_{Z_V} \overset{\rightrightarrows}{\times} Y = (V \times Y) \cup (Z \times \{\top\}) \cup (X \times \{\bot\})$$

and define $\mathbf{X_{Z_V}} \overset{\rightrightarrows}{\times} \mathbf{Y}$, the type III partial lexicographic product of $\mathbf{X}, \mathbf{Z}, \mathbf{V}$ and $\mathbf{Y}$ as follows:

$$\mathbf{X_{Z_V}} \overset{\rightrightarrows}{\times} \mathbf{Y} = \left(X_{Z_V} \overset{\rightrightarrows}{\times} Y, \leq, \circledast, \to_\circledast, (t_X, t_Y), (f_X, f_Y) \right),$$

where $\leq$ is the restriction of the lexicographical order of $\leq_X$ and $\leq_{Y \cup \{\top,\bot\}}$ to $X_{Z_V} \overset{\rightrightarrows}{\times} Y$, $\circledast$ is defined coordinatewise, and the operation $\to_\circledast$ is given by $(x_1, y_1) \to_\circledast (x_2, y_2) = \left((x_1, y_1) \circledast (x_2, y_2)'\right)'$, where

$$(x, y)' = \begin{cases} (x'^*, \bot) & \text{if } x \notin Z \\ (x'^*, y'^\star) & \text{if } x \in Z \end{cases}.$$

In the particular case when $\mathbf{V} = \mathbf{Z}$, we use the simpler notation $\mathbf{X_Z} \overset{\rightrightarrows}{\times} \mathbf{Y}$ for $\mathbf{X_{Z_V}} \overset{\rightrightarrows}{\times} \mathbf{Y}$ and call it the type I partial lexicographic product of $\mathbf{X}$, $\mathbf{Z}$, and $\mathbf{Y}$.

B. Assume that X_{gr} is discretely embedded into X. Add a new element $\top$ to Y as a top element and annihilator. Let $\mathbf{V} \leq \mathbf{X_{gr}}$. Let

$$X_V \overset{\longrightarrow}{\times} Y = (V \times Y) \cup (X \times \{\top\})$$

and define $\mathbf{X_V} \overset{\longrightarrow}{\times} \mathbf{Y}$, the type IV partial lexicographic product of $\mathbf{X}$, $\mathbf{V}$ and $\mathbf{Y}$ as follows:

$$\mathbf{X_V} \overset{\longrightarrow}{\times} \mathbf{Y} = \left(X_V \overset{\longrightarrow}{\times} Y, \leq, \circledast, \to_\circledast, (t_X, t_Y), (f_X, f_Y) \right),$$

where $\leq$ is the restriction of the lexicographical order of $\leq_X$ and $\leq_{Y \cup \{\top\}}$ to $X_V \overset{\longrightarrow}{\times} Y$, $\circledast$ is defined coordinatewise, and the operation $\to_\circledast$ is given by $(x_1, y_1) \to_\circledast (x_2, y_2) = \left((x_1, y_1) \circledast (x_2, y_2)'\right)'$, where $'$ is defined coordinatewise[2] by

$$(x, y)' = \begin{cases} (x'^*, \top) & \text{if } x \notin X_{gr} \text{ and } y = \top \\ ((x'^*)_\downarrow, \top) & \text{if } x \in X_{gr} \text{ and } y = \top \\ (x'^*, y'^\star) & \text{if } x \in V \text{ and } y \in Y \end{cases}. \tag{1}$$

In the particular case when $\mathbf{V} = \mathbf{X_{gr}}$, we use the simpler notation $\mathbf{X} \overset{\longrightarrow}{\times} \mathbf{Y}$ for $\mathbf{X_V} \overset{\longrightarrow}{\times} \mathbf{Y}$ and call it the type II partial lexicographic product of $\mathbf{X}$ and $\mathbf{Y}$.

A 3D plot of a type I extension is in Figure 3, and a 3D plot of a type II extension is in Figure 2 (right)

Theorem 1. *Adapt the notation of Definition 6. $\mathbf{X_{Z_V}} \overset{\rightrightarrows}{\times} \mathbf{Y}$ and $\mathbf{X_V} \overset{\longrightarrow}{\times} \mathbf{Y}$ are involutive FL$_e$-algebras with the same rank as that of $\mathbf{Y}$. In particular, if $\mathbf{Y}$ is odd then so are $\mathbf{X_{Z_V}} \overset{\rightrightarrows}{\times} \mathbf{Y}$ and $\mathbf{X_V} \overset{\longrightarrow}{\times} \mathbf{Y}$. In addition, $\mathbf{X_{Z_V}} \overset{\rightrightarrows}{\times} \mathbf{Y} \leq \mathbf{X_Z} \overset{\rightrightarrows}{\times} \mathbf{Y}$ and $\mathbf{X_V} \overset{\longrightarrow}{\times} \mathbf{Y} \leq \mathbf{X} \overset{\longrightarrow}{\times} \mathbf{Y}$.*

Definition 7. We introduce the following notation. Let $\mathbf{A}_1$, $\mathbf{A}_2, \ldots, \mathbf{A}_n$ and $\mathbf{D}$ be FL$_e$-algebras, and let $\mathbf{D} \leq \mathbf{A}_1 \vec{\times} \mathbf{A}_2$. If ν, the projection operation to the first coordinate maps $\mathbf{D}$ onto A_1, i.e., if $\nu(D) = \{a_1 \in A_1 \ : \ \text{there exists } (a_1, a_2) \in D\} = A_1$ then we denote it by

$$\mathbf{D} \leq_\nu \mathbf{A}_1 \vec{\times} \mathbf{A}_2.$$

Definition 8. Adapt the notation of Definition 6. Let $\mathbf{A} = \mathbf{X_{Z_V}} \overset{\rightrightarrows}{\times} \mathbf{Y}$ and $\mathbf{B} = \mathbf{X_V} \overset{\longrightarrow}{\times} \mathbf{Y}$. Then $A = (V \times Y) \cup (Z \times \{\top\}) \cup (X \times \{\bot\})$, $B = (V \times Y) \cup (X \times \{\top\})$, and the group part $\mathbf{A_{gr}}$ of $\mathbf{A}$, as well as the group part $\mathbf{B_{gr}}$ of $\mathbf{B}$ is the group $\mathbf{V} \vec{\times} \mathbf{Y_{gr}}$, which is the subalgebra of $\mathbf{A}$ and of $\mathbf{B}$ over $V \times Y_{gr}$ in both cases. Let

$$\mathbf{H} \leq_\nu \mathbf{V} \vec{\times} \mathbf{Y_{gr}}. \tag{2}$$

Replace $\mathbf{A_{gr}}$ in $\mathbf{A}$ and $\mathbf{B_{gr}}$ in $\mathbf{B}$ by $\mathbf{H}$ to obtain $\mathbf{A_H}$ and $\mathbf{B_H}$, respectively. More formally, let

$$A_H = H \cup (V \times (Y \setminus Y_{gr}) \cup (Z \times \{\top\}) \cup (X \times \{\bot\}),$$

$$B_H = H \cup (V \times (Y \setminus Y_{gr})) \cup (X \times \{\top\}).$$

Then A_H and B_H are closed under all operations of $\mathbf{A}$ and $\mathbf{B}$, respectively, and hence $\mathbf{A_H}$ is a subalgebra of $\mathbf{A}$, and $\mathbf{B_H}$ is a subalgebra of $\mathbf{B}$. Call them the partial *sub*lex product of their respective algebras (like in Definition 6) and $\mathbf{H}$.

A half-line proof shows (see [27, Section 2]) that in any odd involutive FL$_e$-chain (in particular, in any group-like uninorm) the residual complement of any negative idempotent element is a positive idempotent element. Therefore, for a group-like uninorm, having finitely many idempotent elements is equivalent to having finitely many positive idempotent elements. The main motivating theorem of the present paper asserts that up to isomorphism, any odd involutive FL$_e$-chain which has only finitely many positive idempotent

[2] Note that intuitively it would make up for a coordinatewise definition, too, in the second line of (1) to define it as $(x'^*, \bot)$. But $\bot$ is not among the set of possible second coordinates. However, since X_{gr} is discretely embedded into X, if $(x'^*, \bot)$ would be an element of the algebra then it would be equal to $((x'^*)_\downarrow, \top)$.

elements can be built by iterating finitely many times the type I and type II partial *sub*lex product constructions using only linearly ordered abelian groups as building blocks. We will refer to this fact as follows: every odd involutive FL_e-chain (in particular, any group-like uninorm algebra) which has only finitely many positive idempotent elements has a partial *sub*lex product group-representation.

Theorem 2. *[26, 27] If $\mathbf{X}$ is an odd involutive FL_e-chain, which has only $n \in \mathbb{N}$, $n \geq 1$ positive idempotent elements then it has a partial sublex product group representation, i.e., for $i = 2, \dots, n$ there exist totally ordered abelian groups $\mathbf{H}_1$, $\mathbf{H}_i$, $\mathbf{G}_i$, $\mathbf{Z}_{i-1}$, $\mathbf{V}_{i-1}$ along with $\iota_i \in \{I, II\}$ such that $\mathbf{X} \cong \mathbf{X}_n$, where for $i \in \{2, \dots, n\}$,*

$$\mathbf{X}_1 = \mathbf{H}_1 \text{ and } \mathbf{X}_i = \begin{cases} \left(\mathbf{X}_{i-1\mathbf{Z}_{i-1}\mathbf{V}_{i-1}} \overset{\rightharpoonup\!\!\!\rightharpoonup}{\times} \mathbf{G}_i\right)_{\mathbf{H}_i} & \text{if } \iota_i = I \\ \left(\mathbf{X}_{i-1\mathbf{V}_{i-1}} \overset{\longrightarrow}{\times} \mathbf{G}_i\right)_{\mathbf{H}_i} & \text{if } \iota_i = II \end{cases}.$$

Notice that Theorem 2 claims isomorphism between $\mathbf{X}$ and $\mathbf{X}_n$ hence $\mathbf{X}_n$ and consequently for $i = n-1, \dots, 2$, the $\mathbf{X}_i$'s are claimed implicitly to exist (to be well defined). Hence, by Definitions 6 and 8, using that $(\mathbf{X}_i)_{\mathbf{gr}} = \mathbf{H}_i$ holds for $i \in \{1, \dots, n\}$, it is necessarily that

for $i = 2, \dots, n$, $\mathbf{Z}_{i-1} \leq \mathbf{H}_{i-1}$, $\mathbf{H}_i \leq \mathbf{Z}_{i-1}\vec{\times}\mathbf{G}_i$ and
for $i = 2, \dots, n$, if $\iota_i = II$ then $\mathbf{H}_{i-1}$ is discretely embedded into $\mathbf{X}_{i-1}$.

As said before, by Theorem 2 all odd involutive FL_e-chains with finitely many positive idempotent elements (in particular, all group-like uninorm algebras with finitely many positive idempotent elements) have a partial *sub*lex product group-representation. We define a subclass of them in Definition 9. Recall that by definition a type III partial lex product is a particular instance of a type I partial *sub*lex product. For instance, $\mathbf{X}_{i-1\mathbf{Z}_{i-1}\mathbf{V}_{i-1}} \overset{\rightharpoonup\!\!\!\rightharpoonup}{\times} \mathbf{G}_i$ is a particular instance of $\left(\mathbf{X}_{i-1\mathbf{Z}_{i-1}\mathbf{V}_{i-1}} \overset{\rightharpoonup\!\!\!\rightharpoonup}{\times} \mathbf{G}_i\right)_{\mathbf{H}_i}$ where $\mathbf{H}_i = \mathbf{V_{i-1}}\vec{\times}\mathbf{G_i}$.

Definition 9. An odd involutive FL_e-chain $\mathbf{X}$ is called square it it possesses a partial lex product group-representation. More formally, if for $i = 2, \dots, n$ there exist linearly ordered abelian groups $\mathbf{H}_1, \mathbf{G}_i, \mathbf{Z}_{i-1}, \mathbf{V}_{i-1}$ along with $\iota_i \in \{I, II\}$ such that $\mathbf{X} \cong \mathbf{X}_n$, where for $i \in \{2, \dots, n\}$,

$$\mathbf{X}_1 = \mathbf{H}_1 \text{ and } \mathbf{X}_i = \begin{cases} \mathbf{X}_{i-1\mathbf{Z}_{i-1}\mathbf{V}_{i-1}} \overset{\rightharpoonup\!\!\!\rightharpoonup}{\times} \mathbf{G}_i & \text{if } \iota_i = I \\ \mathbf{X}_{i-1\mathbf{V}_{i-1}} \overset{\longrightarrow}{\times} \mathbf{G}_i & \text{if } \iota_i = II \end{cases}.$$

Like in Theorem 2, Definition 9 asserts isomorphism between $\mathbf{X}$ and $\mathbf{X}_n$ hence $\mathbf{X}_n$ and consequently for $i = n-1, \dots, 2$, the $\mathbf{X}_i$'s are claimed implicitly to exist. By Definition 6, using that $(\mathbf{X}_i)_{\mathbf{gr}} = \mathbf{V_{i-1}}\vec{\times}\mathbf{G_i}$ holds for $i \in \{2, \dots, n\}$, it is necessarily that (by letting $\mathbf{V}_0$ be the trivial group)

for $i = 2, \dots, n$, $\mathbf{Z}_{i-1} = \mathbf{V_{i-2}}\vec{\times}\mathbf{G_{i-1}}$, $\mathbf{V_{i-1}} = \mathbf{Z}_{i-1}$ and
for $i = 2, \dots, n$, if $\iota_i = II$ then $\mathbf{V_{i-2}}\vec{\times}\mathbf{G_{i-1}}$ is discretely embedded in to $\mathbf{X}_{i-1}$.

We call a group-like uninorm square if its induced odd involutive FL_e-chain is square.

3. STRUCTURAL DESCRIPTION

The characterization theorem for square group-like uninorms in Theorem 3 will be achieved via a series of lemmas.

3.1. Groupping Extensions Together

The proofs in this section do not require much more than tedious verification using the rather complex Definition 6. First we show that two (and thus also finitely many) consecutive type II extensions can be replaced by a single type II extension. More formally, we claim that

Lemma 1. For any odd involutive FL_e-algebras $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, it holds true that

$$(\mathbf{A} \overset{\longrightarrow}{\times} \mathbf{B}) \overset{\longrightarrow}{\times} \mathbf{C} \cong \mathbf{A} \overset{\longrightarrow}{\times} (\mathbf{B} \overset{\longrightarrow}{\times} \mathbf{C}),$$

i.e., if the algebra on one side is well-defined[3] then the algebra on the other side is well-defined, too, and the two algebras are isomorphic.

Proof. By Definition 6, the universe of $\mathbf{A} \overset{\longrightarrow}{\times} \mathbf{B}$ is $(A_{gr} \times B) \cup (A \times \{\top\})$, and hence the universe of $(\mathbf{A} \overset{\longrightarrow}{\times} \mathbf{B})_{\mathbf{gr}}$ is $A_{gr} \times B_{gr}$. Therefore, the universe of $(\mathbf{A} \overset{\longrightarrow}{\times} \mathbf{B}) \overset{\longrightarrow}{\times} \mathbf{C}$ is $(A_{gr} \times B_{gr} \times C) \cup (A_{gr} \times B \times \{\top\}) \cup (A \times \{\top\} \times \{\top\})$. On the other hand, the universe of $\mathbf{B} \overset{\longrightarrow}{\times} \mathbf{C}$ is $(B_{gr} \times C) \cup (B \times \{\top\})$. Therefore, by denoting $(\top, \top)$ the new top element added to $\mathbf{B} \overset{\longrightarrow}{\times} \mathbf{C}$, the universe of $\mathbf{A} \overset{\longrightarrow}{\times} (\mathbf{B} \overset{\longrightarrow}{\times} \mathbf{C})$ is $(A_{gr} \times B_{gr} \times C) \cup (A_{gr} \times B \times \{\top\}) \cup (A \times \{\top\} \times \{\top\})$, so the underlying universes of $(\mathbf{A} \overset{\longrightarrow}{\times} \mathbf{B}) \overset{\longrightarrow}{\times} \mathbf{C}$ and $\mathbf{A} \overset{\longrightarrow}{\times} (\mathbf{B} \overset{\longrightarrow}{\times} \mathbf{C})$ coincide. Clearly, the unit elements are the same. Since the monoidal operation of a type II extension is defined coordinatewise, the respective monoidal operations coincide, too. Since both algebras are residuated, and the monoidal operation and the universe uniquely determine the residual operation, it follows that the residual operations coincide, too, hence so do the residual complement operations.

Finally, the left-hand side is well-defined if and only if A_{gr} is discretely embedded into A, and $A_{gr} \times B_{gr}$ is discretely embedded into $(A_{gr} \times B) \cup (A \times \{\top\})$. It cannot be the case that $|B_{gr}| = 1$, since then for $a \in A_{gr}$, $(a, 1_B) \in A_{gr} \times B_{gr}$ but $(a, 1_B)_{\uparrow} = (a, \top) \notin A_{gr} \times B_{gr}$. Therefore $|B_{gr}| = \infty$, and hence, using that $A_{gr} \times B_{gr}$ is discretely embedded into $(A_{gr} \times B) \cup (A \times \{\top\})$, it follows that B_{gr} is discretely embedded into B. Thus $\mathbf{B} \overset{\longrightarrow}{\times} \mathbf{C}$ and also the right-hand side is well-defined, too. On the other hand, the right-hand side is well-defined if and only if A_{gr} is discretely embedded into A, and B_{gr} is discretely embedded into B. But then clearly $A_{gr} \times B_{gr}$ is discretely embedded into $(A_{gr} \times B) \cup (A \times \{\top\})$, and hence the left-hand side is well-defined, too.

Next we show that a type I partial lex extension followed by a type II partial lex extension can be replaced by a single type I partial lex extension. More formally, we claim that

Lemma 2. For any odd involutive FL_e-algebras $\mathbf{A}$, $\mathbf{H}$, $\mathbf{L}$, $\mathbf{B}$, such that $\mathbf{H} \leq \mathbf{A_{gr}}$, it holds true that

$$(\mathbf{A_H} \overset{\rightharpoonup\!\!\!\rightharpoonup}{\times} \mathbf{L}) \overset{\longrightarrow}{\times} \mathbf{B} \cong \mathbf{A_H} \overset{\rightharpoonup\!\!\!\rightharpoonup}{\times} (\mathbf{L} \overset{\longrightarrow}{\times} \mathbf{B})$$

i.e., if the algebra on one side is well-defined then the algebra on the other side is well-defined, too, and the two algebras are isomorphic.

[3] Here and also in the sequel we mean that the respective group parts should be discretely embedded into their algebras, as required in the definition of type II partial lex products.

Proof. For both sides to be well-defined, $\mathbf{H} \le \mathbf{A_{gr}}$ is needed.

That apart, the left-hand side is well-defined if and only if the group part of $\mathbf{A_H} \overset{\rightharpoonup\!\!\rightharpoonup}{\times} \mathbf{L}$ is discretely embedded into the universe of $\mathbf{A_H} \overset{\rightharpoonup\!\!\rightharpoonup}{\times} \mathbf{L}$, i.e., if and only if

i. $H \times L_{gr}$ is discretely embedded into $(H \times L) \cup (H \times \{\top_L\}) \cup (A \times \{\bot_L\})$,
and the right-hand side is well-defined if and only if

ii. L_{gr} is discretely embedded into L.

Clearly, (ii) implies (i). Now, assume (i). Then L_{gr} cannot be finite. Indeed, if L_{gr} were finite then by taking its largest element $l \in L_{gr}$, the element $(1_H, l)_\uparrow$ must be greater than $(1_H, l)$ since $H \times L_{gr}$ is discretely embedded. Therefore, $(1_H, l)_\uparrow$ is either equal to $(1_H, l_\uparrow)$ which is not in $H \times L_{gr}$ since l was chosen to be the greatest element in L_{gr}, or equal to $(1_H, \top)$ which is not in $H \times L_{gr}$ either. Hence L_{gr} is infinite, and therefore for any $(h, l) \in H \times L_{gr}$ it holds true that $(h, l)_\uparrow = (h, l_\uparrow) \in H \times L_{gr}$ and $(h, l)_\downarrow = (h, l_\downarrow) \in H \times L_{gr}$, that is, (ii) holds.

Denote $\mathbf{C} = \mathbf{A_H} \overset{\rightharpoonup\!\!\rightharpoonup}{\times} \mathbf{L}$ for short. As we have seen above, $C = (H \times L) \cup (H \times \{\top_L\}) \cup (A \times \{\bot_L\})$, and $C_{gr} = H \times L_{gr}$. Therefore, the universe of $(\mathbf{A_H} \overset{\rightharpoonup\!\!\rightharpoonup}{\times} \mathbf{L}) \overset{\longrightarrow}{\times} \mathbf{B}$ is

$$(C_{gr} \times B) \cup (C \times \{\top_B\}) = (H \times L_{gr} \times B) \cup (H \times L \times \{\top_B\}) \cup (H \times \{\top_L\} \times \{\top_B\}) \cup (A \times \{\bot_L\} \times \{\top_B\})$$

On the other hand, the universe of $\mathbf{L} \overset{\longrightarrow}{\times} \mathbf{B}$ is $(L_{gr} \times B) \cup (L \times \{\top_B\})$. Let $\top_L$, $\bot_L$, and $\top_B$ be the new top, bottom, and top element added to L, L, and B, respectively, according to Definition 6. Then it is easily verified that $(\top_L, \top_B)$ and $(\bot_L, \top_B)$ satisfy the requirements of Definition 6 to be the new top and bottom elements of $\mathbf{L} \overset{\longrightarrow}{\times} \mathbf{B}$. Hence the universe of $\mathbf{A_H} \overset{\rightharpoonup\!\!\rightharpoonup}{\times} (\mathbf{L} \overset{\longrightarrow}{\times} \mathbf{B})$ is

$$(H \times L_{gr} \times B) \cup (H \times L \times \{\top\}) \cup (H \times \{\top\} \times \{\top\}) \cup (A \times \{\bot\} \times \{\top\})$$

so the underlying universes of the two algebras coincide.

Clearly, the unit elements are the same. Since the monoidal operation of a partial lex product is defined coordinatewise, the respective monoidal operations coincide, too. Since both algebras are residuated and the monoidal operation and the universe uniquely determines its residual operation, it follows that the residual operations coincide, too, hence so do the residual complements.

3.2. When Do Partial Lex Extensions $\cong_o \mathbb{R}$?

It is apparent from Definition 6 that type I and type II lex extensions are simpler particular instances of type III and type IV lex extension, respectively. First we show that all the extensions in the group-representation of a square group-like uninorm must be type I or II; otherwise the constructed algebra cannot be $\cong_o \mathbb{R}$.

Lemma 3. The following statements hold true.

1. Any type III lex extension which is not of type I has a gap outside its group part.
2. Any type IV lex extension which is not of type II has a gap outside its group part.
3. If an odd involutive FL_e-algebra has a gap outside its group part then any type III lex extension of it has a gap outside its group part, too.
4. If an odd involutive FL_e-algebra has a gap outside its group part then any type IV lex extension of it has a gap outside its group part, too.

Proof. The statements are direct consequences of the definition of partial lex products:

1. Consider $\mathbf{A_{H_K}} \overset{\rightharpoonup\!\!\rightharpoonup}{\times} \mathbf{B}$, where $H \setminus K \neq \emptyset$. For any $a \in H \setminus K$ it holds true that $(a, \bot) < (a, \top)$ is a gap in $\mathbf{A_{H_K}} \overset{\rightharpoonup\!\!\rightharpoonup}{\times} \mathbf{B}$, and neither $(a, \bot)$ nor $(a, \top)$ is invertible.
2. Consider $\mathbf{A_H} \overset{\longrightarrow}{\times} \mathbf{B}$, where $A_{gr} \setminus H \neq \emptyset$. For any $a \in A_{gr} \setminus H$ it holds true that $(a, \top) < (a_\uparrow, \top)$ is a gap in $\mathbf{A_{H_K}} \overset{\rightharpoonup\!\!\rightharpoonup}{\times} \mathbf{B}$, and neither $(a, \top)$ nor $(a_\uparrow, \top)$ is invertible.
3. Consider $\mathbf{A}$ with a gap $r < s$ in $A \setminus A_{gr}$. Then $(r, \bot) < (s, \bot)$ is a gap in $\mathbf{A_{H_K}} \overset{\rightharpoonup\!\!\rightharpoonup}{\times} \mathbf{B}$, and neither $(r, \bot)$ nor $(s, \bot)$ is invertible.
4. Consider $\mathbf{A}$ with a gap $r < s$ in $A \setminus A_{gr}$. Then $(r, \top) < (s, \top)$ is a gap in $\mathbf{A_H} \overset{\longrightarrow}{\times} \mathbf{B}$, and neither $(r, \top)$ nor $(s, \top)$ is invertible.

In the next lemma it is characterized when a type I partial lex extension is $\cong_o \mathbb{R}$. Since the partial lex product construction is inherently quite complex, and since there are four properties to be checked when proving $\cong_o$ to $\mathbb{R}$ (see Theorem A), the proof (just like the proofs of all lemmas in this section) will be quite technical and tedious.

Lemma 4. Let $\mathbf{A}$ and $\mathbf{D}$ be odd involutive FL_e-chains, $\mathbf{H} \le \mathbf{A_{gr}}$. The following statements are equivalent.

1. $\mathbf{A_H} \overset{\rightharpoonup\!\!\rightharpoonup}{\times} \mathbf{D} \cong_o \mathbb{R}$,
2. $\mathbf{A} \cong_o \mathbb{R}$, $\mathbf{D} \cong_o \mathbb{R}$, and $\mathbf{H}$ is countable.

Proof. Denote $\mathbf{C} = \mathbf{A_H} \overset{\rightharpoonup\!\!\rightharpoonup}{\times} \mathbf{D}$ for short. It is well-defined since $\mathbf{H} \le \mathbf{A_{gr}}$.

Sufficiency.

By Definition 6, the universe of $\mathbf{C}$ is

$$\begin{aligned} C &= (H \times (D \cup \{\top\})) \cup (A \times \{\bot\}) \\ &= (H \times (D \cup \{\top, \bot\})) \,\dot\cup\, \big((A \setminus H) \times \{\bot\}\big). \end{aligned}$$

i. C has no least neither greatest element since $A \times \{\bot\} \subseteq C$ and A has no least neither greatest element since it is order isomorphic to $\mathbb{R}$.

ii. C is densely ordered. Indeed, let $(p, q) < (r, s)$. If $p < r$ then there exists an element $v \in]p, r[$ since A is densely ordered, and thus $(v, \bot) \in C$ and $(p, q) < (v, \bot) < (r, s)$ holds. If $p = r$ then $q < s$ and hence $p \in H$ follows. Therefore, there is an element $v \in]q, s[$ such that $(p, q) < (p, v) < (p, s)$ holds since $D \cup \{\top, \bot\}$ is densely ordered.

iii. Let A_1 and D_1 be countable dense subsets of A and D, respectively. It follows that $C_1 = (H \times (D_1 \cup \{\top\})) \cup (A_1 \times \{\bot\})$ is a dense subset of $C = (H \times (D \cup \{\top\})) \cup (A \times \{\bot\})$. Moreover, C_1 is countable, since so is H.

iv. Finally we prove that C is Dedekind complete. Take any nonempty subset of $V \subseteq C$ which has an upper bound $(b_1, b_2) \in C$. Let $V_1 = \{v_1 \mid (v_1, v_2) \in V\}$. Then $V_1 \subseteq A$ is nonempty and bounded from above by b_1. Since A is order isomorphic to $\mathbb{R}$, and $\mathbb{R}$ is Dedekind complete, there exists the supremum m_1 of V_1 in A. If $m_1 \notin V_1$ then $(m_1, \bot) \in C$ is the supremum of V. If $m_1 \in V_1$ and $m_1 \in A \setminus H$ then $(m_1, \bot) \in C$ is the supremum of V. Finally, if $m_1 \in V_1$ and $m_1 \in H$ then $V_2 := \{v_2 \mid (m_1, v_2) \in V\} \subseteq D \cup \{\top, \bot\}$ is nonempty and bounded from above by b_2. Since D is Dedekind complete, so does $D \cup \{\top, \bot\}$, and hence V_2 has a supremum m_2 in $D \cup \{\top, \bot\}$, yielding that (m_1, m_2) is the supremum of V.

Necessity.

Assume $\mathbf{C} \cong_o \mathbb{R}$.

i. If D has a least element l then for some $h \in H$, $(h, \bot) < (h, l)$ would make a gap in C, a contradiction. An analogous argument shows that D cannot have a greatest element either. Next, if A had a least or a greatest element (l or g) then $(l, \bot)$ or $(g, \bot)$ would be the least or the greatest element of C, a contradiction.

ii. If D is not densely ordered then there exists a gap $a < b$ in D. Then for $h \in H$, $(h, a) < (h, b)$ is a gap in C, a contradiction. If A is not densely ordered then there exists a gap $a < b$ in A. Then $(a, \top) < (b, \bot)$ is a gap in C when $a \in H$, and $(a, \bot) < (b, \bot)$ is a gap in C when $a \in A \setminus H$, contradiction.

iii. Let Q_2 be a countable and dense subset of C.

We prove that $Q := \{v_1 \in A \mid (v_1, v_3) \in Q_2\}$ is a countable and dense subset of A. Indeed, Q is clearly nonempty and countable, too, since so is Q_2. To show that Q is a dense subset of A, let $a \in A \setminus Q$ arbitrary. Take an arbitrary open interval $]b, c[$ containing a. Since Q_2 is a dense subset of C, we can choose an element $(r, s) \in Q_2$ such that $(b, \bot) < (r, s) < (c, \bot)$ if $b \in A \setminus H$, or we can choose $(r, s) \in Q_2$ such that $(b, \top) < (r, s) < (c, \bot)$ if $b \in H$. In both cases, $r \in Q$ and $b < r < c$ holds, so we are done.

Next, we prove that H is countable. It suffices to prove $H \subseteq Q$ since Q is a countable. Assume there exists $h \in H \setminus Q$. Then $\{h\} \times (D \cup \{\top, \bot\}) \subset C \setminus Q_2$ would follow, showing that there is no element in Q_2 between $(h, \bot)$ and $(h, \top)$. However there should be, since D is nonempty. It is a contradiction to Q_2 being a dense subset of C.

Finally, we prove that $Q_3 := \{v_3 \in D \mid (v_1, v_3) \in Q_2\}$ is a countable and dense subset of D. Indeed, Q_3 is clearly nonempty and countable, too, since so is Q_2. To show that Q_3 is a dense subset of D, let $a_3 \in D \setminus Q_3$ arbitrary. Take an arbitrary open interval $]b_3, c_3[$ containing a_3, and let $h \in H$. Since Q_2 is a dense subset of C and $(h, b_3), (h, c_3) \in C$, we can choose an element $(h, s_3) \in Q_2$ such that $(h, b_3) < (h, s_3) < (h, c_3)$. Thus $s_3 \in Q_3$ follows and $b_3 < s_3 < c_3$ holds, so we are done.

(iv) To prove that A is Dedekind complete we proceed as follows. Take any nonempty subset of $V \subseteq A$ which has an upper bound $b \in A$. Then $V_2 = \{(v, \bot) \mid v \in V\}$ is a nonempty subset of C which has an upper bound $(b, \bot) \in C$. Since C is Dedekind complete, there exists the supremum (m, m_3) of V_2 in C. Because the second coordinate of any element of V_2 is $\bot$, it follows that when (m, m_3) is an upper bound of V_2 then also $(m, \bot)$ is an upper bound of it. Therefore, $(m, \bot) \in C$ is the supremum of V_2, and hence m is the supremum of V.

To prove that D is Dedekind complete we proceed as follows. Take any nonempty subset of $V_3 \subseteq D$ which has an upper bound $b_3 \in D$. Choose an element h from H. Then $V_2 = \{(h, v_3) \mid v_3 \in V_3\}$ is a nonempty subset of C which has an upper bound $(h, b_3) \in C$. Since C is Dedekind complete, there exists the supremum (m, m_3) of V_2 in C. Clearly, $(m, m_3) \leq (h, b_3)$ holds, therefore, $m = h$ follows, and since (h, m_3) is the supremum of V_2, it follows that m_3 is the supremum of V_3.

In the next lemma we characterize when a type I partial lex extension followed by a type II partial lex extension is $\cong_o \mathbb{R}$.

Lemma 5. Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{L}$ be odd involutive FL_e-chains, $\mathbf{H} \leq \mathbf{A_{gr}}$. The following statements are equivalent:

1. $(\mathbf{A_H} \overset{\twoheadrightarrow}{\times} \mathbf{L}) \overset{\longrightarrow}{\times} \mathbf{B}$ is well-defined and $\cong_o \mathbb{R}$
2. • $\mathbf{A} \cong_o \mathbb{R}$ and $\mathbf{B} \cong_o \mathbb{R}$,
   • H and L_{gr} are countable,
   • L is Dedekind complete, has a countable dense subset, and has neither least nor greatest element,
   • L_{gr} is discretely embedded into L, and
   • there exists no gap in L formed by two elements of $L \setminus L_{gr}$.

Proof. Denote $\mathbf{C} = \mathbf{A_H} \overset{\twoheadrightarrow}{\times} \mathbf{L}$ and $\mathbf{D} = (\mathbf{A_H} \overset{\twoheadrightarrow}{\times} \mathbf{L}) \overset{\longrightarrow}{\times} \mathbf{B}$ for short.

Sufficiency.

Note that since L is unbounded and L_{gr} is discretely embedded into L, it follows that $C_{gr} = H \times L_{gr}$ is discretely embedded into $C = (H \times (L \cup \top_L)) \cup (A \times \{\bot_L\})$. Thus $\mathbf{D}$ is well-defined. By definition,

$$D = (C_{gr} \times B) \cup (C \times \{\top_B\}) =$$

$$(H \times L_{gr} \times B) \cup (C \times \{\top_B\}) =$$

$$[H \times L_{gr} \times B] \cup [H \times (L \cup \{\top_L\}) \times \{\top_B\}] \cup [A \times \{\bot_L\} \times \{\top_B\}] =$$

$$[H \times L_{gr} \times B] \cup [H \times L \times \{\top_B\}] \cup [H \times \{\top_L\} \times \{\top_B\}] \\ \cup [A \times \{\bot_L\} \times \{\top_B\}] =$$

$$[H \times L_{gr} \times (B \cup \{\top_B\})] \cup [H \times (L \setminus L_{gr}) \times \{\top_B\}] \\ \cup [H \times \{\top_L\} \times \{\top_B\}] \cup [A \times \{\bot_L\} \times \{\top_B\}].$$

D has no least neither greatest element, since partial lex products clearly inherit the boundedness of their first component, and A has neither least nor greatest element.

We prove that D is densely ordered. Let $x = (x_1, x_2, x_3) < (y_1, y_2, y_3) = y$, $x, y \in D$.

i. Assume $x_1 < y_1$.
Then there exists $z_1 \in A$ such that $x_1 < z_1 < y_1$ since $\mathbf{A}$ is order isomorphic to $\mathbb{R}$, and hence $\mathbf{A}$ is densely ordered, hence $(z_1, \bot_L, \top_B)$ is strictly in between x and y.

ii. Next, assume $x_1 = y_1$ and $x_2 < y_2$.
Since $x_2, y_2 \in L \cup \{\bot_L, \top_L\}$ and $x_2 < y_2$ excludes $x_2 = \top_L$, it follows that $x_2 \in L \cup \{\bot_L\}$.
 - Assume $x_2 = \bot_L$.
 Then $y_2 \in L \cup \{\top_L\}$ follows from $x_2 < y_2$, and we can choose $c \in L$ such that $c < y_2$ since L is nonempty and has no least element; resulting in $(x_1, c, \top_B)$ being strictly in between x and y.
 - Assume $x_2 \in L_{gr}$.
 Then $(x_2)_\uparrow \in L_{gr}$ follows[4] since L_{gr} is discretely embedded into L. In addition, $x_2 < (x_2)_\uparrow \le y_2$ holds. If $(x_2)_\uparrow < y_2$ then the element $(x_1, (x_2)_\uparrow, \top_B) \in D$ is strictly in between x and y, whereas if $(x_2)_\uparrow = y_2$ then $y_2 \in L_{gr}$, too, and hence $y_3 \in B \cup \{\top_B\}$ follows. Since $\mathbf{B}$ has no least element, we can choose $c \in B$ such that $c < y_3$. Hence $(x_1, (x_2)_\uparrow, c)$ is strictly in between x and y.
 - Assume $x_2 \in L \setminus L_{gr}$.
 Then $x_2 < y_2$ excludes $y_2 = \bot_L$, hence $y_2 \in L \cup \top_L$.
   - If $y_2 = \top_L$ then, since L has no greatest element, there is an element $c \in L$ such that $x_2 < c < y_2$.
   - If $y_2 \in L_{gr}$ then since L_{gr} is discretely embedded into L, it follows that $L \setminus L_{gr} \ni x_2 \neq (y_2)_\downarrow \in L_{gr}$, hence letting $c = (y_2)_\downarrow$, $x_2 < c < y_2$ holds.
   - If $y_2 \in L \setminus L_{gr}$ then there is an element $c \in L$ such that $x_2 < c < y_2$ since there exists no gap in L formed by two elements of $L \setminus L_{gr}$.

 In all the three previous cases, the element $(x_1, c, \top_B)$ is strictly in between x and y, and we are done.

iii. Finally, assume $x_1 = y_1$, $x_2 = y_2$ and $x_3 < y_3$.
It follows that $x_1 \in H$, $x_2 \in L_{gr}$, and $x_3 \in B \cup \{\top_B\}$. Since $\mathbf{B}$ is order isomorphic to $\mathbb{R}$ (and hence $\mathbf{B}$ is densely ordered and has no greatest element), it follows that $B \cup \{\top_B\}$ is densely ordered, too, hence there is an element $c \in L$ such that $x_3 < c < y_3$, and thus the element (x_1, x_2, c) is strictly in between x and y.

Next we prove that D is Dedekind complete. Take any nonempty subset of $V \subseteq D$ which has an upper bound $(b_1, b_2, b_3) \in D$. Then

$$V_1 = \{v_1 \mid (v_1, v_2, v_3) \in D\}$$

is a subset of A, it is nonempty and bounded from above by b_1. Since A is order isomorphic to $\mathbb{R}$, and thus A is Dedekind complete, there exists the supremum m_1 of V_1.

i. If $m_1 \notin V_1$ then $(m_1, \bot_L, \top_B) \in D$ is the supremum of V.

ii. If $m_1 \in V_1$ and $m_1 \in A \setminus H$ then $(m_1, \bot_L, \top_B) \in D$ is the supremum of V.

iii. Finally, assume that $m_1 \in V_1$ and $m_1 \in H$. Then

$$V_2 = \{v_2 \mid (m_1, v_2, v_3) \in V\}$$

is a subset of $L \cup \{\top_L\}$, it is nonempty and bounded from above by b_2. Since L is Dedekind complete, so does $L \cup \{\top_L\}$, hence there exists the supremum m_2 of V_2 in $L \cup \{\top_L\}$.

 - If $m_2 \in V_2$ and $m_2 \in (L \cup \top_L) \setminus L_{gr}$ then $(m_1, m_2, \top_B) \in D$ is the supremum of V.
 - If $m_2 \in V_2$ and $m_2 \in L_{gr}$ then

 $$V_3 = \{v_3 \mid (m_1, m_2, v_3) \in V\}$$

 is a subset of $B \cup \{\top_B\}$, it is nonempty and bounded from above by b_3. Since B is order isomorphic to $\mathbb{R}$, and hence B is Dedekind complete, there exists the supremum m_3 of V_3 in $B \cup \{\top_B\}$. Then $(m_1, m_2, m_3) \in D$, and it is the supremum of V.
 - If $m_2 \notin V_2$ then it follows that m_2 cannot be an element of L_{gr}, since L_{gr} is discretely embedded into L. Indeed, if $m_2 \in L_{gr}$ were an upper bound of V_2, and $m_2 \notin V_2$ then $(m_2)_\downarrow (< m_2)$ would be an upper bound of V_2, too, and hence m_2 cannot be the smallest upper bound. Hence, $m_2 \in (L \cup \{\top_L\}) \setminus L_{gr}$, and thus $(m_1, m_2, \top_B) \in D$ is the supremum of V.

Finally, let D and D_4 be countable dense subsets of A and B, respectively. Then, since H and L_{gr} are countable, and L has a countable dense subset D_L,

$$[H \times L_{gr} \times D_4] \cup [H \times (D_L \cup \{\top_L\}) \times \{\top_B\}] \cup [D \times \{\bot_L\} \times \{\top_B\}]$$

is clearly a countable, dense subset of D.

Necessity.

Assume that $\mathbf{D}$ is well-defined and $D \cong_o \mathbb{R}$.

- First we prove that $\mathbf{A} \cong_o \mathbb{R}$ and $\mathbf{B} \cong_o \mathbb{R}$.
  - Since partial lex products clearly inherit the boundedness of their first component, and since D has neither least nor greatest element, it follows that C, and in turn, A has neither least nor greatest element. If B had a greatest element g then, since $t_C \in C_{gr}$, it would yield $(t_C, g) < (t_C, \top_B)$ be a gap in D, a contradiction. Since B is involutive, it cannot have a least element either, because then it would have a greatest one, too.
  - We prove that A and B are densely ordered. Assume A isn't. Then there exists a gap $a < b$ in A. If $a \in H$ then $(a, \top_L) < (b, \bot_L)$ is a gap in C, whereas if $a \in A \setminus H$ then $(a, \bot_L) < (b, \bot_L)$ is a gap in C. In all cases, there is a gap $c < d$ in C such that $c, d \in C \setminus C_{gr}$, thus yielding a gap $(c, \top_B) < (d, \top_B)$ in D, a contradiction. If B were not densely ordered witnessed by a gap $a_4 < b_4$ then for $t_C \in C_{gr}$, $(t_C, a_4) < (t_C, b_4)$ would be a gap in D, a contradiction.

[4] $\uparrow$ is computed in L.

- Next we prove that A and B are Dedekind complete. First assume B isn't. Then there exists a nonempty subset X_4 of B bounded above by $b_4 \in B$ such that X does not have a supremum in B. Then for any $a_2 \in C_{gr}$, the set $\{(a_2, x) \mid x \in X_4\} \subseteq D$ is nonempty, it is bounded from above by (a_2, b_4), and it does not have a supremum in D, a contradiction. Second, we assume that A is not Dedekind complete, i.e., there exists a nonempty subset X of A bounded above by $b \in A$ such that X does not have a supremum in A. Let $X_3 = X \times \{\bot_L\} \times \{\top_B\}$. Then $\emptyset \neq X_3 \subseteq D$, X_3 is bounded from above by $(b, \bot_L, \top_B)$. Let $(c, d, e) \in D$ be an upper bound of X_3. Clearly, $(c, \bot_L, \top_B)$ is an upper bound of X_3, too, hence $c \in A$ is an upper bound of X. Therefore, there exists $A \ni s < c$ such that also s is an upper bound of X. Thus $(s, \bot_L, \top_B) < (c, \bot_L, \top_B)$ is an upper bound of X_3, too, showing that D is not Dedekind complete, a contradiction.
- We prove that both A and B have a countable and dense subset. Let D_3 be a countable and dense subset of D. Let $D = \{a \mid (a, l, a_4) \in D_3\}$. We claim that D is a countable and dense subset of A. Indeed, D is nonempty and countable, since so is D_3. Assume that there exists $d \in A \setminus D$ such that d is not an accumulation point of D. Since A has neither least nor greatest element, it follows that there exists $b, c \in A$ such that $b < d < c$ and there is no element of D in between b and c. Then it follows that $(d, \bot_L, \top_B) \in D \setminus D_3$ is not an accumulation point of D_3, as shown by the neighborhood $D \ni (b, \bot_L, \top_B) < (d, \bot_L, \top_B) < (c, \bot_L, \top_B) \in D$, a contradiction to D_3 being a dense subset of D. Next, we claim that $D_4 = \{a_4 \in B \mid (t_C, a_4) \in D_3\}$ is a countable and dense subset of B. Indeed, D_4 is countable, since so is D_3. D_4 is nonempty, since if for any $a_4 \in B$, $(t_C, a_4) \notin D_3$ then using that B has neither least nor greatest element and thus B is infinite, it follows that there exists $s, v, w \in B$ such that $(t_C, s) < (t_C, v) < (t_C, w)$, showing that $(t_C, v) \in D \setminus D_3$ is not and accumulation point of D_3, a contradiction. Assume that there exists $d_4 \in B \setminus D_4$ such that d_4 is not an accumulation point of D_4. Since B has neither least nor greatest element, it follows that there exists $b_4, c_4 \in B$ such that $b_4 < d_4 < c_4$ and there is no element of D_4 in between b_4 and c_4. Then it follows that $(t_C, d_4) \in D \setminus D_3$ is not an accumulation point of D_3, witnessed by the neighborhood $D \ni (t_C, b_4) < (t_C, d_4) < (t_C, c_4) \in D$, a contradiction to D_3 being a dense subset of D.

- We prove that H and L_{gr} are countable. Assume than any of them isn't. Then, since $C_{gr} = H \times L_{gr}$, it follows that C_{gr} is uncountable, too. In the preceding item, we proved that D_4 is nonempty. In complete analogy, we can prove that for any $c_2 \in C_{gr}$, $D_{c_2} = \{a_4 \in B \mid (c_2, a_4) \in D_3\}$ is nonempty either. But it means that for any $c_2 \in C_{gr}$, there is an element (c_2, y_{c_2}) in D_3. Since $c_2 \mapsto (c_2, y_{c_2})$ is injective, it follows that D_3 is uncountable, a contradiction.
- Finally we prove the statements about L.
  - Since $(\mathbf{A_H} \overrightarrow{\times} \mathbf{L}) \overrightarrow{\times} \mathbf{B}$ is well-defined, $\left(A_H \overrightarrow{\times} L\right)_{gr} = H \times L_{gr}$ is discretely embedded into $(H \times (L \cup \top_L)) \cup (A \times \{\bot_L\})$. Let $l \in L_{gr}$ be arbitrary. Then $H \times L_{gr} \ni (t_H, l)_\uparrow$ cannot be $(t_H, \top_L)$ since it is not in $H \times L_{gr}$. Therefore, $(t_H, l)_\uparrow$ is equal to $(t_H, l_\uparrow)$ and it is in $H \times L_{gr}$. Thus, $l_\uparrow \in L_{gr}$. Summing up, L_{gr} is discretely embedded into L.
  - If L has a greatest element g then $(t_H, g, \top_B) < (t_H, \top_L, \top_B)$ were a gap in D, a contradiction. Since L is involutive, it cannot have a least element either, because then it would have a greatest one, too.
  - Next we prove that L is Dedekind complete. Let an arbitrary $\emptyset \neq L_1 \subset L$ be bounded above by $l \in L$. Then $\{(t_H, l_1, \top_B) \mid l_1 \in L_1\} \subset D$ is nonempty, it is bounded from above by $(t_H, l, \top_B)$, and since D is Dedekind complete, there exists a supremum (x, y, z) of it in D. Clearly, $x = t_H$ and for any $l_1 \in L_1$, $_1 \leq y \leq l$ holds. The latest implies $y \in L$. But then y is the supremum of L_1. Indeed, if for any $l_1 \in L_1$, $_1 \leq z < y$ would hold then $(t_H, z, \top_B)$ would also be an upper bound of L_1, a contradiction.
  - If there were a gap $l_1 < l_2$ in L formed by two noninvertible elements then $D \supset H \times (L \setminus L_{gr}) \times \{\top_B\} \ni (t_H, l_1, \top_B) < (t_H, l_2, \top_B) \in H \times (L \setminus L_{gr}) \times \{\top_B\} \subset D$ would be a gap in D, a contradiction.
  - Let $D_L = \{l \in L \mid (a, l, a_4) \in D_3\}$. We prove that D_L is a countable and dense subset of L. Indeed, D_L is clearly countable since so is D_3, and $D_L \subseteq [H \times L_{gr} \times (B \cup \{\top_B\})] \cup [H \times (L \setminus L_{gr}) \times \{\top_B\}]$ holds. Assume that there is $l_1 \in D_L \setminus L$ such that l_1 is not an accumulation point of D_L. Since L has neither least nor greatest element, there are $s, v \in L$ such that $s < l_1 < v$ and there is no element of D_L strictly in between s and v. If $l_1 \in L_{gr}$ then choose $a, b, c \in B$ such that $a < b < c$; then $(t_H, l_1, b) \in D$ is not an accumulation point of D_3 witnessed by its neighborhood $D \ni (t_H, l_1, a) < (t_H, l_1, c) \in D$, a contradiction. Hence we can assume $l_1 \in L \setminus L_{gr}$. If $v \in L \setminus L_{gr}$ then there exists $w \in L$ such that $l_1 < w < v$ since there exists no gap in L formed by two elements of $L \setminus L_{gr}$, whereas if $v \in L_{gr}$ then $w := v_\downarrow < v$ holds since L_{gr} is discretely embedded into L, and $L \setminus L_{gr} \ni l_1 \neq v_\downarrow \in L_{gr}$. In both cases $s < l_1 < w < v$ follows. Therefore, $(t_H, l_1, \top_B)$ is not an accumulation point of D_3 witnessed by its neighborhood $D \ni (t_H, s, \top_B) < (t_H, w, \top_B) \in D$, a contradiction.

The next lemma characterizes when a type II partial lex extension (the first component of which is a linearly ordered abelian group) is $\cong_o \mathbb{R}$.

Lemma 6. Let $\mathbf{A}$ be a linearly ordered abelian group, $\mathbf{B}$ be an odd involutive FL_e-chain. The following statements are equivalent:

1. $\mathbf{A} \overrightarrow{\times} \mathbf{B} \cong_o \mathbb{R}$
2. $\mathbf{A} \cong \mathbb{Z}$ and $\mathbf{B} \cong_o \mathbb{R}$.

Proof. (2 ⇒1) is straightforward. (1 ⇒2): denote $\mathbb{1}$ the trivial one-element subalgebra of $\mathbb{R}$. By Lemma 4, $\mathbb{R}_{\mathbb{1}} \overrightarrow{\times} (\mathbf{A} \overrightarrow{\times} \mathbf{B}) \cong_o \mathbb{R}$. Therefore, by Lemma 2, $\mathbb{R}_{\mathbb{1}} \overrightarrow{\times} (\mathbf{A} \overrightarrow{\times} \mathbf{B}) \cong_o \mathbb{R}$. Hence, by Lemma 5, $\mathbf{B} \cong_o \mathbb{R}$ follows. Also by Lemma 5 it follows that $\mathbf{A}$ is Dedekind complete and A_{gr} is discretely embedded into A. Since $\mathbf{A} = \mathbf{A_{gr}}$ the latest condition is equivalent to saying that $x_\downarrow < x < x_\uparrow$ for $x \in A$. Thus, by Lemma A, $\mathbf{A} \cong \mathbb{Z}$.

3.3. Representation of Square Group-Like Uninorms by Basic Group-Like Uninorms

Below we define the so-called basic group-like uninorms, one for each natural number. These uninorms will serve as the building blocks in the main representation theorem in Theorem 3.

Definition 10. Let $\mathbb{U}_0 = \mathbb{R}$ and for $n \in \mathbb{N}$ let $\mathbb{U}_{n+1} = \mathbb{Z} \overrightarrow{\times} \mathbb{U}_n$. Call $\mathbb{U}_n$ the n^{th} basic uninorm. (see Figure 2 for the first two examples). Lemma 1 yields that $\mathbb{U}_n$ can equivalently be written without brackets as

$$\underbrace{\mathbb{Z} \overrightarrow{\times} \dots \overrightarrow{\times} \mathbb{Z}}_{n} \overrightarrow{\times} \mathbb{R}.$$

We are ready to give a complete characterization for basic group-like uninorms: If a group-like uninorm can be built by subsequent applications of only type II partial lex enlargements from linearly ordered abelian groups it is one of the basic uninorms. Note that already in the statement of this lemma we shall implicitly use Lemma 1 when we omit brackets.

Lemma 7. Let $n \in \mathbb{N}$, $n \geq 1$. For $i = 1, \dots, n$, let $\mathbf{G}_i$ be linearly ordered abelian groups. If $\mathbf{U} = \mathbf{G}_1 \overrightarrow{\times} \mathbf{G}_2 \overrightarrow{\times} \dots \overrightarrow{\times} \mathbf{G}_n \simeq_o \mathbb{R}$ then $\mathbf{G}_i \simeq \mathbb{Z}$ holds for $1 \leq i \leq n-1$, $\mathbf{G}_n \simeq \mathbb{R}$, and thus $\mathbf{U} \simeq \mathbb{U}_{n-1}$.

Proof. Induction on n. If $n = 1$ then $\mathbf{G}_1$ is a linearly ordered abelian group and $\mathbf{G}_1 \simeq_o \mathbb{R}$. By Lemma A, $\mathbf{G}_1 \simeq \mathbb{R}$. The case $n = 2$ is concluded by Lemma 6 and Lemma A. Assume the statement holds for $k-1$. An application of Lemma 6 to $\mathbf{G}_1 \overrightarrow{\times} (\mathbf{G}_2 \overrightarrow{\times} \dots \overrightarrow{\times} \mathbf{G}_k)$ yields that (qua FL_e-algebras) $\mathbf{G}_1 \simeq \mathbb{Z}$ and $\mathbf{G}_2 \overrightarrow{\times} \dots \overrightarrow{\times} \mathbf{G}_k \simeq \mathbb{R}_o$. By the induction hypothesis, for $2 \leq i \leq k-1$, $\mathbf{G}_i \simeq \mathbb{Z}$ and $\mathbf{G}_n \simeq \mathbb{R}$, and we are done.

We are ready to prove the main theorem. Theorem 3 is a representation theorem for those square group-like uninorms which have finitely many idempotent elements, by means of basic group-like uninorms and the type I partial lex product construction. Alternatively, one may view Theorem 3 together with Lemma 7 as a representation theorem by means of $\mathbb{Z}$ and $\mathbb{R}$ and the type I and type II partial lex product constructions.

Theorem 3. ***(Representation by basic group-like uninorms)*** *If* $\mathbf{U}$ *is a square group-like uninorm, which has* $n \in \mathbb{N}$, $n \geq 1$ *positive idempotent elements, and out of their residual complements there are* $m \in \mathbb{N}$ *strictly negative idempotent elements then there exists a sequence* $k \in \mathbb{N}^{\{0,\dots,m\}}$ *such that*

$$\mathbf{U} \cong \mathbf{X}_m,$$

where for $i \in \{0, \dots, m\}$,

$$\mathbf{X}_i = \begin{cases} \mathbb{U}_{k_0} & \text{if } i = 0 \\ \mathbf{X}_{i-1}{}_{\mathbf{Z}_{i-1}} \overrightarrow{\times} \mathbb{U}_{k_i} & \text{if } 1 \leq i \leq m \end{cases},$$

where for $2 < i \leq m$, $\mathbf{Z}_{i-1}$ *is a countable subgroup of* $(\mathbf{X}_{i-1})_{\mathbf{gr}}$.

Proof. Consider a representation of $\mathbf{U}$ by subsequent application of type III and type IV partial lex extensions. Since $\mathbf{X}_n \cong \mathbf{U} \cong_o \mathbb{R}$, $\mathbf{X}_n$ cannot have any gaps, and by Lemma 3 it follows that all the extensions in the representation of $\mathbf{U}$ must be either of type I or type II. More formally, for $i = 2, \dots, n$ there exist linearly ordered abelian groups $\mathbf{G}_1$, $\mathbf{G}_i$, $\mathbf{Z}_{i-1}$ along with $\iota_i \in \{I, II\}$ such that $\mathbf{X} \cong \mathbf{X}_n$, where for $i \in \{2, \dots, n\}$,

$$\mathbf{X}_1 = \mathbf{G}_1 \text{ and } \mathbf{X}_i = \begin{cases} \mathbf{X}_{i-1}{}_{\underrightarrow{\mathbf{Z}_{i-1}}} \overrightarrow{\times} \mathbf{G}_i & \text{if } \iota_i = I \\ \mathbf{X}_{i-1} \overrightarrow{\times} \mathbf{G}_i & \text{if } \iota_i = II \end{cases}.$$

Induction on l, the number of type I extensions in the group representation.

If $l = 0$ then $\mathbf{U} \simeq \mathbf{G}_1 \overrightarrow{\times} \mathbf{G}_2 \overrightarrow{\times} \dots \overrightarrow{\times} \mathbf{G}_n$. By denoting $k_1 = n - 1$, Lemma 7 confirms $\mathbf{U} \simeq \mathbb{U}_{k_1}$.

Let $l \geq 1$ and assume that the statement holds for $l - 1$, and that $\mathbf{U}$ has l type I extensions in its group representation. There are two cases:

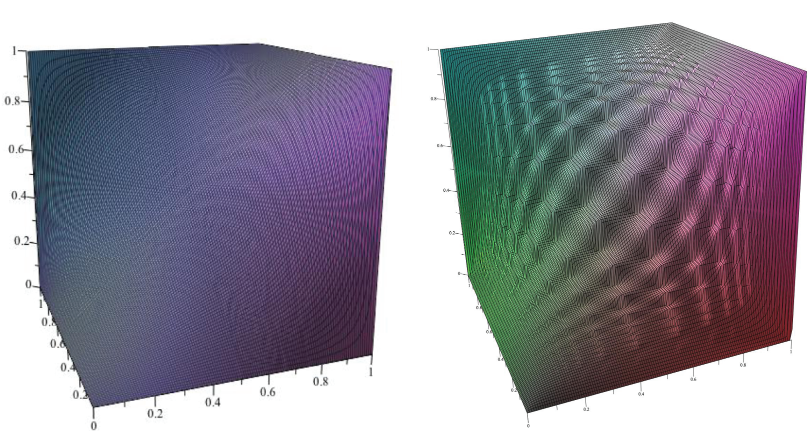

Figure 2 Visualization: the graphs of two basic group-like uninorms, $\mathbb{U}_0 = \mathbb{R}$ and $\mathbb{U}_1 = \mathbb{Z} \overrightarrow{\times} \mathbb{R}$ shrank into $]0; 1[$. One can describe $\mathbb{U}_1$ as infinitely many $\mathbb{U}_0$ components. Imagine $\mathbb{U}_2$ in the same way: as infinitely many $\mathbb{U}_1$ components, etc.

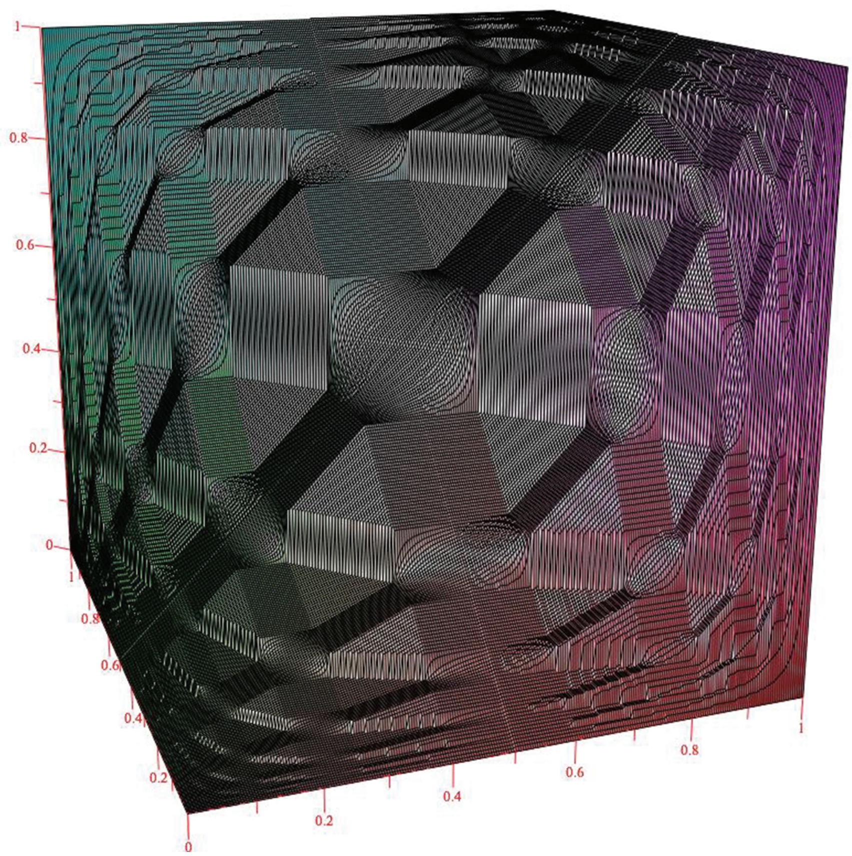

Figure 3 | Visualization: an example for a type I extension, $\mathbb{R}_{\mathbb{Z}} \overset{\twoheadrightarrow}{\times} \mathbb{R}$ shrank into $]0;1[$

If $\iota_n = I$ then $\mathbf{U} \simeq \mathbf{X}_{n-1\mathbf{Z}_{n-1}} \overset{\twoheadrightarrow}{\times} \mathbf{G}_n$ and by Lemma 4, $\mathbf{X}_{n-1} \simeq_o \mathbb{R}$, $\mathbf{G}_n \simeq_o \mathbb{R}$, and $\mathbf{Z}_{n-1}$ is countable. By Lemma A, $\mathbf{G}_n \simeq \mathbb{R} = \mathbb{U}_0$. Applying the induction hypothesis to $\mathbf{X}_{n-1}$ concludes the proof.

If $\iota_n = II$ then let $j = \max\{i \in \{1, \dots, n\} \mid \iota_i = I\}$. Note that this set in nonempty, since $l \geq 1$, that is, there is at least one type I extension in the group representation. Then

$$\mathbf{U} \simeq \left(\dots \left(\left(\mathbf{X}_{j-1\mathbf{Z}_{j-1}} \overset{\twoheadrightarrow}{\times} \mathbf{G}_j \right) \overrightarrow{\times} \mathbf{G}_{j+1} \right) \overrightarrow{\times} \dots \overrightarrow{\times} \mathbf{G}_n \right).$$

By Lemma 1 it is isomorphic to

$$\left(\mathbf{X}_{j-1\mathbf{Z}_{j-1}} \overset{\twoheadrightarrow}{\times} \mathbf{G}_j \right) \overrightarrow{\times} \left(\mathbf{G}_{j+1} \overrightarrow{\times} \dots \overrightarrow{\times} \mathbf{G}_n \right),$$

and by Lemma 2 it is isomorphic to

$$\mathbf{X}_{j-1\mathbf{Z}_{j-1}} \overset{\twoheadrightarrow}{\times} \left(\mathbf{G}_j \overrightarrow{\times} \dots \overrightarrow{\times} \mathbf{G}_n \right)$$

Applying Lemma 4 it follows that $\mathbf{X}_{j-1} \simeq_o \mathbb{R}$ and $\mathbf{G}_j \overrightarrow{\times} \dots \overrightarrow{\times} \mathbf{G}_n \simeq_o \mathbb{R}$, and $\mathbf{Z}_{j-1}$ is countable. Thus, by Lemma 7 it follows that $\mathbf{G}_j \overrightarrow{\times} \dots \overrightarrow{\times} \mathbf{G}_n \simeq \mathbb{U}_{n-j}$, and the induction hypothesis applied to $\mathbf{X}_{j-1}$ ends the proof.

Remark 1. One possible application of square group-like uninorms seems particularly manifest and interesting. In preference modeling and decision-making sometimes there are primary and secondary preferences. It means that in case two objects score equal with respect to the primary preferences, then (and only then) the secondary preferences come into play. This phenomenon is exactly imitated by a lexicographic ordering: the ranking of two objects is determined by the first ordering, and in case the two objects rank equally with respect to the first ordering then (and only then) comes into play the second ordering to determine their ranking. For such a scenario of having both primary and secondary preferences in a real-world problem, the uninorm (monoidal) operation of. e.g.. $\mathbb{Z} \overrightarrow{\times} \mathbb{R}$ works just fine. Indeed, the universe of $\mathbb{Z} \overrightarrow{\times} \mathbb{R}$ is $\mathbb{Z} \times (\mathbb{R} \cup \{\top\})$, and it is equipped with the lexicographic ordering. If the addition operation of $\mathbb{Z}$ plays the role of the aggregation operation for the primary preferences, and the addition operation of $\mathbb{R}$ plays the role of aggregating the secondary preferences then the coordinatewise-defined monoidal operation of $\mathbb{Z} \overrightarrow{\times} \mathbb{R}$ together with the lexicographic ordering on its universe does just what is needed.

3.4. Open Problem

By relying on Theorem 2 develop a corresponding characterization for the whole class of group-like uninorms having finitely many idempotent elements.

CONFLICT OF INTEREST

The authors have no conflict of interest to declare.

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