ATLANTIS PRESS

Advances in Social Science, Education and Humanities Research, volume 467

Proceedings of the SEMANTIK Conference of Mathematics Education (SEMANTIK 2019)

k-Metric Dimension of Generalized Fan Graph and $C_m *_2 K_n$ Graph

Nur Hidayah[1], Tri Atmojo Kusmayadi[2]

[1]*Faculty of Mathematics and Natural Science, Sebelas Maret University, Surakarta, Indonesia*
E-mail: nurhidayah26427@gmail.com

[2]*Faculty of Mathematics and Natural Science, Sebelas Maret University, Surakarta, Indonesia*
E-mail: tri.atmojo.kusmayadi@staff.uns.ac.id

Abstract

Let G be a simple connected graph with the vertex set V(G) and the set edge E(G). The set $S \subseteq V(G)$ is called the k-metric generator on G if and only if every two different vertices in G are distinguished by at least k elements in S, in other words for every two different vertices $u, v \in V(G)$, there are at least k vertices $w_1, w_2, \dots, w_k \in S$ such that $d(u, w_i) \neq d(v, w_i)$, for every $i \in \{1, \dots, k\}$. The k-metric generator with the smallest cardinality is called the k-metric base and the cardinality of the k-metric base is on call the k-metric dimension of the graph G denoted $\dim_k(G)$. In this study, the k-metric dimension was obtained in generalized fan $F_{m,n}$ graph with $m \geq 1$, $n \geq 3$ and $C_m *_2 K_n$ graph with $m \geq 3$, $n \geq 2$.

Keywords: k-metric dimension, k-metric generator, basis of k-metric, generalized fan $F_{m,n}$ graph, $C_m *_2 K_n$ graph

1. Introduction

Mathematics is a science that has developed and can be applied in various fields, one of which is graph theory. Various in this field include science, computing, and networking. Graph theory can be presenting real problems in the form of graphs. A graph G according to Chartrand et al. [1] is a finite non-empty set $V(G) = \{v_1, v_2, \dots, v_n\}$ called vertices and with a possibly empty set $E(G) = \{e_1, e_2, \dots, e_n\}$ are non-consecutive pairings of members of $V(G)$ incall edge.

One concept in graph theory is the metric dimension. Metric dimension was first introduced by Slater [2] in 1975, which was later in continued by Harary and Melter [3] in 1976. Chartrand *et al*. [4] defining the differentiating set W as the set of vertices in a graph G such that each vertex in G produces a different distance to a point in W. The metric dimension is the smallest cardinality of the differentiating set. Until now the concept of the metric dimension is still being studied and studied develop. Therefore, a new metric dimension concept was developed namely the k-metric dimension presented by Estrada-Moreno *et al*. [5].

Let G be a connected and simple graph, the set $S \subseteq V(G)$ is called the k-metric generator on G if and only if every two different vertices in G are distinguished by at least k elements in S, in other words for every two different vertices $u, v \in V(G)$ there are at least k vertices $w_1, w_2, w_3, \dots, w_k \in S$ such that $d(u, w_i) \neq d(v, w_i)$, for every $i \in \{1, \dots, k\}$. The k-metric generator with the smallest cardinality is called the k-metric base and the cardinality of the k-metric base is called the k dimension the metric of the notated G graph $dim_k(G)$.

In 2015, Estrada-Moreno *et al*. [5] on the path graph, cycle graph, tree graph, and graph the result of a join operation. In 2016, Estrada Moreno *et al*. [2] have found the k-metric dimension on the corona operation of two graphs. In 2017 Geetha and Sooryanarayana [6] have found the k-metric dimension in the graph of cartesian product operation result. In 2018 Rahmadi and Susanti [7] found the k-metric dimension in double fan graph, double cones graph, double fan snake graph, centralized double fan graph, generalized parachute graph, and graph parachute generalized with the upper path. The results of these studies can be a reference in finding dimensions k-metric in another graph class. This study discusses the k-metric dimensions in generalized fan ($F_{m,n}$) graph and $C_m *_2 K_n$ graph.

2. Main Results

Before going to the main result, first will give the following lemma taken from Estrada-Moreno *et al*. [8].

Lemma 1. Let *G be a connected graph of order* $n \geq 2$, *G is a 2-metric dimension if and only if G has twin vertices.*
A. k-metric dimension of $F_{m,n}$ graph

Intaja and Sitthiwarttham [9] define a generalized fan graph. Generalized fan graph $F_{m,n} \cong \overline{K_m} + P_n$ is a graph with $V(F_{m,n}) = V(K_m) \cup V(P_n)$ and $E(F_{m,n}) = E(P_n) \cup \{u, v | u \in V(\overline{K_m}), v \in V(P_n)\}$ with $m \geq 1$ and $n \geq 2$. $F_{m,n}$ the graph is illustrated in Fig 1.

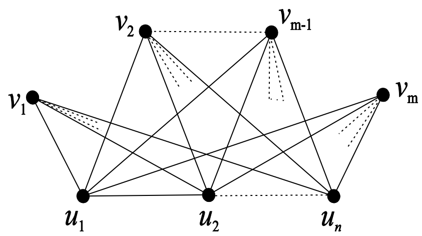

Fig 1. Generalized Fan Graph

From Fig 1, it can be seen that the graph $F_{m,n}$ has $m + n$ vertices. The following TABLE I gives the distance of every two vertices in the $F_{m,n}$ graph.

Table 1. The distance of each two vertices is different on $F_{m,n}$ graph

Dist	v_1	v_2	$\cdots$	v_m	u_1	u_2	$\cdots$	$\boldsymbol{u_n}$
v_1	0	2	$\cdots$	2	1	1	$\cdots$	1
v_2	2	0	$\cdots$	2	1	1	$\cdots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_m	2	2	$\cdots$	0	1	1	$\cdots$	1
u_1	1	1	$\cdots$	1	0	1	$\cdots$	2
u_2	1	1	$\cdots$	1	1	0	$\cdots$	2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\boldsymbol{u_n}$	1	1	$\cdots$	1	2	2	$\cdots$	0

Following are the result regarding the k-metric dimensions in a graph $F_{m,n}$.

Lemma 2. *Let $F_{m,n}$ be a generalized fan graph with $m \geq 2$ and $n \geq 3$, then $F_{m,n}$ is a 2-metric dimension graph.*

Proof. It is known that $F_{m,n}$ is a generalized fan graph of order $m + n$. There are $N_{F_{m,n}}(v_1) = N_{F_{m,n}}(v_2) = \cdots = N_{F_{m,n}}(v_n)$, such that $(v_1, v_2, \ldots, v_n)$ is twin vertices. Then $F_{m,n}$ is a 2-metric dimension graph.

The following is given a lower bound for the 2-metric generator cardinality of $F_{m,n}$ graph.

Lemma 3. *Let $F_{m,n}$ be a generalized fan graph with $m \geq 2$ and n = 3, 4. If S is a 2-metric generator of $F_{m,n}$, then* $|S| \geq (m + n) - 1$.

Proof. It is known that S is a 2-metric generator, meaning that for each $u, v \in V(F_{m,n})$ has $W \subset S$ such that $r(u|W) \neq r(v|W)$ with $|W| = 2$. Suppose S is a 2-metric generator with $|S| < (m + n) - 1$. Let $V_1 = \{u_1, u_2, \ldots, u_n\}$ and $V_2 = \{v_1, v_2, \ldots, v_m\}$. Write $S_1 = S \cap V_1$ and $S_2 = S \cap V_2$, because $|S_1| + |S_2| < (m + n) - 1$, then there are $u, v \in V_1 \backslash S$ such that $r(u|W) = r(v|W)$ for each $W \subset S$ with $|W| = 2$. This contradicts with the statement that S is a 2-metric generator. Presupposition wrong and must be denied, thus S is not a 2-metric generator. Hence $|S| \geq (m + n) - 1$.

Lemma 4. *Let $F_{m,n}$ be a generalized fan graph with $m \geq 2$ and $n \geq 5$. If S is a 2-metric generator of $F_{m,n}$, then* $|S| \geq (m + n) - 2$.

Proof. It is known that S is a 2-metric generator, meaning that for each $u, v \in V(F_{m,n})$ has $W \subset S$ such that $r(u|W) \neq r(v|W)$ with $|W| = 2$. Suppose S is a 2-metric generator with $|S| < (m + n) - 2$. Let $V_1 = \{u_1, u_2, \ldots, u_n\}$ and $V_2 = \{v_1, v_2, \ldots, v_m\}$. Write $S_1 = S \cap V_1$ and $S_2 = S \cap V_2$, because $|S_1| + |S_2| < (m + n) - 2$, then there are $u, v \in V_1 \backslash S$ such that $r(u|W) = r(v|W)$ for each $W \subset S$ with $|W| = 2$. This contradicts with the statement that S is a 2-metric generator. Presupposition wrong and must be denied, thus S is not a 2-metric generator. Hence $|S| \geq (m + n) - 2$.

■

Theorem 1. *Let $F_{m,n}$ be a graph with $m \geq 1$ and $n \geq 2$, then*

$$dim_2(F_{m,n}) = \begin{cases} (m + n) - 1, & m \geq 2 \quad n = 3, 4; \\ (m + n) - 2, & m \geq 2 \quad n \geq 5. \end{cases}$$

Proof. Let $F_{m,n}$ be a 2-metric dimension graph for $m \geq 2$ and $n \geq 3$, meaning that there is a 2-metric base on $F_{m,n}$. In this case, the evidence is divided into two cases, namely for $m \geq 2$ with $n = 3, 4$ and for $m \geq 2$ with $n \geq 5$.
Case 1. For $m \geq 2$ and $n = 3, 4$.

- For $m \geq 2$ and $n = 3$.

Take $S = \{v_1, v_2, \ldots, v_m, u_1, u_3\}$, will show that S is a 2-metric basis. The following are given a representation of each vertex in $F_{m,3}$ with respect to S.

$$r(v_1|S) = \{0, 2, \ldots, 2, 1, 1\};$$
$$r(v_2|S) = \{2, 0, \ldots, 2, 1, 1\};$$
$$\vdots \quad = \quad \vdots$$
$$r(v_m|S) = \{2, 2, \ldots, 0, 1, 1\};$$
$$r(u_1|S) = \{1, 1, \ldots, 1, 0, 2\};$$
$$r(u_2|S) = \{1, 1, \ldots, 1, 1, 1\};$$
$$r(u_3|S) = \{1, 1, \ldots, 1, 2, 0\}.$$

Based on this representation, if taken a $W \subset S$ with $|W| = 2$, then for each $u, v \in V(F_{m,3})$ applies $r(u|W) \neq r(v|W)$.

- For $m \geq 2$ and $n = 3$.

Take $S = \{ v_1, v_2, \dots, v_m, u_1, u_2, u_3\}$, will show that S is a 2-metric basis. The following are given a representation of each vertex in $F_{m,4}$ with respect to S.

$$r(v_1|S) = \{0, 2, \dots, 2, 1, 1, 1\};$$
$$r(v_2|S) = \{2, 0, \dots, 2, 1, 1, 1\};$$
$$\vdots \quad = \quad \vdots$$
$$r(v_m|S) = \{2, 2, \dots, 0, 1, 1, 1\};$$
$$r(u_1|S) = \{1, 1, \dots, 1, 0, 1, 2\};$$
$$r(u_2|S) = \{1, 1, \dots, 1, 1, 0, 1\};$$
$$r(u_3|S) = \{1, 1, \dots, 1, 2, 1, 0\};$$
$$r(u_4|S) = \{1, 1, \dots, 1, 2, 2, 1\}.$$

Based on this representation, if taken a $W \subset S$ with $|W| = 2$, then for each $u, v \in V(F_{m,4})$ applies $r(u|W) \neq r(v|W)$.

Case 2. $m \geq 2$ and $n \geq 5$.

Take $S_1 = \{ v_1, v_2, \dots, v_m, u_2, \dots, u_{n-1}\}$ or $S_2 = \{ v_1, v_2, \dots, v_m, u_1, \dots, u_{n-4}, u_{n-2}, u_n\}$ will show that S_1 and S_2 is a 2-metric basis. The following are given a representation of each vertex in $F_{1,n}$ with respect to S_1.

$$r(v_1|S_1) = \{0, 2, \dots, 2, 1, \dots, 1\};$$
$$r(v_2|S_1) = \{2, 0, \dots, 2, 1, \dots, 1\};$$
$$\vdots \quad = \quad \vdots$$
$$r(v_m|S_1) = \{2, 2, \dots, 0, 1, \dots, 1\};$$
$$r(u_1|S_1) = \{1, 1, \dots, 1, 1, \dots, 2\};$$
$$r(u_2|S_1) = \{1, 1, \dots, 1, 0, \dots, 2\};$$
$$\vdots \quad = \quad \vdots$$
$$r(u_n|S_1) = \{1, 1, \dots, 1, 2, \dots, 1\}.$$

and

$$r(v_1|S_2) = \{0, 2, \dots, 2, 1, \dots, 1, 1, 1\};$$
$$r(v_2|S_2) = \{2, 0, \dots, 2, 1, \dots, 1, 1, 1\};$$
$$\vdots \quad = \quad \vdots$$
$$r(v_m|S_2) = \{2, 2, \dots, 0, 1, \dots, 1, 1, 1\};$$
$$r(u_1|S_2) = \{1, 1, \dots, 1, 0, \dots, 2, 2, 2\};$$
$$r(u_2|S_2) = \{1, 1, \dots, 1, 1, \dots, 2, 2, 2\};$$
$$\vdots \quad = \quad \vdots$$
$$r(u_n|S_2) = \{1, 1, \dots, 1, 2, \dots, 2, 2, 0\}.$$

Based on this representation, if taken a $W \subset S$ with $|W| = 2$, then for each $u, v \in V(F_{m,n})$ applies $r(u|W) \neq r(v|W)$.

Thus it is obtained that S is a 2-metric generator. Furthermore, based on Lemma 3, it is obtained that S is a 2-metric basis, so $dim_2(F_{m,n}) = (m+n) - 1$ and based on Lemma 4, it is obtained that S is a 2-metric basis, so $dim_2(F_{m,n}) = (m+n) - 2$.

■

K-metric dimension of $C_m *_2 K_n$ graph

Let $C_m *_2 K_n$ be a graph with $m \geq 3$ and $n \geq 3$ is a graph resulting from operations edge amalgamation or combining one edge on C_m and one edge on K_n so that it becomes one edge that incident with vertex x and vertex y. The graph of $C_m *_2 K_n$ the graph is illustrated in Fig 2.

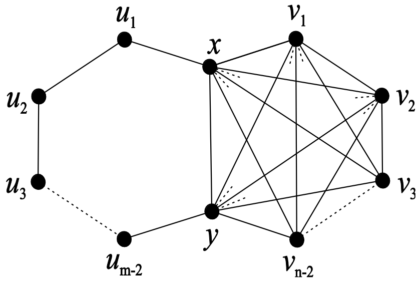

Fig 2. $C_m *_2 K_n$ graph

From Fig 2, it can be seen that the graph $C_m *_2 K_n$ has $m + n - 2$ vertices. The following table is given a distance of every two vertices in the graph $C_m *_2 K_n$ presented in Table 2.

Table 2. The distance between every two vertices is different on the graph $C_m *_2 K_n$

Dist	x	y	u_1	u_2	$\cdots$	u_{m-2}	v_1	v_2	$\cdots$	v_{n-2}
x	0	1	1	2	$\cdots$	2	1	1	$\cdots$	1
y	1	0	2	3	$\cdots$	1	1	1	$\cdots$	1
u_1	1	2	0	1	$\cdots$	3	2	2	$\cdots$	2
u_2	2	3	1	0	$\cdots$	4	3	3	$\cdots$	3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
u_{m-2}	2	1	3	4	$\cdots$	0	2	2	$\cdots$	2
v_1	1	1	2	3	$\cdots$	2	0	1	$\cdots$	1
v_2	1	1	2	3	$\cdots$	2	1	0	$\cdots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_{n-2}	1	1	2	3	$\cdots$	2	1	1	$\cdots$	0

Following are the result regarding the dimensions of *k*-metrics in $C_m *_2 K_n$ graph.

Lemma 5. *Let $C_m *_2 K_n$be a graph obtained from the results edge amalgamation operation between the C_m graph and the K_n graph with $m \geq 3$ and $n \geq 3$, then $C_m *_2 K_n$ is a 2-metric dimension graph.*

Proof. It is known that $C_m *_2 K_n$ is a graph obtained from the operation results edge amalgamation between the C_m graph and K_n graph of order $m + n + 2$. Obtained that $N_{C_m *_2 K_n}[v_1] = N_{C_m *_2 K_n}[v_2] = \cdots = N_{C_m *_2 K_n}[v_{n-2}]$, such that $(v_1, v_2, \dots, v_{n-2})$ is twin vertices. Then $C_m *_2 K_n$ is a 2-metric dimension graph.

The following is given a bound limit for the 2-metric generator cardinality of $C_m *_2 K_n$ graph.

Lemma 6. *Let $C_m *_2 K_n$be a graph obtained from the results edge amalgamation operation between C_m graph and K_n*

graph with $m \geq 3$ and $n \geq 3$. If S is a 2-metric generator of $C_m *_2 K_n$, then $|S| \geq n$..

Proof. It is known that S is a 2-metric generator, meaning that for each $u, v \in V(C_m *_2 K_n)$ has $W \subset S$ such that $r(u|\mathrm{W}) \neq r(v|\mathrm{W})$ with $|W| = 2$. Suppose S is a 2-metric generator with $|S| < n$, then there is the possible set of S that is,

1. $S \subseteq \{x, y, v_{k_i} | 1 \leq \mathrm{i} \leq \mathrm{n} - 3\}$ with $k_i \in \{1, \dots, n\}$.
2. $S \subseteq \{x, v_{k_i} | 1 \leq \mathrm{i} \leq \mathrm{n} - 2\}$ with $k_i \in \{1, \dots, n\}$.
3. $S \subseteq \{y, v_{k_i} | 1 \leq \mathrm{i} \leq \mathrm{n} - 2\}$ with $k_i \in \{1, \dots, n\}$.
4. $S \subseteq \{u_h, u_j, v_{k_i} | 1 \leq \mathrm{i} \leq \mathrm{n} - 3\}$ with $k_i \in \{1, \dots, n\}$ for some $h, j \in [1, m], i \neq j$.
5. $S \subseteq \{u_j, v_{k_i} | 1 \leq \mathrm{i} \leq \mathrm{n} - 2\}$ with $k_i \in \{1, \dots, n\}$ for some $h, j \in [1, m]$.

Note that for each $u, v \in V(C_m *_2 K_n)$, there is a set

$W \subset S$ with $|W| = 2$ which must satisfy $r(u|\mathrm{W}) \neq r(v|\mathrm{W})$ for S to be 2-metric generator. Based on Table 4.3 it is found that there are $u, v \in V(C_m *_2 K_n)$, such that $r(u|\mathrm{W}) = r(v|\mathrm{W})$ for each $W \subset S$ with $|W| = 2$. This contradicts with the statement that S is a 2-metric generator. Assumption wrong and must be denied, thus S is not a 2-metric generator. So $|\mathrm{S}| \geq n$.

Theorem 2. *Let $C_m *_2 K_n$ be a graph with $m \geq 3$ and $n \geq 3$, then*

$$dim_2(C_m *_2 K_n) = \begin{cases} 4, & m = 3 \ \ n = 3; \\ n, & m \geq 3 \ n \geq 4. \end{cases}$$

Proof. It is known that $C_m *_2 K_n$ is a 2-metric dimension graph for $m \geq 3$ and $n \geq 3$, meaning that there is a 2-metric base on $C_m *_2 K_n$. The proof is divided into two cases, namely for $m = 3$ with $n = 3$ and $m \geq 3$ with $n \geq 4$.

Case 1. For $m = 3$ and $n = 3$.

Take $S = \{x, y, u_1, v_1\}$, will show that S is a 2-metric basis. The following are given a representation of each vertex in $C_3 *_2 K_3$ with respect to S.

$$r(x|S) = \{0, 1, 1, 1\};$$
$$r(y|S) = \{1, 0, 1, 1\};$$
$$r(u_1|S) = \{1, 1, 0, 2\};$$
$$r(v_1|S) = \{1, 1, 2, 0\}.$$

Based on the representation obtained, selected $W_1 = \{x, y\}$, $W_2 = \{x, u_1, \}$, $W_3 = \{y, u_1\}$ dan $W_4 = \{u_1, v_1\}$. Obtained that

$$r(x|W_1) \neq r(y|W_1);$$
$$r(x|W_2) \neq r(u_1|W_2);$$
$$r(x|W_2) \neq r(v_1|W_2);$$
$$r(y|W_3) \neq r(u_1|W_3);$$
$$r(y|W_3) \neq r(v_1|W_3);$$
$$r(u_1|W_4) \neq r(v_1|W_4).$$

It is thus obtained that for every two vertices in $V(C_3 *_2 K_3)$ distinguished by two elements in S, in other words, S is generator 2-metric.

It will be further shown that S is a 2-metric basis. Suppose S is not a 2-metric basis, meaning that there is $S_1 < 4$ which is a 2-metric generator. For example $|\mathrm{S}| = 3$ obtained four possible S_1, that is

(a) $S_1 = \{x, y, u_1\}$;
(b) $S_1 = \{x, y, v_1\}$;
(c) $S_1 = \{y, u_1, v_1\}$;
(d) $S_1 = \{x, u_1, v_1\}$.

From the possibilities (a) - (b), it can be seen that $r(u_1|W) = r(v_1|W)$. Furthermore, from the possibility of (c) - (d) it can be seen that $r(x|W) = r(y|W)$ with $W \subset S_1$ with $|\mathrm{W}| = 2$, then for each $W \subset S_1$. Thus there is no 2-metric generator with a cardinality of 3.

Furthermore, if $|S_1| = 2$, then S_1 is not a 2-metric base for $C_3 *_2 K_3$, because $C_3 *_2 K_3 \ncong P$ for each $n \geq 2$. It is thus obtained that $S = \{x, y, u_1, v_1\}$ is a 2-metric basis. So $dim_2(C_3 *_2 K_3) = 4$.

Case 2. For $m \geq 3$ $n \geq 4$.

Take $S = \{x, y, v_1, \dots, v_{n-2}\}$ will be indicated that S is a 2-metric base. The following are given a representation of each vertex at $C_m *_2 K_n$ with respect to S.

$$\begin{aligned} r(x|S) &= \{0, 1, 1, \dots, 1\}; \\ r(y|S) &= \{1, 0, 1, \dots, 1\}; \\ r(u_1|S) &= \{1, 2, 2, \dots, 2\}; \\ \vdots &= \vdots \\ r(u_{m-2}|S) &= \{2, 1, 2, \dots, 2\}; \\ r(v_1|S) &= \{1, 1, 0, \dots, 1\}; \\ \vdots &= \vdots \\ r(v_{n-2}|S) &= \{1, 1, 1, \dots, 0\}. \end{aligned}$$

Based on this representation, if taken a $W \subset S$ with $|\mathrm{W}| = 2$, then for each $u, v \in V(C_m *_2 K_2)$ applies $r(u|\mathrm{W}) \neq r(v|\mathrm{W})$. Thus it was found that S is a 2-metric generator. Furthermore, based on Lemma 6, it is obtained that S is a 2-metric basis, so $dim_2(C_m *_2 K_n)\ ) = n$.

3. Conclusion

The k-metric generator with the smallest cardinality is called the k-metric base and the cardinality of the k-metric base is on call the k-metric dimension of the graph G denoted dim_k (G). In this study, the k-metric dimension was obtained in generalized fan F_(m,n) graph with m ⩾1, n⩾3 and C_m *_2 K_n graph with m ⩾3, n⩾2.

Acknowledgment

The authors would thank the Mathematics Department of Mathematics and Natural Sciences Faculty, Sebelas Maret University for all the supports.

References

[1] G. Chartrand, L. Lesniak, and P. Zhang, *Graphs and Digraphs*, 6th ed., CRC Press, New York, 2016.

[2] P. J. Slater, "Leave of trees," *Congressus Numerantium*, vol. 14, pp.549-559, 1975.

[3] F. Harary and R. A. Melter, "On the metric dimension of a graph," *Ars Combinatoria*, vol. 2, pp.191-195, 1976.

[4] G. Chartrand, L. Eroh, M. Johnson, and O. Oellermann, "Resolvability in graphs and the metric dimension of graph," *Discrete Appl. Math*, vol. 105, pp.98-113, 2000.

[5] A. Estrada-Moreno, J. A. Rodriguez-Velazquez, and I. G. Yero, "The k-metric dimension of a graph," *Applied Mathematics and Information Sciences*, vol. 9, pp.2829-2840, 2015.

[6] K. N. Geetha and B Sooryanarayana, "2-metric dimension of the cartesian product of graphs," *International Journal of Pure and Applied Mathematics*, vol. 112, no. 1, pp.27-45, 2017.

[7] D. Rahmadi, and Y. Susanti, "Dimension of *k*-metric on double fan graph and multiple graph-related", Thesis Department of Mathematics, Faculty of Mathematics and Natural Science UGM, Yogyakarta, 2018.

[8] A. Estrada-Moreno, J. A. Rodriguez-Velazquez, and I. G. Yero, "The k-metric dimension of corona product graph," *Bull. Malays. Math. Sci.*, vol. 39, no. 1, pp.135-156, 2016.

[9] S. Intaja and T. Sitthiwarttham, "Some graph parameters of fan graph," *International Journal of Pure and Applied Mathematics*, vol. 80, pp.217-223, 2012.