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The k-Metric Dimension of $N_k + P_n$ Graph and Starbarbell Graph

Citra Ayu Ratna Saidah[1*], Tri Atmojo Kusmayadi[2]

[1]*Department of Mathematics, Faculty of Mathematics and Natural Sciences, Sebelas Maret University, Surakarta, Indonesia*
Email: citraratna2603@gmail.com

[2]*Department of Mathematics, Faculty of Mathematics and Natural Sciences, Sebelas Maret University, Surakarta, Indonesia*

Abstract

Let G be a simple connected graph with a set of vertices $V(G)$ and set of edges $E(G)$. The distance between two vertices u and v in a graph G are the shortest path length between two vertices u and v denoted by $d(u,v)$. Let k be a positive integer, $S \subseteq V$ with S is a k-metric generator if and only if for each different vertex pair $u, v \in V$ there are at least k vertices $w_1, w_2, \dots, w_k \in S$ and fulfill $d(u, w_i) \neq d(v, w_i)$ with $i \in \{1, 2, \dots, k\}$. Minimum cardinality of a k-metric generator of a graph G is called the basis k-metric of graph G. The number of elements on the basis of k-metric graph G are called k-metric dimension of graph G and denoted by $dim_k(G)$. $N_k + P_n$ is the result of a join operation between null graph N_k and path graph P_n with $k, n \geq 2$. Starbarbell graph denoted by $SB_{m_1, m_2, \dots, m_n}$ is a graph formed from a star graph $K_{1,n}$ and n complete graph K_{m_i} then merge one vertex from each K_{m_i} with i^{th} leaf of $K_{1,n}$ with $m_i \geq 3, 1 \leq i \leq n$, and $n \geq 2$. In this paper, we determine the k-metric dimension of $N_k + P_n$ graph and starbarbell graph.

Keywords: k-metric dimension, k-metric generator, basis of k-metric, $N_k + P_n$ graph, starbarbell graph

1. Introduction

The branch of mathematics that is now developing rapidly is graph theory. According to Chartrand [5], a graph G is a finite non-empty set with $V(G) = \{v_1, v_2, \dots, v_n\}$ is the set of vertices and $E(G) = \{e_1, e_2, \dots, e_n\}$ is the set of edges that connects members of $V(G)$ in sequence.

One topic of graph theory is the k-metric dimension. Dimension of k-metric is one of the concepts in graph theory obtained from the expansion of metric dimension. The metric dimension was introduced by Slater [10] in 1975, then Harary and Melter [4] in 1976 also introduced the same concept. While the k-metric dimension was first developed by Estrada-Moreno et al. [1] in 2015. Let G be a simple connected graph with a set of vertices $V(G)$ and set of edges $E(G)$. The distance between two vertices u and v in a graph G are the shortest path length between two vertices u and v denoted by $d(u,v)$. Let k be a positive integer, $S \subseteq V$ with S is a k-metric generator if and only if for each different vertex pair $u, v \in V$ there are at least k vertices $w_1, w_2, \dots, w_k \in S$ and fulfill $d(u, w_i) \neq d(v, w_i)$ with $i \in \{1, 2, \dots, k\}$. Minimum cardinality of a k-metric generator of a graph G is called the basis k-metric of graph G. The number of elements on the basis of k-metric graph G are called k-metric dimension of graph G and denoted by $dim_k(G)$. In 2015 Estrada-Moreno et al. [1] have found the k-metric dimension in the path graph, cycle graph, tree graph, and join operation between two graphs. Then in 2016 Estrada-Moreno et al. [2] have found k-metric dimension on the corona operation between two graphs. In 2017 Geetha and Sooryanarayana [9] have found the

k-metric dimension in the graph of cartesian product operation results. In 2018 Rahmadi and Susanti [3] have found the k-metric dimension in double fan graph, double conesgraph, double fan snake graph, centralized double fan graph, generalized parachute graph, and generalized parachute graph with the upper path. The results of the research that has been done become a reference in finding the k-metric dimension on other graph classes. In this paper, we discussed the k-metric dimension of $N_k + P_n$ graph and starbarbell graph.

2. Main Results

Before giving the main results, we give the following definition and lemma due to Estrada-Moreno et al. [**1**].

Definition 1. *Let G be a* graph*. Two vertices x, y are called false twins if $N(x) = N(y)$ and x, y are called true twins if $N[x] = N[y]$. Two vertices x, y are twins if they are false twins or true twins. A vertex x is said to be a twin if there exists a vertex $y \in V(G) - \{x\}$ such that x and y are twins in G.*

Lemma 1. *A connected graph G with orde $n \geq 2$ is 2-metric dimension if and only if G has twin vertices.*

2.1. The k-metric dimension of $N_k + P_n$ graph

$N_k + P_n$ is a graph obtained from join operation between null graph N_k and path graph P_n with $k, n \geq 2$. So, $N_k + P_n$ graph has $k + n$ vertices. Chartrand and Lesniak [6] define a null graph, path graph, and join operation between two graphs. Null graph N_k is a graph whose set of edges are an empty set and the set of vertices are k vertices. Null graph is called an empty graph. While the path graph P_n is walk that does not repeat any vertices and the set of vertices are n vertices. Next, the definition of join operation in a graph. Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be a graph. Join operation between $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ denoted by $G_1 + G_2$ is a graph with $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1 \cup G_2) \cup \{uv | u \in V(G_1), v \in V(G_2)\}$. The $N_k + P_n$ graph can be depicted as in **Figure 1**.

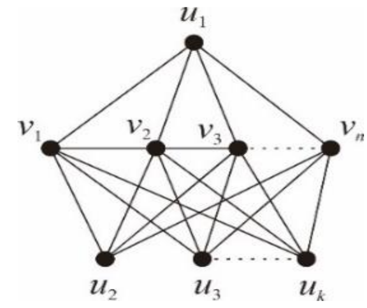

Fig. 1. $N_k + P_n$ graph

The following is given lemma of k-metric dimension on $N_k + P_n$ graph.

Lemma 2. *Let $N_k + P_n$ be a graph from join operation between null graph N_k and path graph P_n, with $k, n \geq 2$, then $N_k + P_n$ is a* 2*-metric dimension graph. Proof.* It is known that $N_k + P_n$ is a graph from join operation between null graph N_k and path graph P_n. $N_k + P_n$ graph has $k + n$ vertices.

Based on **Figure 1**, obtained that $N_{N_k+P_n}(u_1) = N_{N_k+P_n}(u_2) = N_{N_k+P_n}(u_3) = \cdots = N_{N_k+P_n}(u_k)$, so $u_1, u_2, \cdots, u_k$ are twin vertices. Based on **Lemma 1**, $N_k + P_n$ is a 2-metric dimension graph.

□

Lemma 3. *Let $N_k + P_n$ be a graph from join operation between null graph N_k and path graph P_n with $k, n \geq 2$. If S is a* 2*-metric generator of $N_k + P_n$, then $|S| \geq k + \left\lceil \frac{n+1}{2} \right\rceil$.*

Proof. It is known that S is a 2-metric generator, it means that for every $u, v \in V\ (N_k + P_n)$ there are $W \subset S$ such that $r(u|W) \neq r(v|W)$ with $|W| = 2$. Suppose S is a 2-metric generator with $|S| < k + \left\lceil \frac{n+1}{2} \right\rceil$. Assume that $V_1 = \{u_1, u_2, u_3, \ldots, u_k\}$ and $V_2 = \{v_1, v_2, v_3, \ldots, v_n\}$. Defined $S_1 = S \cap V_1$ and $S_2 = S \cap V_2$ Therefore $|S_1| + |S_2| < k + \left\lceil \frac{n+1}{2} \right\rceil$, there are $u, v \in V_1 \setminus S$ such that $r(u|W) = r(v|W)$ for each $W \subset S$ with $|W| = 2$. Therefore S is not a 2-metric generator, a contradiction. So, it is obtained that $|S| \geq k + \left\lceil \frac{n+1}{2} \right\rceil$.

□

Theorem 1. *Let $N_k + P_n$ is a graph with $k, n \geq 2$ then,*

$$dim_2(N_k + P_n) = k + \left\lceil \frac{n+1}{2} \right\rceil. \qquad (1)$$

Proof. It is known that $N_k + P_n$ is a 2-metric dimension graph for $k, n \geq 2$, it means that there is a basis of 2-metric on $N_k + P_n$. In this case, the proof is divided into two cases according the value of n.

a. For n odd numbers.

Assume that $S = \{u_1, u_2, \dots, u_k, v_1, v_3, v_5, \dots, v_n\}$, it will be shown that S is a basis of 2-metric. The representations of every vertex in $N_k + P_n$ with respect to S are

$$\begin{aligned} r(u_1|S) &= (0, 2, \dots, 2, 1, 1, 1, \dots, 1);\\ r(u_2|S) &= (2, 0, \dots, 2, 1, 1, 1, \dots, 1);\\ &\vdots\\ r(u_k|S) &= (2, 2, \dots, 0, 1, 1, 1, \dots, 1);\\ r(v_1|S) &= (1, 1, \dots, 1, 0, 2, 2, \dots, 2);\\ r(v_2|S) &= (1, 1, \dots, 1, 1, 1, 2, \dots, 2);\\ &\vdots\\ r(v_n|S) &= (1, 1, \dots, 1, 2, 2, 2, \dots, 0). \end{aligned}$$

Based on this representation, if taken a $W \subset S$ with $|W| = 2$, then for every $u, v \in V\,(N_k + P_n)$ applies $r(u|W) \neq r(v|W)$.

b. For n even numbers.

Assume that $S = \{u_1, u_2, \dots, u_k, v_1, v_3, v_5, \dots, v_{n-1}, v_n\}$, it will be shown that S is a basis of 2-metric. The representations of every vertex in $N_k + P_n$ with respect to S are

$$\begin{aligned} r(u_1|S) &= (0, 2, \dots, 2, 1, 1, 1, \dots, 1, 1);\\ r(u_2|S) &= (2, 0, \dots, 2, 1, 1, 1, \dots, 1, 1);\\ &\vdots\\ r(u_k|S) &= (2, 2, \dots, 0, 1, 1, 1, \dots, 1, 1);\\ r(v_1|S) &= (1, 1, \dots, 1, 0, 2, 2, \dots, 2, 2);\\ r(v_2|S) &= (1, 1, \dots, 1, 1, 1, 2, \dots, 2, 2);\\ &\vdots\\ r(v_n|S) &= (1, 1, \dots, 1, 2, 2, 2, \dots, 1, 0). \end{aligned}$$

Based on this representation, if taken a $W \subset S$ with $|W| = 2$, then for every $u, v \in V\,(N_k + P_n)$ applies $r(u|W) \neq r(v|W)$.

From (a) and (b), it is obtained that S is a 2-metric generator. Based on **Lemma 3**, it is obtained that S is a basis of 2-metric. So, it concludes that $dim_2(N_k + P_n) = k + \left\lceil \frac{n+1}{2} \right\rceil$.

□

2.2. The k-metric dimension of starbarbell graph

Budianto and Kusmayadi [11] define the starbarbell graph. Starbarbell graph denoted by $SB_{m_1,m_2,\dots,m_n}$ is a graph formed from a star graph $K_{1,n}$ and n complete graph K_{m_i} then merge one vertex from each K_{m_i} with i^{th} leaf of $K_{1,n}$ with $m_i \geq 3, 1 \leq i \leq n$, and $n \geq 2$. According to Chartrand et al. [8], complete graph is a graph which every pair of distinct vertices is connected by an edge. According to Chartrand and Zhang [7] complete bipartite graph $K_{m,n}$ with $m = 1$ is called star graph $K_{1,n}$ with orde $n + 1$ and size n. The starbarbell graph can be depicted as in **Figure 2**. It looks that the starbarbell graph has $\sum_{i=1}^{n} m_i + 1$ vertices.

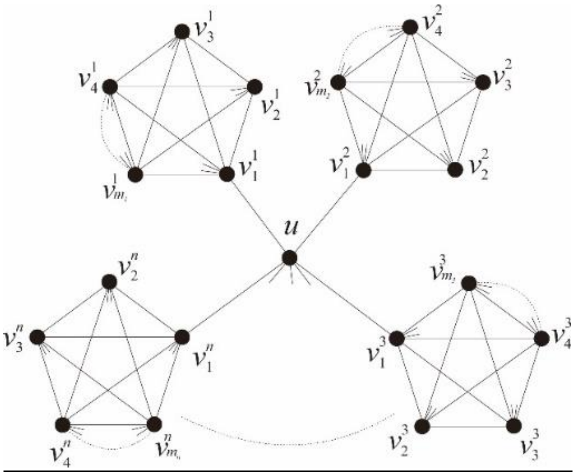

Fig. 2. Starbarbell graph $SB_{m_1,m_2,\dots,m_n}$

The distance for each of the two vertices in the starbarbell graph presented in **Table 1** (in Appendix).

The following is given lemma of k-metric dimension on starbarbell graph.

Lemma 4. The central vertex u is not contained in any basis on a starbarbell graph $SB_{m_1,m_2,\dots,m_n}$.

Proof. Proven by contradiction. Suppose that P is the basis of the starbarbell graph $SB_{m_1,m_2,\dots,m_n}$ which contains the center vertex u. Because $P \setminus \{u\}$ is not a basis, there are vertices $v, v' \in V(SB_{m_1,m_2,\dots,m_n})$ such that $d(v, x) = d(v', x)$ for every $x \in P \setminus \{u\}$. It is clear that $P = \{u\}$ is not a basis, so $P \setminus \{u\} \neq \emptyset$. If it does not apply $v = u$ and $v' = u$ then $d(v, u) = d(v', u)$ and P are not basis on the starbarbell graph $SB_{m_1,m_2,\dots,m_n}$. Without reducing the generality, we assume that $v' = u$ and v is vertex v_2^2. In this case, we obtain $d(v_2^2, x) = d(u, x)$ for each $x \in P$. This means that P is not a basis. So it is proven that the central vertex u is not contained in any basis.

□

Lemma 5. Let $SB_{m_1,m_2,\dots,m_n}$ be a starbarbell graph with $m_i \geq 3$ for every $1 \leq i \leq n$ and $n \geq 2$, then $SB_{m_1,m_2,\dots,m_n}$ is a 2-metric dimension graph.

Proof. It is known that $SB_{m_1,m_2,\dots,m_n}$ is a starbarbell graph with orde $\sum_{i=1}^{n} m_i + 1$. Based on **Figure 2**, obtained that $N_{SB_{m_1,m_2,\dots,m_n}}(v_y^x) = N_{SB_{m_1,m_2,\dots,m_n}}(v_t^s)$ with $x = s = 1, 2, \dots, n$ and $y \neq t$ for $y, t = 2, 3, \dots, m_i$. So, v_y^x and v_t^s are twin vertices. Based on **Lemma 1**, starbarbell graph $SB_{m_1,m_2,\dots,m_n}$ is a 2-metric dimension graph.

□

Lemma 6. Let $SB_{m_1,m_2,\dots,m_n}$ be a starbarbell graph with $m_i \geq 3$ for every $1 \leq i \leq n$ and $n \geq 2$. If S is a 2-metric generator of $SB_{m_1,m_2,\dots,m_n}$, then $|S| \geq \sum_{i=1}^{n}(m_i - 1)$.

Proof. It is known that S is a 2-metric generator, it means that for every $u, v \in V\ (SB_{m_1,m_2,\dots,m_n})$ there are $W \subset S$ such that $r(u|W) \neq r(v|W)$ with $|W| = 2$. Suppose S is a 2-metric generator with $|S| < \sum_{i=1}^{n}(m_i - 1)$. Assume that $V_1 = \{v_1^1, v_2^1, v_3^1, \dots, v_{m_1}^1\}$, $V_2 = \{v_1^2, v_2^2, v_3^2, \dots, v_{m_2}^2\}$, ..., $V_n = \{v_1^n, v_2^n, v_3^n, \dots, v_{m_n}^n\}$. Defined $S_1 = S \cap V_1$, $S_2 = S \cap V_2, \dots$, $S_n = S \cap V_n$. Therefore $|S_1| + |S_2| + \cdots + |S_n| < \sum_{i=1}^{n}(m_i - 1)$, there are $u, v \in V_1 \setminus S$ such that $r(u|W) = r(v|W)$ for each $W \subset S$ with $|W| = 2$. Therefore S is not a 2-metric generator, a contradiction. So, it is obtained that $|S| \geq \sum_{i=1}^{n}(m_i - 1)$.

□

Theorem 2. *Let $SB_{m_1,m_2,\dots,m_n}$ be a starbarbell graph with $m_i \geq 3$ for every $1 \leq i \leq n$ and $n \geq 2$ then,*

$$dim_2(SB_{m_1,m_2,\dots,m_n}) = \sum_{i=1}^{n}(m_i - 1). \qquad (2)$$

Proof. Let starbarbell graph $SB_{m_1,m_2,\dots,m_n}$ be a 2-metric dimension graph with $m_i \geq 3$ for every $1 \leq i \leq n$ and $n \geq 2$.

a. We will show that $S \leq \sum_{i=1}^{n}(m_i - 1)$.

Assume, the set $S = \{v_j^i\}$ with $1 \leq i \leq n$ and $2 \leq j \leq m_i$, cardinality of S is $\sum_{i=1}^{n}(m_i - 1)$. The following are given a representation of each vertex in $SB_{m_1,m_2,\dots,m_n}$ with respect to S.

$$r(v_1^1|S) = (1, 1, \dots, 1, 3, 3, \dots, 3, \dots, 3, 3, \dots, 3);$$
$$r(v_2^1|S) = (0, 1, \dots, 1, 4, 4, \dots, 4, \dots, 4, 4, \dots, 4);$$
$$\vdots$$
$$r(v_{m_1}^1|S) = (1, 1, \dots, 1, 4, 4, \dots, 4, \dots, 4, 4, \dots, 4);$$
$$\vdots$$
$$r(v_1^n|S) = (3, 3, \dots, 3, 3, 3, \dots, 3, \dots, 1, 1, \dots, 1);$$
$$r(v_2^n|S) = (4, 4, \dots, 4, 4, 4, \dots, 4, \dots, 0, 1, \dots, 1);$$
$$\vdots$$
$$r(v_{m_n}^n|S) = (4, 4, \dots, 4, 4, 4, \dots, 4, \dots, 1, 1, \dots, 1);$$
$$r(u|S) = (2, 2, \dots, 2, 2, 2, \dots, 2, \dots, 2, 2, \dots, 2).$$

Based on this representation, if taken a $W \subset S$ with $|W| = 2$, then for every $u, v \in V\ (SB_{m_1,m_2,\dots,m_n})$ applies $r(u|W) \neq r(v|W)$.

b. We will show that $S \geq \sum_{i=1}^{n}(m_i - 1)$.

Assume, the set $S = \{v_j^i\}$ with $1 \leq i \leq n$ and $2 \leq j \leq m_i$, cardinality of S is $\sum_{i=1}^{n}(m_i - 1)$. Suppose S is a 2-metric generator of the starbarbell graph $SB_{m_1,m_2,\dots,m_n}$ with $S < \sum_{i=1}^{n}(m_i - 1)$. If the set $S = \{v_x^1, v_z^y\}$ with $2 \leq x \leq m_i - 1$, $2 \leq y \leq n$, and $2 \leq z \leq m_y$. Note that for each $u, v \in V\ (SB_{m_1,m_2,\dots,m_n})$, there are set of $W \subset S$ with $|W| = 2$ which must fulfill $r(u|W) \neq r(v|W)$ for S to be 2-metric generator. Based on **Table 1 (**in Appendix), it is found there are $u, v \in V\ (SB_{m_1,m_2,\dots,m_n})$ such that $r(u|W) = r(v|W)$ for each $W \subset S$ with $|W| = 2$. Therefor S is not 2-metric generator, a contradiction. So, it is obtained that $S \geq \sum_{i=1}^{n}(m_i - 1)$. Based on **Lemma 6**, it is obtained that S is a basis of 2-metric.

From (a) and (b), it concludes that $dim_2(SB_{m_1,m_2,\dots,m_n}) = \sum_{i=1}^{n}(m_i - 1)$.

□

3. Conclusion

It can be concluded that the k-metric dimension of $N_k + P_n$ graph and starbarbell graph $SB_{m_1,m_2,\dots,m_n}$ are as stated in **Theorem 1** and **Theorem 2**.

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Apendix

Table ***1. The Distance From Each of the Two Different Vertices on the Starbarbell Graph***

Distance	v_1^1	v_2^1	⋯	$v_{m_1}^1$	v_1^2	v_2^2	⋯	$v_{m_2}^2$	⋯	v_1^n	v_2^n	⋯	$v_{m_n}^n$	u
v_1^1	0	1	⋯	1	2	3	⋯	3	⋯	2	3	⋯	3	1
v_2^1	1	0	⋯	1	3	4	⋯	4	⋯	3	4	⋯	4	2
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$v_{m_1}^1$	1	1	⋯	0	3	4	⋯	4	⋯	3	4	⋯	4	2
v_1^2	2	3	⋯	3	0	1	⋯	1	⋯	2	3	⋯	3	1
v_2^2	3	4	⋯	4	1	0	⋯	1	⋯	3	4	⋯	4	2
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$v_{m_2}^2$	3	4	⋯	4	1	1	⋯	0	⋯	3	4	⋯	4	2
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
v_1^n	2	3	⋯	3	2	3	⋯	3	⋯	0	1	⋯	1	1
v_2^n	3	4	⋯	4	3	4	⋯	4	⋯	1	0	⋯	1	2
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$v_{m_n}^n$	3	4	⋯	4	3	4	⋯	4	⋯	1	1	⋯	0	2
u	1	2	⋯	2	1	2	⋯	2	⋯	1	2	⋯	2	0