On the *k*-Metric Dimension of a Barbell Graph and a *t*-fold Wheel Graph

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Abstract

Let G be a connected and simple graph with the vertex set V(G) and the edge set E(G). The set S ⊆ V (G) is called a k-metric generator for G if and only if for every two pairs different vertices u,v ∈ V(G), there are at least k vertices w1,w2, . . .,wk ∈ S such that d(u,wi) ≠ d(v,wi) for every i ∈ {1, 2, …, k}, with d(u,v) is the length of shortest u – v path. A minimum k-metric generator is called a k-metric basis and its cardinality is called the k-metric dimension of G, denoted by dimk(G). A barbell graph Bn,n for n ≥ 3 is the simple graph obtained from two complete graph Kn connected by a bridge. A t-fold wheel graph Wnt for t ≥ 2 and n ≥ 3 is the simple graph that contain the central t vertex which are adjacent to each vertex in a cycle, but not adjacent to each other. In this paper, we determine the k-metric dimension of a barbell graph and a t-fold wheel graph.

Keywords: k-metric dimension, k-metric generator, barbell graph, t-fold wheel graph

1. Introduction

Graph theory is one branch of mathematics that deals with a network of point connected by lines. According to Chartrand et al. [1], A graph *G* is a finite nonempty set *V* of objects called vertices together with a possibly empty set *E* of 2-element subsets of *V* called edges. The development of research on graph theory has given rise to new concepts. One of the new concepts in graph theory is the metric dimension. The metric dimension was introduced by Slater [2] in 1975, then Harary and Melter [3] in 1976 also introduced the same concept. Along with development of research in graph theory, new concept was emerged to expand the concept of the metric dimension, that is the k-metric dimension. In 2015, Estrada-Moreno et al. [3] introduced the k-metric dimension of a graph. Let G be a connected and simple graph, the set S ⊆ V (G) is called a k-metric generator for G if and only if for every two pairs different vertices u,v ∈ V (G), there are at least k vertices w1,w2,…,wk ∈ S such that d(u,wi) ≠ d(v,wi) for every i ∈ {1, 2,..., k}. A minimum k-metric generator is called a k-metric basis and its cardinality is called the k-metric dimension of G, denoted by dimk(G). In 2015, Estrada-Moreno et al. [4] discovered the k-metric dimension of path graphs, cycle graphs, tree graphs and graphs resulting from joint operations with vertices on each graph are twin vertices. In 2016, Estrada-Moreno et al. [5] discovered the k-metric dimension of corona product graphs. In 2017, Geetha and

Sooryanarayana [6] discovered the 2-metric dimension of Cartesian product graphs. In 2017, Yero et al. [7] investigated computing the k-metric dimension of a graphs. In 2018, Rahmadi [8] discovered the k-metric dimension of double fan graph and some related graphs. In this paper, we determine the k-metric dimension of a barbell graph and a t-fold wheel graph.

1.1. Our Contribution

This paper presents the k-metric dimensions in several graph classes that have never been examined. This paper presents the k-metric dimensions on the barbell graph and t-fold wheel graph that refer to Estrada-Moreno et al. [3].

1.2. Paper Structure

The rest of the paper is organized as follows. Section 2 presents the results of the k-metric dimensions on the barbell graph and t-fold wheel graph. Finally, Section 3 concludes the paper and presents direction for future research.

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2. Main Result

Before strarting the main results we give the following definition and lemma due to Estrada-Moreno et al. [4].

Definition 1. Let G be a graph. Two vertices x,y are called false twins if $N(x) = N(y)$ and x,y are called true twins if $N[x] = N[y]$. Two vertices x,y are twins if they are false twins or true twins. A vertex x is said to be a twin if there exists a vertex $y \in V(G)$ - $\{x\}$ such that x and y are twins in G.

Lemma 1. *A connected graph G of order $n \geq 2$ is 2-metric dimensional if and only if G has twin vertices.*

2.1. k-Metric Dimension of a Barbell Graph

Ghosh et al. [9] defined a barbell graph is the simple graph obtained from two complete graph K_n connected by a bridge.

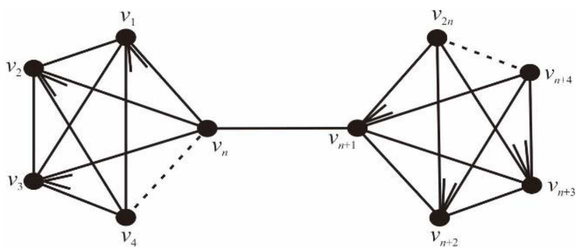

Figure 1 Barbell Graph $B_{n,n}$

The following Table 1 is the distance of every two different vertices on the barbell graph.

Table 1 the distance of every two different vertices on the barbell graph.

Distance	v_1	v_2	v_3	…	v_n	v_{n+1}	v_{n+2}	v_{n+3}	…	v_{2n}
v_1	0	1	1	…	1	2	3	3	…	3
v_2	1	0	1	…	1	2	3	3	…	3
v_3	1	1	0	…	1	2	3	3	…	3
v_4	1	1	1	…	1	2	3	3	…	3
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
v_n	1	1	1	…	0	1	2	2	…	2
v_{n+1}	2	2	2	…	1	0	1	1	…	1
v_{n+2}	3	3	3	…	2	1	0	1	…	1
v_{n+3}	3	3	3	…	2	1	1	0	…	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
v_{2n}	3	3	3	…	2	1	1	1	…	1

Lemma 2. *Let $B_{n,n}$ be a barbell graph with $n \geq 3$, then $B_{n,n}$ is a 2-metric dimension graph.*

Proof. Let $B_{n,n}$ be a barbell graph with order $2n$. Based on Figure 1 obtained that $N_{Bn,n}[v_1] = N_{Bn;n}[v_2] = ... = N_{Bn,n}[v_{n-1}]$ and $N_{Bn,n}[v_{n+2}] = N_{Bn,n}[v_{n+3}] = ... = N_{Bn,n}[v_{2n}]$, so $v_1, v_2, ..., v_{n-1}$ and $v_{n+2}, v_{n+3}, ..., v_{2n}$ is a *twin vertices.* Based on Lemma 1 obtained that $B_{n,n}$ is a 2-metric dimension graph.

□

Lemma 3. *Let $B_{n,n}$ be a barbell graph with $n \geq 3$. If S is a 2-metric generator for $B_{n,n}$ so $|S| \geq 2n - 2$.*

Proof. Let S be a 2-metric generator, meaning that for each $u, v \in V(B_{n;n})$ has $W \subset S$ such that $r(u \mid W) \neq r(v \mid W)$ with $|W| = 2$. Suppose S is 2-metric generator with $|S| < 2n - 2$, then there is set S that is $S \subseteq \{v_{pi} \mid 1 \leq i \leq (2n-2)-1\}$ with $1 \leq p_i \leq 2n$. Based on table 1 there is $r(u \mid W) = r(v \mid W)$ for each $W \subset S$ with $|W| = 2$. This contradicts with the statement that S is a 2-metric generator. The presumption is false and must be denied, thus S is not 2-metric generator. So, obtained that $|S| \geq 2n - 2$.

□

Theorem 1. *Let $B_{n,n}$ be a barbell graph. Then for $n \geq 3$* $dim_2(B_{n,n}) = 2n - 2$.

Proof. Based on Lemma 2, given $B_{n,n}$ is a 2-metric dimension graph for $n \geq 3$, it means there is 2-metric basis on $B_{n,n}$.

Let $S = \{v_1, v_2, v_3, v_4, \ldots, v_{n-1}, v_{n+2}, v_{n+3}, \ldots, v_{2n}\}$ it will be show that S set is 2-metric basis. The following is given a representation of each vertex in $B_{n,n}$ with respect to S are

$$r(v_1|S) = (0, 1, 1, 1, \ldots, 1, 3, 3, \ldots, 3);$$
$$r(v_2|S) = (1, 0, 1, 1, \ldots, 1, 3, 3, \ldots, 3);$$
$$r(v_3|S) = (1, 1, 0, 1, \ldots, 1, 3, 3, \ldots, 3);$$
$$r(v_4|S) = (1, 1, 1, 0, \ldots, 1, 3, 3, \ldots, 3);$$

⋮ ⋮

$$r(v_n|S) = (1, 1, 1, 1, \ldots, 1, 2, 2, \ldots, 2);$$
$$r(v_{n+1}|S) = (2, 2, 2, 2, \ldots, 2, 1, 1, \ldots, 1);$$
$$r(v_{n+2}|S) = (3, 3, 3, 3, \ldots, 3, 0, 1, \ldots, 1);$$
$$r(v_{n+3}|S) = (3, 3, 3, 3, \ldots, 3, 1, 0, \ldots, 1);$$

⋮ ⋮

$$r(v_{2n}|S) = (3, 3, 3, 3, \ldots, 3, 1, 1, \ldots, 0).$$

Based on the representation obtained if taken $W \subset S$ with $|W|= 2$ for each $u, v \in V(B_{n;n})$ applies $r(u|W) \neq r(v|W)$. Thus it is obtained that S is 2-metric generator. Furthermore, based on Lemma 3, it is obtained that S is 2-metric basis, and so $dim_2(B_{n,n}) = 2n - 2$. □

2.2. k-Metric Dimension of a t-fold Wheel Graph

Wallis [10], defined a t-fold wheel graph is the simple graph that contains the central t vertex which are adjacent to each vertex in cycle, but not adjacent to each other.

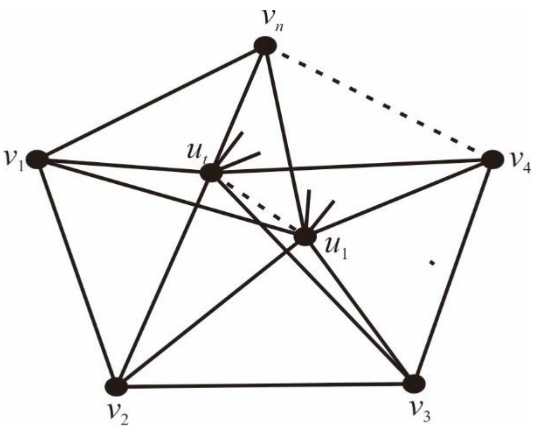

Figure 2 *t*-fold wheel Graph W_n^t

Lemma 4. *Let W_n^t be a t-fold wheel graph with $t \geq 2$ and $n \geq 3$, then W_n^t is a 2-metric dimension graph.*

Proof. Let W_n^t is *t*-fold wheel graph with order $t + n$. Based on Figure 2 obtained that $N_{W_n^t}(u_1) = N_{W_n^t}(u_2) = N_{W_n^t}(u_3) = ... = N_{W_n^t}(u_t)$, so $u_1, u_2, u_3, \ldots, u_t$ is a *twin vertices.* Based on Lemma 1 obtained that W_n^t is a 2-metric dimension graph.

□

Lemma 5. *Let W_n^t be a t-fold wheel graph with $t \geq 2$ and $n = 3,4$. If S is a 2-metric generator for W_n^t so $|S| \geq t + n$.*

Proof. Let S be a 2-metric generator, meaning that for each $u, v \in V(W_n^t)$ has $W \subset S$ such that $r(u \mid W) \neq r(v \mid W)$ with $|W|= 2$. Suppose S is 2-metric generator with $|S| < t+n \left\lceil\frac{n}{2}\right\rceil$. Let $V_1 = \{u_1, u_2, u_3, ..., u_t\}$and $V_2 = \{v_1, v_2, v_3, ..., v_n\}$. Defined $S_1 = S \cap V_1$ and $S_2 = S \cap V_2$. Because $|S_1| + |S_2| < t + n \left\lceil\frac{n}{2}\right\rceil$, there are $u, v \in V_1 \setminus S$ such that $r(u|W) \neq r(v|W)$ for each $W \subset S$ with $|W| = 2$. This contradicts with the statement that S is a 2-metric generator. The presumption is false and must be denied, thus S is not 2-metric generator. So, $|S| \geq t + n$.

□

Lemma 6. *Let W_n^t be a t-fold wheel graph with $t \geq 2$ and $n \geq 5$. If S is a 2-metric generator for W_n^t so $|S| \geq t + \left\lceil\frac{n}{2}\right\rceil$.*

Proof. . Let S be a 2-metric generator, meaning that for each $u, v \in V(W_n^t)$ has $W \subset S$ such that $r(u \mid W) \neq r(v \mid W)$ with $|W| = 2$. Suppose S is 2-metric generator with $|S| < t + \left\lceil\frac{n}{2}\right\rceil$. Let $V_1 = \{u_1, u_2, u_3, ..., u_t\}$and $V_2 = \{v_1, v_2, v_3, ..., v_n\}$. Defined $S_1 = S \cap V_1$ and $S_2 = S \cap V_2$. Because $|S_1| + |S_2| < t + \left\lceil\frac{n}{2}\right\rceil$, there are $u, v \in V_1 \setminus S$ such that $r(u|W) \neq r(v|W)$ for each $W \subset S$ with $|W| = 2$. This contradicts with the statement that S is a 2-metric generator. The presumption is false and must be denied, thus S is not 2-metric generator. So, obtained that $|S| \geq t + \left\lceil\frac{n}{2}\right\rceil$.

□

Theorem 2. *Let W_n^t be a t-fold wheel graph. Then for $t \geq 2$ and $n \geq 3$*

$$dim_2(W_n^t) = \begin{cases} t + n, t \geq 2 \text{ and } n = 3,4 \\ t + \left\lceil\frac{n}{2}\right\rceil, t \geq 2 \text{ and } n \geq 5 \end{cases}$$

Proof. Based on Lemma 4, given W_n^t is a 2-metric dimension graph for $t \geq 2$ *and* $n \geq 3$, it means there is 2-metric basis on W_n^t. In this case, the proof is divided into two cases according to the value of t and n.

Case 1. $t \geq 2$ and $n = 3,4$.

Let $S = \{u_1, u_2, ..., u_t, v_1, v_2, ..., v_n\}$ it will be show that S is 2-metric basis. The following is given a representation of each vertex in W_n^t with respect to S are

$r(u_1|S) = (0, 2, \ldots, 2, 1, 1, \ldots, 1)$;
$r(u_2|S) = (2, 0, \ldots, 2, 1, 1, \ldots, 1)$;
⋮ ⋮
$r(u_t|S) = (2, 2, \ldots, 0, 1, 1, \ldots, 1)$;
$r(v_1|S) = (1, 1, \ldots, 1, 0, 1, \ldots, 2)$;
$r(v_2|S) = (1, 1, \ldots, 1, 1, 0, \ldots, 1)$;
⋮ ⋮
$r(v_n|S) = (1, 1, \ldots, 1, 1, 2, \ldots, 0)$.

Based on this representation, if taken $W \subset S$ with $|W|= 2$, then for every $u, v \in V(W_n^t)$ applies $r(u|W) \neq r(v|W)$. Thus it is obtained that S is a 2-metric generator. Furthermore, based on Lemma 5, it is obtained that S is 2-metric basis, and so $dim_2(W_n^t) = t + n$.

Case 2. $t \geq 2$ and $n \geq 5$.

The proof for $t \geq 2$ and $n \geq 5$ divided into two cases, that are n odd and n even.

(1) For n odd.

Let $S = \{u_1, u_2, ..., u_t, v_1, v_3, v_5, ..., v_n\}$ it will be show that S is 2-metric basis. The following is given a representation of each vertex in W_n^t with respect to S are

$r(u_1|S) = (0, 2, \ldots, 2, 1, 1, 1, \ldots, 1)$;
$r(u_2|S) = (2, 0, \ldots, 2, 1, 1, 1, \ldots, 1)$;

$\vdots$ $\vdots$

$r(u_t|S) = (2, 2, \dots, 0, 1, 1, 1, \dots, 1)$;
$r(v_1|S) = (1, 1, \dots, 1, 0, 2, 2, \dots, 1)$;
$r(v_2|S) = (1, 1, \dots, 1, 1, 1, 2, \dots, 2)$;
$r(v_3|S) = (1, 1, \dots, 1, 2, 0, 2, \dots, 2)$;

$\vdots$ $\vdots$

$r(v_n|S) = (1, 1, \dots, 1, 1, 2, 2, \dots, 0)$.

Based on this representation, if taken $W \subset S$ with $|W| = 2$, then for every $u, v \in V(W_n^t)$ applies $r(u|W) \neq r(v|W)$.

(2) For n even.

Let $S = \{u_1, u_2, \dots, u_t, v_2, v_4, v_6, \dots, v_n\}$ it will be show that S is 2-metric basis. The following is given a representation of each vertex in W_n^t with respect to S are

$r(u_1|S) = (0, 2, \dots, 2, 1, 1, 1, \dots, 1)$;
$r(u_2|S) = (2, 0, \dots, 2, 1, 1, 1, \dots, 1)$;

$\vdots$ $\vdots$

$r(u_t|S) = (2, 2, \dots, 0, 1, 1, 1, \dots, 1)$;
$r(v_1|S) = (1, 1, \dots, 1, 1, 2, 2, \dots, 1)$;
$r(v_2|S) = (1, 1, \dots, 1, 0, 2, 2, \dots, 2)$;
$r(v_3|S) = (1, 1, \dots, 1, 1, 1, 2, \dots, 2)$;

$\vdots$ $\vdots$

$r(v_n|S) = (1, 1, \dots, 1, 1, 2, 2, \dots, 0)$.

Based on this representation, if taken $W \subset S$ with $|W| = 2$, then for every $u, v \in V(W_n^t)$ applies $r(u|W) \neq r(v|W)$. Thus it is obtained that S is a 2-metric generator. Furthermore, based on Lemma 6, it is obtained that S is 2-metric basis, and so $dim_2(W_n^t) = t + \left\lceil \frac{n}{2} \right\rceil$.

□

3. Conclusion

Based on the main result, it can be concluded that 1) The k-metric dimension of $B_{n,n}$ with $n \geq 3$, then $dim_2(B_{n,n}) = 2n - 2$, and 2) The k-metric dimension of W_n^t with $t \geq 2$ and $n \geq 3$, then

$$dim_2(W_n^t) = \begin{cases} t + n, t \geq 2 \text{ and } n = 3,4 \\ t + \left\lceil \frac{n}{2} \right\rceil, t \geq 2 \text{ and } n \geq 5 \end{cases}$$

Acknowledgment

This work was supported by Department of Mathematics, Faculty of Mathematics and Natural Sciences, Sebelas Maret University

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