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Inferences for the Type-II Exponentiated Log-Logistic Distribution Based on Order Statistics with Application

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ABSTRACT

In this paper, we first derive the exact explicit expressions for the single and product moments of order statistics from the type-II exponentiated log-logistic distribution, and then use these results to compute the means, variances, skewness and kurtosis of rth order statistics. Besides, best linear unbiased estimators (BLUEs) for the location and scale parameters for the type-II exponentiated log-logistic distribution with known shape parameters are studied. Finally, the results are illustrated with a real data set.

1. INTRODUCTION

Rao *et al.* (2012) suggested a generalization of the log-logistic distribution called type-II exponentiated log-logistic (TIIELL) distribution with probability density function (pdf)

$$f(x) = \frac{\tau\eta\left(\frac{x}{\xi}\right)^{\eta-1}}{\xi\left[1+\left(\frac{x}{\xi}\right)^{\eta}\right]^{\tau+1}}, \quad x > 0, \quad (\tau,\xi) > 0, \eta > 1. \tag{1}$$

The cumulative distribution function (cdf) and quantile function are, respectively given by

$$F(x) = 1 - \left[1+\left(\frac{x}{\xi}\right)^{\eta}\right]^{-\tau}, \quad x > 0, \quad (\tau,\xi) > 0, \eta > 1 \tag{2}$$

and

$$F^{-1}(x) = \xi\left(\left(\frac{1}{1-x}\right)^{\frac{1}{\tau}} - 1\right)^{\frac{1}{\eta}}, \tag{3}$$

where ξ is the scale parameter and η and τ are the shape parameters of the distribution. If $\tau = 1$, then equation (1) becomes log-logistic distribution, and if $\eta = 1$, then TIIELL distribution becomes Pareto type-II distribution. The kth moment of the TIIELL distribution in equation (1) can be easily derived as

$$E(X^k) = \xi^k \tau B\left(\tau - \frac{k}{\eta}, 1 + \frac{k}{\eta}\right), \tag{4}$$

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where $B(.,.)$ is the beta function. Note that the kth moment exists if $\eta > \max\{1, k/\tau\}$. A more compact form of equation (4) can be derived using the fact that $\Gamma(z)\Gamma(1-z) = \pi csc(\pi z)$ (Abramowitz and Stegun [1]) as follows:

$$\begin{aligned} E(X^k) &= \frac{\xi^k \tau\, \Gamma\left(\tau - \frac{k}{n}\right) \Gamma\left(1 + \frac{k}{n}\right)}{\Gamma(\tau+1)} \\ &= \frac{\xi^k \tau \left(\tau - \frac{k}{n} - 1\right)\left(\tau - \frac{k}{n} - 2\right) \ldots \left(\tau - \frac{k}{n} - (\tau-1)\right) \Gamma\left(\tau - \frac{k}{n} - (\tau-1)\right)\left(\frac{k}{n}\right)\Gamma\left(\frac{k}{n}\right)}{\tau\Gamma(\tau)} \\ &= \frac{\xi^k \left(\frac{k}{n}\right)\Gamma\left(\frac{k}{n}\right)\Gamma\left(1 - \frac{k}{n}\right)}{\Gamma(\tau)} \prod_{i=1}^{\tau-1}\left(\tau - \frac{k}{\tau} - i\right) \\ &= \frac{\xi^k \frac{k}{\eta}\pi csc\frac{k\pi}{\eta}}{\Gamma(\tau)} \prod_{i=1}^{\tau-1}\left(\tau - \frac{k}{\eta} - i\right). \end{aligned} \tag{5}$$

Therefore,

$$E(X) = \frac{\xi \frac{\pi}{\eta} csc\frac{\pi}{\eta}}{\Gamma(\tau)} \prod_{i=1}^{\tau-1}\left(\tau - \frac{1}{\eta} - i\right),$$

$$E(X^2) = \frac{\xi^2 \frac{2\pi}{\eta} csc\frac{2\pi}{\eta}}{\Gamma(\tau)} \prod_{i=1}^{\tau-1}\left(\tau - \frac{2}{\eta} - i\right)$$

and

$$Var(X) = \left[\frac{\xi^2 \frac{2\pi}{\eta} csc\frac{2\pi}{\eta}}{\Gamma(\tau)} \prod_{i=1}^{\tau-1}\left(\tau - \frac{2}{\eta} - i\right) - \left(\frac{\xi \frac{\pi}{\eta} csc\frac{\pi}{\eta}}{\Gamma(\tau)} \prod_{i=1}^{\tau-1}\left(\tau - \frac{1}{\eta} - i\right)\right)^2\right].$$

To the best of our knowledge, the work on type-II generalized log-logistic model is scanty in literature. The few works available in literature on this distribution which includes: Rao *et al.* [2,3] developed the reliability and an economic reliability test plans for this distribution. Kumar [4] studied exact moments of generalized order statistics from TIIELL distribution. Recently, Rao *et al.* [5] studied Bootstrap confidence intervals (CIs) of the process capability index, C_{Npk} for type-II generalized log-logistic distribution.

Order statistics and functions of these statistics occupy a place of great significance in diverse field of studies involving theoretical and practical problems such as characterization of probability distributions, entropy estimation, analysis of censored samples, reliability analysis and quality control (see Arnold *et al.* [6] and David and Nagaraja [7]). The moments of order statistics have applicability in areas such as quality control, reliability, etc. For example, it is seen that when the duration of the failed items is high, the reliability of an item is also high, which in turn makes the product too costly, both in terms of time and money. In such a situation the one may not know enough about the item in a short period of time and hence would require few early failures data for predicting the failure of future items. Thus moments of order statistics is useful in making these kinds of prediction in such situations.

In recent past several authors have tabulated the moments of order statistics quite extensively for several distributions and also obtained maximum likelihood estimates (MLEs) and best linear unbiased estimators (BLUEs) for the scale and location parameters of the distributions based on complete and type-II censored samples. Further, they developed point prediction and goodness-of-fit tests. In this regard, readers may refer to the works of Balakrishnan and Cohan [8], Balakrishnan and Sultan [8], Sultan and Balakrishnan [9,10], Genç [11], Jabeen *et al.* [12], Mir Mostafaee [13], Balakrishnan *et al.* [14], Sultan and AL-Thubyani [15], Kumar *et al.* [16], Kumar and Dey [17,18], Ahsanullah and Alzaatreh [19], Kumar and Goyal [20,21], Kumar *et al.* [22] and many others.

In this paper, we derive the exact expressions for the single and product moments of order statistics from TIIELLD in Sections 2 and 3. In Section 4, we obtain BLUEs for θ and ξ by using these moments. These BLUEs are then used in Section 5 to obtain $(1-\alpha)100\%$ CIs for the location and scale parameters of the BLUEs based on the pivotal quantities. Besides, lower and upper percentage points of pivotal quantities through Edgeworth approximations are obtained and compare the results with simulated percentage points. A real data application is provided in Section 6. Finally, in Section 7, we draw a conclusion for the paper.

2. SOME RELATIONS FOR THE MOMENTS OF ORDER STATISTICS

Let $X_1, X_2, \ldots, X_n$ be n independent copies of a random variable X that follows TIIELL distribution. Let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ be the corresponding order statistics, then the pdf of the rth order statistic is

$$f_{X_{r:n}}(x) = C_{r:n}\, F^{r-1}(x)\,[1 - F(x)]^{n-r} f(x), x > 0, \tag{6}$$

where $C_{r:n} = \dfrac{n!}{(r-1)!\,(n-r)!}$. From equations (1), (2) and (6), the pdf of rth order statistic from TIIELL distribution is given by

$$f_{X_{r:n}}(x) = \frac{\tau\eta}{\xi} C_{r:n} \sum_{i=0}^{r-1} \binom{r-1}{i} (-1)^i \frac{\left(\frac{x}{\xi}\right)^{\eta-1}}{(1+\left(\frac{x}{\xi}\right)^{\eta})^{\tau i+\tau+1+\tau(n-r)}}, \quad x \geq 0. \tag{7}$$

The *kth* moment of $X_{r:n}$ can be derived by the following Theorem:

Theorem 1. *For the TIIELL distribution given in equation (1) and for* $1 \leq r \leq n$ *and* $k = 1, 2, ...,$

$$\alpha_{r:n}^{(k)} = \tau\, \xi^k\, C_{r:n} \sum_{i=0}^{r-1} \binom{r-1}{i} (-1)^i B\left(\tau(i+n-r+1) - \frac{k}{\eta}, 1 + \frac{k}{\eta}\right), \tag{8}$$

where $B(a,b)$ *denotes the beta function defined by* $B(a,b) = \int_0^\infty \frac{x^{a-1}}{(1+x)^{a+b}} dx$.

Proof. From equation (7), we have

$$\begin{aligned}
\alpha_{r:n}^{(k)} &= E(X_{r:n}^k) = \frac{\tau\eta}{\xi} C_{r:n} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \int_0^\infty x^k \frac{\left(\frac{x}{\xi}\right)^{\eta-1}}{(1+\left(\frac{x}{\xi}\right)^{\eta})^{\tau i+\tau+1+\tau(n-r)}} dx \\
&= \tau\, \xi^k\, C_{r:n} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \int_0^\infty \frac{u^{\frac{k}{\eta}}}{(1+u)^{\tau(n-r+i+1)+1}} du,
\end{aligned}$$

where $u = \left(\frac{x}{\xi}\right)^{\eta}$. The result follows from the definition of the beta function

Theorem 2. *For the TIIELL distribution given in equation (1) and for* $1 \leq r \leq n$ *and* $k = 1, 2, ...,$ *the kth moment of rth order statistics can be expressed as*

$$\alpha_{r:n}^{(k)} = \frac{\xi^k\, k\, \tau\, n!\, \pi\, csc\frac{k\pi}{\eta}}{\eta (r-1)!\,(n-r)!} \sum_{i=0}^{r-1} \binom{r-1}{i} (-1)^i \prod_{j=1}^{\tau(i+n-r+1)-1} \frac{\tau(i+n-r+1) - j - \frac{k}{\eta}}{\Gamma\left(\tau(i+n-r+1)-1\right)}. \tag{9}$$

Proof. The proof is straightforward and omitted for brevity.

Note that from equation (9), the first and second moments of $X_{r:n}$ are, respectively, given by

$$\alpha_{r:n}^{(1)} = \frac{\xi\tau n!\,\pi csc\frac{\pi}{\eta}}{\eta(r-1)!\,(n-r)!} \sum_{i=0}^{r-1} \binom{r-1}{i} (-1)^i \prod_{j=1}^{\tau(i+n-r+1)-1} \frac{\tau(i+n-r+1) - j - \frac{1}{\eta}}{\Gamma\left(\tau(i+n-r+1)-1\right)},$$

and

$$\alpha_{r:n}^{(2)} = \frac{\xi^2\tau n!\,\pi csc\frac{2\pi}{\eta}}{\eta(r-1)!\,(n-r)!} \sum_{i=0}^{r-1} \binom{r-1}{i} (-1)^i \prod_{j=1}^{\tau(i+n-r+1)-1} \frac{\tau(i+n-r+1) - j - \frac{2}{\eta}}{\Gamma\left(\tau(i+n-r+1)-1\right)},$$

Some special cases from equation (9) are

1. For $\tau = 1$, in equation (9), we get the explicit expression for order statistic of log-logistic distribution

$$\alpha_{r:n}^{(k)} = \frac{\xi^k k n!\,\pi csc\frac{k\pi}{\eta}}{\eta(r-1)!\,(n-r)!} (-1)^r \prod_{j=1}^{n} \frac{n-r+1-j-\frac{k}{\eta}}{\Gamma\left((n-r+1)-1\right)}.$$

2. If $r = n = 1$, we get

$$\alpha_{1:1}^{(k)} = \xi^k \tau B\left(\tau - \frac{k}{\eta}, 1 + \frac{k}{\eta}\right)$$

which agrees with equation (4).

3. If $k = r = 1$ in equation (9), we get

$$\alpha_{1:n}^{(1)} = \frac{\xi\tau n\pi csc\frac{\pi}{\eta}}{\eta}\prod_{j=1}^{\tau n-1}\frac{\left[\tau n-j-\frac{1}{\eta}\right]}{\Gamma(\tau n-1)},$$

4. If $k = 1, r = n$ in equation (9), we get

$$\alpha_{n:n}^{(1)} = \frac{\xi\tau n\pi csc\frac{\pi}{\eta}}{\eta}\sum_{i=0}^{n-1}\binom{n-1}{i}(-1)^i\prod_{j=1}^{\tau(i+1)-1}\frac{\left[\tau(i+1)-j-\frac{1}{\eta}\right]}{\Gamma(\tau(i+1)-1)},$$

5. If $k = r = n = 1$ in equation (9), we get

$$\alpha_{1:1}^{(1)} = \frac{\xi\tau\pi csc\frac{\pi}{\eta}}{\eta}\prod_{j=1}^{\tau-1}\frac{\left[\tau-j-\frac{1}{\eta}\right]}{\Gamma(\tau-1)}, \tag{10}$$

which agree with equation (5) for $k = 1$.

It is interesting to note that the equation (9) can be used easily to derive several recurrence relations for the moments of order statistics. Some of these recurrence relations already exist in the literature. Below, we provide some of these recurrence relations.

I. From equation (9), we can write

$$\alpha_{r:n}^{(k)} = \frac{\xi^k\,\tau\,k\,n(n-1)!\;\pi\;csc\frac{k\pi}{\eta}}{\eta(r-1)(r-2)!\,(n-r)!}\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^{r-1+i}\prod_{j=1}^{\tau(i+n-r+1)-1}\frac{\tau(i+n-r+1)-j-\frac{k}{\eta}}{\Gamma(\tau(i+n-r+1)-1)}.$$

Let us define $\Delta(\tau,\eta) = \frac{\xi^k\,\tau\,k\,(n-1)!\,\pi\,csc\frac{k\pi}{\eta}}{\eta(r-2)!(n-r)!}$, we have

$$\begin{aligned}
\alpha_{r-1:n-1}^{(k)} &= \Delta(\tau,\eta)\sum_{i=0}^{r-2}\binom{r-2}{i}(-1)^i\prod_{j=1}^{\tau(i+n-r+1)-1}\frac{\tau(i+n-r+1)-j-\frac{k}{\eta}}{\Gamma(\tau(i+n-r+1)-1)}\\
&= \Delta(\tau,\eta)\binom{r-2}{0}(-1)^0\prod_{j=1}^{\tau(n-r+1)-1}\frac{\tau(n-r+1)-j-\frac{k}{\eta}}{\Gamma(\tau(i+n-r+1)-1)}\\
&\quad+\Delta(\tau,\eta)\binom{r-2}{1}(-1)^1\prod_{j=1}^{\tau(n-r+2)-1}\frac{\tau(n-r+2)-j-\frac{k}{\eta}}{\Gamma(\tau(n-r+2)-1)}\\
&\quad\vdots\\
&\quad+\Delta(\tau,\eta)\binom{r-2}{r-2}(-1)^{r-2}\prod_{j=1}^{\tau(n-1)-1}\frac{\tau(n-1)-j-\frac{k}{\eta}}{\Gamma(\tau(n-1)-1)},
\end{aligned}$$

which can be written in vector form as

$$\alpha_{r-1:n-1}^{(k)} = 1'\,{}_i\alpha_{r-1:n-1}^{(k)},$$

where $1' = (1, 1, \dots, 1)$ and ${}_i\alpha_{r-1:n-1}^{(k)}$ denotes a vector of order $(1\times r-2)$ and $((r-2)\times 1)$, respectively, where

$${}_i\alpha_{r-1:n-1}^{(k)} = \begin{pmatrix}\Delta(\tau,\eta)\binom{r-2}{0}(-1)^0\prod_{j=1}^{\tau(n-r+1)-1}\frac{\tau(n-r+1)-j-\frac{k}{\eta}}{\Gamma(\tau(n-r+1)-1)}\\ \vdots\\ \vdots\\ \Delta(\tau,\eta)\binom{r-2}{r-2}(-1)^{r-2}\prod_{j=1}^{\tau(n-1)-1}\frac{\tau(n-1)-j-\frac{k}{\eta}}{\Gamma(\tau(n-1)-1)}.\end{pmatrix}$$

Therefore, we can write $\alpha_{r:n}^{(k)}$ as

$$\begin{aligned}
\alpha_{r:n}^{(k)} &= \frac{n}{r-1}\Delta(\tau,\eta)\left[\binom{r-1}{r-1}(-1)^{r-1}\prod_{j=1}^{\tau n-1}\frac{\tau n-j-\frac{k}{\eta}}{\Gamma(\tau n-1)}\right.\\
&\quad\left.+\sum_{i=0}^{r-2}\frac{r-1}{r-1-i}\binom{r-2}{i}(-1)^{i}\prod_{j=1}^{\tau(i+n-r+1)-1}\frac{\tau(i+n-r+1)-j-\frac{k}{\eta}}{\Gamma(\tau(i+n-r+1)-1)}\right]\\
&= \frac{n}{r-1}\Delta(\tau,\eta)(-1)^{r-1}\prod_{j=1}^{\tau n-1}\frac{\tau n-j-\frac{k}{\eta}}{\Gamma(\tau n-1)}\\
&\quad+n\Delta(\tau,\eta)\sum_{i=0}^{r-2}\frac{1}{(r-1-i)}\binom{r-2}{i}(-1)^{i}\prod_{j=1}^{\tau(i+n-r+1)-1}\frac{\tau(i+n-r+1)-j-\frac{k}{\eta}}{\Gamma(\tau(i+n-r+1)-1)}\\
&= \frac{n}{r-1}\Delta(\tau,\eta)(-1)^{r-1}\prod_{j=1}^{\tau n-1}\frac{\tau n-j-\frac{k}{\eta}}{\Gamma(\tau n-1)}+n\,v'\,{}_i\alpha_{r-1:n-1}^{(k)},
\end{aligned}$$

where $v'=\left(\frac{1}{r-1},\frac{1}{r-2},\frac{1}{r-3},\ldots,1\right)$ is vector of order $(1\times(r-2))$ **II** If $r=1$ in equation (9), we get

$$\alpha_{1:n}^{(k)} = \prod_{u=1}^{\tau}\frac{n\left(u-\frac{k}{\eta}\right)}{\eta(n-1)\prod_{h=1}^{\tau+1}[\tau(i+n)-h]}\alpha_{1:n-1}^{(k)}.$$

3. PRODUCT MOMENTS OF ORDER STATISTICS

Let $X_{1:n}\le X_{2:n}\le\cdots\le X_{n:n}$ be the order statistics from the TIIELL distribution given in (1) with its cdf in equation (2). Then, the joint pdf of the rth and sth order statistics is

$$f_{X_{(r:n)},X_{(s:n)}}(x,y) = C_{r,s:n}\,F^{r-1}(x)\left[F(y)-F(x)\right]^{s-1-r}\left[1-F(y)\right]^{n-s}f(x)f(y), \tag{11}$$

for $0<x<y$, $r,s=1,2,\ldots,r<s$ and $C_{r,s:n}=\frac{n!}{(r-1)!(s-r-1)!(n-s)!}$. To obtain the covariance between $X_{r:n}$ and $X_{s:n}$, consider the joint pdf of $X_{r:n}$ and $X_{s:n}$, $1\le r<s\le n$ as follows:

$$\begin{aligned}
f_{r,s:n}(x,y) &= \frac{\tau^2\eta^2}{\xi^2}C_{r,s:n}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}\binom{r-1}{i}\binom{s-r-1}{j}(-1)^{i+j}\\
&\quad\times\frac{\left(\frac{x}{\xi}\right)^{\eta-1}\left(\frac{y}{\xi}\right)^{\eta-1}}{\left(1+\left(\frac{x}{\xi}\right)^{\eta}\right)^{\tau(i+s-r-j)+1}\left(1+\left(\frac{y}{\xi}\right)^{\eta}\right)^{\tau(n-s+1+j)+1}}.
\end{aligned}\tag{12}$$

Therefore, the product moments of $X_{r:n}$ and $X_{s:n}$, can be obtain from the following Theorem:

Theorem 3. *For the TIIELL distribution given in (1) and for* $1\le r<s\le n$,

$$\begin{aligned}
\alpha_{r,s:n} &= \xi^2\tau^2C_{r,s:n}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}\binom{r-1}{i}\binom{s-r-1}{j}(-1)^{i+j}\left\{B\left(\tau(i+s-r-j)-\frac{1}{\eta},\frac{1}{\eta}+1\right)\right.\\
&\quad\times B\left(\frac{1}{\eta}+1,\tau(n-s+1+j)-\frac{1}{\eta}\right)-\psi(\tau,\eta)\\
&\quad\left.\times\sum_{k=0}^{\infty}B\left(\frac{1}{\eta}+1,\tau(i+n-r+1)-\frac{2}{\eta}+k+1\right)\right\},
\end{aligned}\tag{13}$$

where $B(a,b)$ denotes the beta function defined by $B(a,b)=\int_0^\infty\frac{x^{a-1}}{(1+x)^{a+b}}\,dx$ and

$$\psi(\tau,\eta) = \frac{\left(\tau(i+s-r-j)-\frac{1}{\eta}\right)_k\left(\frac{1}{\eta}\right)_k}{\left(\tau(i+s-r-j)-\frac{1}{\eta}+1\right)_k\left(\tau(i+s-r-j)-\frac{1}{\eta}+1\right)k!}.$$

Proof. From equation (12), we have

$$\begin{aligned}
\alpha_{r,s:n} &= E(X_{r:n}X_{s:n}) = \tau^2\eta^2 C_{r,s:n}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}\binom{r-1}{i}\binom{s-r-1}{j}(-1)^{i+j} \\
&\quad\times\int_0^\infty\int_0^y \frac{\left(\frac{x}{\xi}\right)^\eta\left(\frac{y}{\xi}\right)^\eta}{\left(1+\left(\frac{x}{\xi}\right)^\eta\right)^{\tau(i+s-r-j)+1}\left(1+\left(\frac{y}{\xi}\right)^\eta\right)^{\tau(n-s+1+j)+1}}dxdy \\
&= \xi\tau^2\eta^2 C_{r,s:n}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}\binom{r-1}{i}\binom{s-r-1}{j}(-1)^{i+j} \\
&\quad\times\int_0^\infty \frac{\left(\frac{y}{\xi}\right)^\eta}{\left(1+\left(\frac{y}{\xi}\right)^\eta\right)^{\tau(n-s+1+j)+1}}\left(\frac{1}{\eta}\int_0^{\left(\frac{y}{\xi}\right)^\eta}\frac{u^{\frac{1}{\eta}}}{(1+u)^{\tau(i+s-r-j)+1}}du\right)dy \\
&= \xi\tau^2\eta^2 C_{r,s:n}\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}\binom{r-1}{i}\binom{s-r-1}{j}(-1)^{i+j} \\
&\quad\times\int_0^\infty \frac{\left(\frac{y}{\xi}\right)^\eta}{\left(1+\left(\frac{y}{\xi}\right)^\eta\right)^{\tau(n-s+1+j)+1}}\delta(t)dy,
\end{aligned} \tag{14}$$

where $x^\eta = u$ and $t = \frac{1}{u+1}$, it is not difficult to show that $\delta(t)$ can be simplified

$$\begin{aligned}
\delta(t) &= B\left(\tau(i+s-r-j)-\frac{1}{\eta},\frac{1}{\eta}+1\right) - \frac{\left(\frac{1}{1+\left(\frac{y}{\xi}\right)^\eta}\right)^{\tau(i+s-r-j)+1-\frac{1}{\eta}}}{\tau(i+s-r-j)+1-\frac{1}{\eta}} \\
&\quad\times{}_2F_1\left[\tau(i+s-r-j)-\frac{1}{\eta},\frac{1}{\eta},\tau(i+s-r-j)-\frac{1}{\eta}+1,\frac{1}{(1+\left(\frac{y}{\xi}\right)^\eta)}\right],
\end{aligned} \tag{15}$$

where ${}_pF_q$ is the generalized hypergeometric function defined as

$${}_pF_q(a_1,\dots,b_1,\dots,b_q;x) = \sum_{k=0}^{\infty}\frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k}\frac{x^k}{k!},$$

where $(f)_k = f(f+1)\cdots(f+k-1)$ denotes the ascending factorial. The result follows by using equations (14) and (15). The proof is complete.

4. ESTIMATION OF PARAMETERS

4.1. Estimation of the Location and Scale Parameters

In this section, we study parameter estimation for the TIIELL distribution based on order statistics. Let $Y_1 \le Y_2 \le \cdots \le Y_m$ be a random sample of size m from TIIELL distribution, the pdf of the scale-parameter given in (1) and the pdf of the location-scale parameter TIIELL distribution is

$$f(x) = \frac{\tau\eta\left(\frac{x-\theta}{\xi}\right)^{\eta-1}}{\xi\left(1+\left(\frac{x-\theta}{\xi}\right)^\eta\right)^{\tau+1}},\quad x>\theta,\quad (\tau,\xi)>0,\quad \eta>1. \tag{16}$$

Let $Y_{r+1:m} \le Y_{r+2:m} \le \dots \le Y_{m-s:m}$ denote the type-II right-censored sample from the location-scale parameter TIIELL distribution in equation (1). Let us denote $X_{i:m} = (Y_{i:m}-\theta)/\xi$, $E(X_{i:m}) = \theta^{(1)}_{i:m}$, $1 \le r \le (m-s)$, and $Cov(X_{i:m}, X_{j:m}) = \xi_{i,j:m} = \theta^{(1,1)}_{i,j:m} - \theta^{(1)}_{i:m}\,\theta^{(1)}_{j:m}$, $1 \le i < j \le (m-s)$. Therefore

$$\mathbf{Y} = (Y_{r+1:m}, Y_{r+2:m}, \dots, Y_{m-s:m})^T,$$

$$\boldsymbol{\theta} = (\theta_{r+1:m}, \theta_{r+2:m}, \dots, \theta_{m-s:m})^T,$$

$$1 = \underbrace{(1,\ 1,\ \dots,\ 1)}_{m-s}{}^T,$$

and

$$\sum = ((\xi_{i,j:m})),\ \ r+1 \le i,\ \ j \le m-s,$$

where $\theta_{i:m} = E(Y_{i:m})$, $\xi_{ii} = Var(Y_{i:m})$ and $\xi_{ij} = Cov(Y_{i:m}, Y_{j:m})$, $i, j = 1, 2, \dots (m-s)$. Then the BLUEs of θ and ξ can be computed as follows [see Balakrishnan and Cohen [23], Sultan *et al.* [9,10]]

$$\theta^* = \sum_{i=r+1}^{m-s} p_i Y_{i:m} \quad \text{and} \qquad \xi^* = \sum_{i=r+1}^{m-s} q_i Y_{i:m},$$

where

$$p_i = \left\{ \frac{\theta^{\mathbf{T}} \sum^{-1} \theta 1^T \sum^{-1} - \theta^{\mathbf{T}} \sum^{-1} 1 \theta^T \sum^{-1}}{(\theta^{\mathbf{T}} \sum^{-1} \theta)(1^{\mathbf{T}} \sum^{-1} 1) - (\theta^{\mathbf{T}} \sum^{-1} 1)^2} \right\}, \tag{17}$$

$$q_i = \left\{ \frac{1^T \sum^{-1} 1 \theta^{\mathbf{T}} \sum^{-1} - 1^{\mathbf{T}} \sum^{-1} \theta 1^{\mathbf{T}} \sum^{-1}}{(\theta^{\mathbf{T}} \sum^{-1} \theta)(1^{\mathbf{T}} \sum^{-1} 1) - (\theta^{\mathbf{T}} \sum^{-1} 1)^2} \right\} \tag{18}$$

and the variances and covariance of these BLUEs can be computed as follows [see Balakrishnan and Cohen [23], Sultan *et al.* [9,10]]

$$Var(\theta^*) = \xi^2 \left\{ \frac{\theta^{\mathbf{T}} \sum^{-1} \theta}{(\theta^{\mathbf{T}} \sum^{-1} \theta)(1^{\mathbf{T}} \sum^{-1} 1) - (\theta^{\mathbf{T}} \sum^{-1} 1)^2} \right\} = \xi^2 W_1, \tag{19}$$

$$Var(\xi^*) = \xi^2 \left\{ \frac{1^T \sum^{-1} 1}{(\theta^T \sum^{-1} \theta)(1^{\mathbf{T}} \sum^{-1} 1) - (\theta^{\mathbf{T}} \sum^{-1} 1)^2} \right\} = \xi^2 W_2, \tag{20}$$

and

$$Cov(\theta^*,\ \xi^*) = \xi^2 \left\{ \frac{-\theta^{\mathbf{T}} \sum^{-1} 1}{(\theta^{\mathbf{T}} \sum^{-1} \theta)(1^{\mathbf{T}} \sum^{-1} 1) - (\theta^{\mathbf{T}} \sum^{-1} 1)^2} \right\} = \xi^2 W_3, \tag{21}$$

for details readers may refer to the works of David [24], Balakrishnan and Cohen [23], and Arnold *et al.* [6]. The values of p_i and q_i are displayed in Tables 3 and 4 for different values of sample sizes $m = 7,\ 10$ and different censoring cases $s = 0(1)([m/2] - 1)$ and for some selected values for $\eta = 2, 3$. The coefficient of the BLUEs p_i and q_i given by equations (17) and (18), respectively satisfies the conditions

$$\sum_{i=r+1}^{m-s} p_i = 1$$

and

$$\sum_{i=r+1}^{m-s} q_i = 0,$$

which are used to check the computations accuracy.

Table 1 Expected values, second moments, variances, skewness and kurtosis of the rth order statistic from type-II exponentiated log-logistic (TIIELL) distribution for $n = 1, 2, \ldots, 10$, $\tau = 2.5$, $\eta = 2$ and $\xi = 0.25$

r	n	$E(X)$	$E(X^2)$	$V(X)$	τ_1	τ_2	γ_1	γ_2
1	1	0.166667	0.041667	0.013889	6.124244	24.00119	2.474721	21.00119
	2	0.107379	0.015625	0.004095	1.480527	5.834856	1.216769	2.834856
	3	0.085248	0.009615	0.002348	0.970243	4.645654	0.985009	1.645654
	4	0.072834	0.006944	0.001639	0.775930	4.337018	0.880869	1.337018
	5	0.064627	0.005435	0.001258	0.690985	4.060399	0.831255	1.060399
	6	0.058687	0.004464	0.001020	0.653052	3.734868	0.808116	0.734868
	7	0.054132	0.003788	0.000858	0.577341	4.350134	0.759829	1.350134
	8	0.050495	0.003289	0.000739	0.576750	3.853975	0.759440	0.853975
	9	0.047505	0.002907	0.000650	0.534243	4.254706	0.730919	1.254706
	10	0.044990	0.002604	0.000580	0.479173	3.332768	0.692223	0.332768
2	2	0.225955	0.067708	0.016652	6.669762	26.89585	2.582588	23.89585
	3	0.151640	0.027644	0.004649	1.311287	5.810189	1.145115	2.810189
	4	0.122490	0.017628	0.002624	0.742933	4.483691	0.861936	1.483691
	5	0.105661	0.012983	0.001819	0.537407	4.080492	0.733081	1.080492
	6	0.094328	0.010287	0.001389	0.445519	3.949799	0.667472	0.949799
	7	0.086020	0.008523	0.001124	0.391159	3.375693	0.625427	0.375693
	8	0.079588	0.007277	0.000943	0.335838	3.046048	0.579516	0.046048
	9	0.074417	0.006349	0.000811	0.307304	3.904693	0.554350	0.904693
	10	0.070140	0.005632	0.000712	0.278758	3.848412	0.527975	0.848412
3	3	0.263112	0.087740	0.018512	7.184192	28.97034	2.680334	25.97034
	4	0.180790	0.037660	0.004975	1.322886	5.987013	1.150168	2.987013
	5	0.147734	0.024596	0.002771	0.707746	4.471739	0.841277	1.471739
	6	0.128326	0.018375	0.001907	0.492489	3.942891	0.701776	0.942891
	7	0.115099	0.014699	0.001451	0.380277	4.011283	0.616666	1.011283
	8	0.105315	0.012261	0.001170	0.312036	3.453741	0.558602	0.453741
	9	0.097688	0.010522	0.000979	0.302289	3.434712	0.549808	0.434712
	10	0.091521	0.009218	0.000842	0.251679	2.882750	0.501676	0.399884
4	4	0.290553	0.104434	0.020013	7.578064	30.47057	2.752828	27.47057
	5	0.202827	0.046370	0.005231	1.363788	6.098496	1.167813	3.098496
	6	0.167141	0.030817	0.002881	0.708859	4.536456	0.841938	1.536456
	7	0.145963	0.023276	0.001971	0.489885	3.966303	0.699918	0.966303
	8	0.131407	0.018762	0.001494	0.363963	3.895946	0.603293	0.895946
	9	0.120569	0.015738	0.001201	0.301321	3.393636	0.548927	0.393636
	10	0.112075	0.013565	0.001004	0.272530	3.072031	0.522044	0.072031
5	5	0.312484	0.118950	0.021304	7.887611	31.61464	2.808489	28.61464
	6	0.220670	0.054146	0.005451	1.404782	6.200086	1.185235	3.200086
	7	0.183024	0.036473	0.002975	0.717814	4.548271	0.847239	1.548271
	8	0.160518	0.027791	0.002025	0.477320	4.057296	0.690884	1.057296
	9	0.144955	0.022541	0.001529	0.368251	3.717102	0.606837	0.717102
	10	0.133308	0.018998	0.001227	0.293470	3.286510	0.541729	0.286501
6	6	0.330847	0.131910	0.022450	8.139275	32.53265	2.852941	29.53265
	7	0.235729	0.061216	0.005648	1.443160	6.280985	1.201316	3.280985
	8	0.196528	0.041682	0.003059	0.735319	4.529456	0.857508	1.529456
	9	0.172968	0.031990	0.002072	0.488891	4.037825	0.699208	1.037825
	10	0.156602	0.026085	0.001561	0.363625	3.532627	0.603013	0.532627
7	7	0.346701	0.143693	0.023492	8.346127	33.28143	2.888966	30.28143
	8	0.248795	0.067727	0.005828	1.477872	6.381171	1.215677	3.381171
	9	0.208308	0.046527	0.003135	0.750557	4.606563	0.866347	1.606563
	10	0.183879	0.035927	0.002116	0.499695	4.026693	0.706891	1.026693
8	8	0.360687	0.154545	0.024450	8.522843	33.91907	2.919391	30.91907
	9	0.260363	0.073784	0.005995	1.505565	6.457144	1.227015	3.457144
	10	0.218777	0.051071	0.003207	0.757008	4.664468	0.870062	1.664468
9	9	0.373227	0.164640	0.025342	8.674506	34.46545	2.945251	31.46545
	10	0.270760	0.079463	0.006152	1.531512	6.518105	1.237543	3.518105
10	10	0.384613	0.174104	0.026177	8.808048	34.94389	2.967835	31.94389

Table 2 Expected values, second moments, variances, skewness and kurtosis of the rth order statistic from type-II exponentiated log-logistic (TIIELL) distribution for $n = 1, 2, \ldots, 10$, $\tau = 5$, $\eta = 2$ and $\xi = 0.5$

r	n	$E(X)$	$E(X^2)$	$V(X)$	τ_1	τ_2	γ_1	γ_2
1	1	0.214757	0.062501	0.016379	1.482629	5.832059	1.217633	2.832059
	2	0.145668	0.027778	0.006559	0.782284	4.113691	0.884468	1.113691
	3	0.117375	0.017857	0.004080	0.626158	3.790735	0.791301	0.790735
	4	0.100991	0.013158	0.002959	0.560859	3.566212	0.748905	0.566212
	5	0.089980	0.010417	0.002321	0.525832	3.442966	0.725143	0.442966
	6	0.081930	0.008621	0.001908	0.500077	3.502186	0.707161	0.502186
	7	0.075714	0.007353	0.001620	0.495238	3.237636	0.703732	0.237636
	8	0.070728	0.006410	0.001408	0.478453	3.227514	0.691703	0.227514
	9	0.066612	0.005682	0.001245	0.456279	3.397797	0.675484	0.397797
	10	0.063141	0.005102	0.001115	0.451449	3.303742	0.671901	0.303742
2	2	0.283846	0.097222	0.016653	1.400052	6.014629	1.183238	3.014629
	3	0.202256	0.047619	0.006712	0.563894	4.001324	0.750929	1.001324
	4	0.166526	0.031955	0.004224	0.391693	3.627183	0.625854	0.627183
	5	0.145032	0.024123	0.003089	0.322388	3.480211	0.567792	0.480211
	6	0.130231	0.019397	0.002437	0.284753	3.357070	0.533623	0.357070
	7	0.119225	0.016227	0.002012	0.265080	3.246232	0.514859	0.246232
	8	0.110621	0.013952	0.001715	0.245458	3.320959	0.495437	0.320959
	9	0.103652	0.012238	0.001494	0.230641	3.469610	0.480251	0.469610
	10	0.097858	0.010901	0.001324	0.233139	2.927740	0.482845	0.218501
3	3	0.324642	0.122024	0.016632	1.466340	6.256670	1.210925	3.256670
	4	0.237985	0.063283	0.006646	0.526732	4.039598	0.725763	1.039598
	5	0.198768	0.043703	0.004194	0.342041	3.552107	0.584842	0.552107
	6	0.174635	0.033575	0.003078	0.263913	3.429646	0.513725	0.429646
	7	0.157746	0.027320	0.002436	0.222956	3.410097	0.472182	0.410097
	8	0.145037	0.023054	0.002018	0.193353	3.346856	0.439719	0.346856
	9	0.135010	0.019951	0.001723	0.176914	3.377494	0.420612	0.377494
	10	0.126829	0.017590	0.001504	0.175914	2.934875	0.419421	0.192158
4	4	0.353527	0.141604	0.016623	1.540551	6.459855	1.241189	3.459855
	5	0.264129	0.076337	0.006573	0.522017	4.094437	0.722507	1.094437
	6	0.222901	0.053831	0.004146	0.322694	3.617912	0.568062	0.617912
	7	0.197152	0.041916	0.003047	0.241729	3.378629	0.491659	0.378629
	8	0.178928	0.034431	0.002416	0.197982	3.283768	0.444952	0.283768
	9	0.165092	0.029259	0.002004	0.179578	3.085929	0.423766	0.085929
	10	0.154098	0.025460	0.001714	0.161099	3.196057	0.401371	0.196057
5	5	0.375877	0.157921	0.016637	1.608648	6.618557	1.268325	3.618557
	6	0.284744	0.087590	0.006511	0.529866	4.110246	0.727919	1.110246
	7	0.242213	0.062767	0.004101	0.317573	3.609708	0.563536	0.609708
	8	0.215376	0.049401	0.003014	0.232813	3.364292	0.482507	0.364292
	9	0.196223	0.040895	0.002392	0.189503	3.326950	0.435320	0.326950
	10	0.181583	0.034958	0.001986	0.166596	3.238532	0.408162	0.238532
6	6	0.394104	0.171987	0.016669	1.668102	6.757334	1.291550	3.757334
	7	0.301756	0.097519	0.006462	0.540625	4.142795	0.735272	1.142795
	8	0.258315	0.070787	0.004060	0.313873	3.650085	0.560244	0.650085
	9	0.230698	0.056205	0.002983	0.229549	3.451351	0.479112	0.451351
	10	0.210864	0.046832	0.002368	0.180792	3.353638	0.425196	0.353638
7	7	0.409495	0.184398	0.016712	1.719701	6.877223	1.311373	3.877223
	8	0.316236	0.106430	0.006425	0.551682	4.167832	0.742753	1.167832
	9	0.272123	0.078077	0.004026	0.318976	3.677382	0.564780	0.677382
	10	0.243921	0.062454	0.002957	0.228410	3.391861	0.477923	0.391861
8	8	0.422818	0.195537	0.016762	1.764809	6.979383	1.328461	3.979383
	9	0.328840	0.114530	0.006394	0.562789	4.217233	0.750193	1.217233
	10	0.284209	0.084773	0.003998	0.320279	3.693316	0.565932	0.693316
9	9	0.434565	0.205663	0.016816	1.805957	7.064301	1.343859	4.064301
	10	0.339998	0.121970	0.006371	0.572928	4.244680	0.756920	1.244680
10	10	0.445072	0.214962	0.016873	1.842618	7.141398	1.357431	4.141398

5. APPROXIMATE INFERENCE

Here, we derive the $(1-\alpha)100\%$confidence intervalCIs for the location and scale parameters of the BLUEs θ^* and ξ^* based on the pivotal quantities

$$U_1 = \frac{\theta^* - \theta}{\xi\sqrt{W_1}}, \quad U_2 = \frac{\xi^* - \xi}{\xi\sqrt{W_2}}, \quad U_3 = \frac{\theta^* - \theta}{\xi^*\sqrt{W_1}}, \tag{22}$$

where θ^* and ξ^* are the BLUEs of θ and ξ with variances $\xi^2 W_1$ and $\xi^2 W_2$, respectively. U_1 is used to draw inference for θ when ξ is known, while U_3 can be used to draw inference for θ when ξ is unknown. Similarly, U_2 can be used to draw inference for θ when ξ is unknown.

To derive the CIs of the location and scale parameters based on the pivotal quantities in equation (22), the moments presented in Section 2, are used.

Hence, U_1 and U_2 can be rewritten as

$$U_1 = \frac{1}{\sqrt{W_1}}\left(\sum_{i=r+1}^{m-s} p_i X_{i:m}\right) = \frac{U_1^*}{\sqrt{W_1}}, \qquad U_2 = \frac{1}{\sqrt{W_2}}\left(\sum_{i=r+1}^{m-s} q_i X_{i:m} - 1\right) = \frac{U_2^* - 1}{\sqrt{W_2}}, \tag{23}$$

where $X_{i:m} = (Y_{i:m} - \theta)/\xi$, $i = 1, 2, \ldots, m-s$, is the standardized form of the available type-II right-censored sample $Y_{i:m}$, $i = 1, 2, \ldots, m-s$. Now, we consider to find the approximate distribution by using Edgeworth approximation for a statistic S (with mean 0 and variance 1) as

$$H(s) \approx \Phi(s) - \phi(s)\left[\frac{\sqrt{\tau_1}}{6}(s^2 - 1) + \frac{\tau_2 - 3}{24}(s^3 - 3s) + \frac{\tau_1}{72}(s^5 - 10s^2 + 15s)\right], \tag{24}$$

where $\sqrt{\tau_1}$ and τ_2 are the coefficients of skewness and kurtosis of S, respectively and $\Phi(s)$, $\phi(s)$ are the cdf and pdf of the standard normal distribution, respectively.

To obtain the coefficients of skewness and kurtosis of linear functions of order statistics, single moments $E(X_{r:n}^{k_1})$ denoted by $\alpha_{r:n}^{(k_1)}$, the double moments $E(X_{r:n}^{k_1} X_{s:n}^{k_2})$, denoted by $\alpha_{r,s:n}^{(k_1,k_2)}$ of the TIIELL distribution for $1 \le r < s \le (m-s)$ are required.

Table 3 displays the values of the mean, variance, coefficients of skewness and kurtosis ($\sqrt{\tau_1}$ and τ_2) of U_1^* and U_2^*. From Table 3 it is observed that the distributions of U_1^* and U_2^* and hence of U_1 and U_2 are positively skewed and heavier tailed than normal. Also, we can see that $\sqrt{\tau_1}$ of U_1^* and U_2^* increases as η increases and decreases as m increases and decreases as s increases. τ_2 of U_1^* decreases as m increases and increases as η increases and decreases as s increases, while τ_2 of U_2^* increases as m and η increases and decreases as s increases. We also obtained the lower and upper 1%, 2.5%, 5%, *and* 10% points of U_1 and U_2 through Edgeworth approximation (see Tables 4 and 5). From Tables 4 and 5, we can observe that the percentage points of U_1 and U_2 increases as η increases for $m = 7$ and decreases for $m = 10$ in most of the cases and increases as n increases in most of the cases for $\eta = 2$, while decreases as m increases for $\eta = 3$ and decreases as s increases. Similarly the percentage points of U_3 increases as n increases.

Table 3 Coefficient of the best linear unbiased estimators (BLUEs) of the p_i for $\tau = 1.5$

η	m	s	p_i									
2	7	0	0.013963	0.101597	0.558978	0.263992	0.051795	0.009676	−5.27E-07			
		1	−0.34963	0.833878	0.401217	0.108408	0.006114	1.22E-05				
		2	−0.11492	0.812707	0.290103	0.012084	2.69E-05					
	10	0	0.026562	0.067891	0.131564	0.228105	0.328165	0.132287	0.060663	0.023448	0.001311	3.53E-06
		1	0.039625	0.100778	0.188777	0.284606	0.265189	0.082232	0.036252	0.002535	4.15E-06	
		2	0.04799	0.127099	0.250121	0.374234	0.143044	0.053643	0.003868	1.68E-06		
		3	0.029534	0.131456	0.511993	0.235	0.087684	0.00432	1.37E-05			
		4	−0.25036	0.761078	0.334894	0.145439	0.008938	9.13E-06				
3	7	0	−0.46467	-0.74389	1.449078	0.545318	0.204818	0.009332	1.17E-05			
		1	−0.29246	-0.60828	1.444277	0.453126	0.003324	1.28E-05				
		2	−1.58802	1.838166	0.728422	0.021388	4.29E-05					
	10	0	−0.01692	−0.03763	−0.06087	−0.04818	0.118261	0.764299	0.199978	0.078079	0.002988	1.05E-06
		1	0.002353	0.007739	0.019326	0.051598	0.170786	0.611624	0.131185	0.005383	5.88E-06	
		2	−0.01192	−0.03172	−0.03954	0.060255	0.846969	0.166772	0.009176	7.58E-06		
		3	−0.06968	−0.16761	−0.15093	1.136136	0.240728	0.011335	2.05E-05			
		4	−0.29882	−0.61856	1.540259	0.349781	0.027336	8.17E-06				

Table 4 Coefficient of the best linear unbiased estimators (BLUEs) of the q_i for $\tau = 1.5$

η	m	s	q_i									
2	7	0	−1.42624	−2.69332	2.763219	1.092811	0.224505	0.039025	−3.63E-06			
		1	−5.05483	3.111726	1.515761	0.403758	0.023545	4.36E-05				
		2	−4.45187	3.226687	1.177263	0.047808	0.000117					
	10	0	−0.40703	−0.92138	−1.2938	−0.64137	2.3615	0.5141	0.274523	0.107245	0.006194	1.47E-05
		1	−0.28225	−0.6877	−1.08679	−0.841	2.287784	0.409765	0.185349	0.014828	1.58E-05	
		2	−0.56606	−1.313	−1.47408	2.431089	0.650346	0.253884	0.017819	7.43E-06		
		3	−1.34466	−2.59115	2.584963	0.96291	0.369273	0.018615	5.26E-05			
		4	−4.68322	2.837779	1.270177	0.540726	0.034506	3.06E-05				
3	7	0	−2.17816	−3.85459	4.049251	1.422057	0.537002	0.024409	3.09E-05			
		1	−1.65382	−3.79265	4.181823	1.255093	0.009517	3.47E-05				
		2	−6.66135	4.733542	1.872912	0.054788	0.000108					
	10	0	−0.21816	−0.51608	−0.92486	−1.24654	−0.76193	2.846231	0.570023	0.242188	0.009133	2.23E-06
		1	−0.12748	−0.32472	−0.62717	−1.01443	−0.95517	2.569284	0.460068	0.019596	1.77E-05	
		2	−0.25649	−0.69056	−1.2704	−1.44404	3.097968	0.533783	0.029706	2.53E-05		
		3	−0.6069	−1.57047	−2.26163	3.697287	0.70758	0.03408	5.81E-05			
		4	−1.6753	−3.83228	4.469613	0.962392	0.075552	2.27E-05				

Table 5 Variances and covariance of the best linear unbiased estimators (BLUEs) when $\tau = 1.5$ $\theta = 0$ and $\xi = 1$

η	n	c	$Var(\theta^*)$	$Var(\xi^*)$	$Cov(\theta^*,\xi^*)$
2	7	0	0.074821	0.970035	0.253528
		1	0.093972	1.102865	0.310651
		2	0.125613	1.597531	0.426752
	10	0	0.044879	0.616469	0.152306
		1	0.052941	0.848929	0.185043
		2	0.060478	0.878651	0.209287
		3	0.073563	0.955970	0.248770
		4	0.087637	1.020583	0.287920
3	7	0	0.153237	0.952101	0.377268
		1	0.214149	1.459957	0.549955
		2	0.248820	1.525382	0.610006
	10	0	0.106367	0.810697	0.284275
		1	0.132891	1.206524	0.379329
		2	0.150195	1.238222	0.414675
		3	0.182625	1.375398	0.488490
		4	0.211345	1.424032	0.539068

The performance of the developed inference can be shown from the simulated average width of CIs in Table 10. We observe that the Edgeworth approximations of the distributions of U_1 and U_2 both work quite satisfactory; this is also clear from the average width of the CIs based on U_1 and U_2 which are presented in Table 6. In addition we can see that average width decreases as η increases for most of the cases.

Tables 8 represent the simulated percentage points at $\tau = 0.5$ and sample sizes $n = 7, 10$. Table 9 displays the values of the mean, variance and the coefficients of skewness and kurtosis and Tables 10 represent the Average width of the Edgeworth and simulated class intervals.

6. DATA ANALYSIS

To demonstrate how the proposed methods can be used in practice, we consider the following real-life data set (see Bhaumik *et al.* [25]). The data set represents vinyl chloride from clean upgradient monitoring wells in mg/L. The data are:

5.1 1.2 1.3 0.6 0.5 2.4 0.5 1.1 8.0 0.8 0.4 0.6 0.9 0.4 2.0 0.5 5.3 3.2 2.7 2.9 2.5 2.3 1.0 0.2 0.1 0.1 1.8 0.9 2.0 4.0 6.8 1.2 0.4 0.2

Now a random sample of size 10 is selected from the given data set and data are 0.1, 1.1, 0.9, 2.3, 1.3, 2.5, 0.4, 2.0, 0.5, 3.2. Figure 1 shows Q-Q plot of the sample. The Kolmogorov–Smirnov (K–S) statistic is 0.17491 and the corresponding p-value is 0.8697. This shows the suitability of the TIIELL distribution for this data set.

Then by using the BLUEs coefficients in Tables 1 and 2, we have

$$\theta^* = \sum_{r=1}^{n} p_r X_{r:n} = 1.012389 \quad \text{and} \quad \xi^* = \sum_{r=1}^{n} q_r X_{r:n} = 2.443834.$$

We note that the average width of the CIs increase as the level of significant increases.

Table 6 Edgeworth approximate and the simulated (*) values of the distribution of U_1 when $\tau = 1.5, \theta = 0$ and $\xi = 1$

η	n	c	1%	2.50%	5%	10%	90%	95%	97.50%	99%
2	7	0	−0.7784	−0.76196	−0.73504	−0.68271	1.083601	1.925702	1.854568	4.255494
			−1.03231*	−0.97438*	−0.90792*	−0.81937*	1.083648*	1.76806*	2.560536*	3.7586*
		1	−2.27628	−2.32369	−0.67909	−0.62752	0.20892	1.889575	1.822893	4.241948
			−1.4765*	−1.41945*	−1.35722*	−1.26558*	0.682143*	1.351194*	2.104918*	3.327379*
		2	−2.34755	−2.41664	−2.56902	−2.86364	0.324397	0.432937	1.791339	3.749634
			−2.0666*	−2.01487*	−1.95338*	−1.8637*	0.126588*	0.818807*	1.556711*	2.749675*
	10	0	−0.63703	−0.62091	−0.59465	−0.54415	1.968333	1.825218	3.499767	4.188481
			−1.19869*	−1.11596*	−1.03013*	−0.90966*	1.176293*	1.81677*	2.499808*	3.605541*
		1	−2.5634	−1.08063	−1.02533	−0.92635	0.773829	1.106087	1.429446	3.896156
			−1.68973*	−1.60117*	−1.51202*	−1.38262*	0.766383*	1.428123*	2.120552*	3.137602*
		2	−2.51757	−2.77431	−2.97307	−2.89647	0.704014	0.993738	1.354706	3.585678
			−2.19815*	−2.10884*	−2.02115*	−1.89652*	0.271168*	0.924273*	1.627683*	2.585719*
		3	−2.53477	−3.16427	−2.9827	−2.98269	0.732411	1.042713	1.382714	1.382703
			−2.72401*	−2.63791*	−2.55189*	−2.42756*	−0.25733*	0.390673*	1.068124*	2.10397*
		4	−3.83866	−3.30443	−2.96218	−2.96219	0.181199	0.835414	1.163553	1.314588
			−3.25567*	−3.17872*	−3.09859*	−2.98289*	−0.81876*	−0.16451*	0.504467*	1.485206*
3	7	0	−0.38025	−0.37311	−0.36123	−0.33769	1.936298	1.888275	1.865233	4.491297
			−0.9566*	−0.90678*	−0.8515*	−0.77025*	1.032663*	1.716643*	2.505057*	3.743581*
		1	−2.28575	−2.32319	−0.66079	−0.61545	0.161453	1.927139	1.87266	4.263209
			−1.45259*	−1.40138*	−1.34743*	−1.268*	0.586968*	1.278905*	2.080461*	3.263277*
		2	−2.30645	−2.35543	−2.45118	−2.81325	0.226085	0.306988	1.837418	3.764807
			−1.991*	−1.94309*	−1.89224*	−1.81329*	0.111492*	0.792065*	1.60848*	2.764848*
	10	0	−0.83021	−0.81163	−0.78146	−0.72326	1.296581	1.916373	1.828839	4.202645
			−1.0873*	−1.0183*	−0.94733*	−0.84676*	1.096046*	1.772315*	2.542991*	3.727137*
		1	−2.18033	−2.21922	−2.29146	−2.39668	0.156439	1.849359	1.797111	4.137826
			−1.65246*	−1.57954*	−1.5031*	−1.39675*	0.607943*	1.262664*	2.027025*	3.137897*
		2	−2.41444	−2.52619	−2.92543	−2.92543	0.480104	0.644207	1.703217	3.567014
			−2.28619*	−2.21093*	−2.12713*	−2.01619*	0.051319*	0.72511*	1.48204*	2.567054*
		3	−2.44641	−2.58122	−2.94305	−2.94304	0.424358	0.740532	0.873089	1.518757
			−2.94442*	−2.86884*	−2.78695*	−2.67421*	−0.57559*	0.095727*	0.809759*	1.841936*
		4	−3.94473	−3.48833	−2.95145	−2.95145	−0.14498	0.510946	0.891869	1.527782
			−3.48544*	−3.41859*	−3.34497*	−3.23734*	−1.14494*	−0.48899*	0.243865*	1.30883*

Table 7 Edgeworth approximate and the simulated (*) values of the distribution of U_2 when $\tau = 1.5, \theta = 0$ and $\xi = 1$

η	n	c	1%	2.50%	5%	10%	90%	95%	97.50%	99%
2	7	0	−0.68625	−0.67387	−0.65342	−0.613	1.120145	1.963351	1.917835	4.386462
			−0.89515*	−0.85571*	−0.81026*	−0.74149*	1.01628*	1.700985*	2.516495*	3.864885*
		1	−2.23387	−2.26766	−0.5896	−0.54756	0.133227	1.900113	1.852216	4.326887
			−1.37336*	−1.33204*	−1.28475*	−1.2124*	0.661287*	1.334092*	2.137424*	3.36625*
		2	−2.28594	−2.32889	−2.41048	−2.68876	0.190544	0.261264	1.847944	3.857855
			−1.90759*	−1.87523*	−1.83573*	−1.7718*	0.101144*	0.809872*	1.64006*	2.857924*
	10	0	−0.43146	−0.42262	−0.408	−0.37914	1.928908	1.865193	1.83512	4.403044
			−1.03058*	−0.97355*	−0.9115*	−0.821*	1.084735*	1.761624*	2.52332*	3.710513*
		1	−2.39482	−0.86625	−0.83388	−0.77177	0.366606	1.928228	1.824014	4.173098
			−1.50745*	−1.44968*	−1.38733*	−1.29935*	0.641782*	1.332129*	2.118734*	3.318632*
		2	−2.25389	−2.31073	−2.42521	−2.8198	0.242052	0.315944	1.774996	3.702989
			−2.09084*	−2.0311*	−1.96779*	−1.87474*	0.139231*	0.812459*	1.597538*	2.70303*
		3	−2.39193	−2.48068	−2.72223	−2.91628	0.413258	0.553318	0.636243	1.67989
			−2.62983*	−2.57057*	−2.5047*	−2.4097*	−0.3817*	0.297366*	1.054344*	2.16912*
		4	−3.95033	−3.49826	−2.94629	−2.94627	0.159258	0.734727	0.863918	0.968469
			−3.14328*	−3.08945*	−3.02804*	−2.93252*	−0.8407*	−0.15407*	0.56838*	1.601999*
3	7	0	−0.37383	−0.36704	−0.35582	−0.33353	1.947225	1.90334	1.882234	4.526831
			−0.89584*	−0.8538*	−0.80822*	−0.73897*	1.00372*	1.702315*	2.507245*	3.772574*

(*continued*)

Table 7 Edgeworth approximate and the simulated (*) values of the distribution of U_2 when $\tau = 1.5, \theta = 0$ and $\xi = 1$ *(Continued)*

η	n	c	1%	2.50%	5%	10%	90%	95%	97.50%	99%
		1	−2.25553	−0.60699	−0.58794	−0.55025	0.094725	1.941317	1.901147	4.364091
			−1.33671*	−1.30106*	−1.26096*	−1.19843*	0.562082*	1.280316*	2.118576*	3.364137*
		2	−2.28995	−2.3318	−2.41096	−2.66945	0.186123	0.257073	1.855683	3.870927
			−1.88758*	−1.85427*	−1.81481*	−1.74995*	0.119499*	0.83215*	1.645907*	2.871001*
	10	0	−0.72809	−0.71409	−0.69103	−0.64574	1.167925	1.949229	1.893921	4.330301
			−0.95512*	−0.90828*	−0.85568*	−0.77823*	1.042522*	1.747703*	2.533224*	3.81236*
		1	−2.12219	−2.13917	−2.16865	−2.23302	0.037221	1.888342	1.864152	4.247133
			−1.4378*	−1.395*	−1.34838*	−1.27871*	0.511022*	1.219516*	2.01116*	3.247213*
		2	−2.33214	−2.38482	−2.48992	−2.88514	0.254338	0.34656	1.841463	3.647901
			−2.11302*	−2.06494*	−2.0116*	−1.93031*	−0.02184*	0.681566*	1.499523*	2.647942*
		3	−2.4204	−2.50059	−3.24692	−2.94216	0.379866	0.576944	0.665016	1.735106
			−2.7661*	−2.71757*	−2.66356*	−2.5804*	−0.62009*	0.089863*	0.870971*	2.015355*
		4	−4.07898	−3.68032	−2.72291	−2.94198	−0.13232	0.561205	0.679897	1.720768
			−3.29861*	−3.25072*	−3.19745*	−3.115*	−1.13227*	−0.43873*	0.318227*	1.532725*

Table 8 Simulated values of the distribution of U_3 when $\tau = 1.5, \theta = 0$ and $\xi = 1$

η	n	c	1%	2.50%	5%	10%	90%	95%	97.50%	99%
2	7	0	−8.58565	−6.00463	−4.36124	−2.92522	0.577351	0.740279	0.891971	1.127982
		1	−22.0015	−14.7982	−10.3426	−7.00606	0.490966	0.754123	0.961019	1.207395
		2	−64.3401	−40.636	−27.6904	−17.7657	0.128992	0.633082	1.003714	1.452917
	10	0	−6.71847	−4.88497	−3.67234	−2.5574	0.671541	0.86046	1.024841	1.245431
		1	−14.416	−10.4421	−7.84722	−5.55788	0.624766	0.977244	1.3142	1.778569
		2	−25.1951	−18.3949	−14.0381	−10.1702	0.274881	0.778888	1.17583	1.693537
		3	−43.5958	−31.043	−23.3486	−16.8049	−0.31343	0.390987	0.902069	1.484084
		4	−80.3106	−55.2902	−40.0114	−27.942	−1.15339	−0.18127	0.463265	1.118295
3	7	0	−8.2837	−5.77654	−4.18322	−2.83497	0.526657	0.65697	0.744066	0.828474
		1	−25.6916	−17.0079	−12.0546	−8.10941	0.431894	0.703867	0.882413	1.061739
		2	−68.5725	−41.7708	−28.1831	−17.9228	0.104877	0.574845	0.890182	1.167284
	10	0	−7.14072	−5.19687	−3.88199	−2.69139	0.589233	0.735267	0.842694	0.964767
		1	−18.5664	−13.4642	−10.1707	−7.19092	0.482042	0.788716	1.012966	1.290519
		2	−34.5303	−25.0795	−19.0127	−13.4866	0.055358	0.591351	0.969128	1.348443
		3	−60.0887	−43.3305	−32.6716	−23.452	−0.75955	0.100226	0.669017	1.197819
		4	−104.838	−72.0204	−52.8649	−36.8763	−1.75691	−0.58387	0.226488	0.958274

Table 9 Mean, Variance and coefficients of skewness and kurtosis of U_1^* and U_2^* when $\tau = 1.5, \theta = 0$ and $\xi = 1$

			U_1				U_2			
η	n	c	Mean	V_1	$\sqrt{\tau_1}$	τ_2	Mean	V_1	$\sqrt{\tau_1}$	τ_2
2	7	0	0.000963	0.074656	1.90405	32.19869	1.005801	0.982068	2.114422	46.51046
		1	−0.08758	0.04176	1.864872	32.9059	0.714386	0.465816	1.986304	42.78576
		2	−0.15359	0.023106	1.763638	25.11762	0.498366	0.258819	1.922601	35.45901
	10	0	0.000786	0.046427	1.746194	31.8938	1.001825	0.652555	2.067376	60.53579
		1	−0.07138	0.026898	1.489195	12.70246	0.722236	0.351057	1.801017	24.7817
		2	−0.12518	0.017567	1.485461	13.29467	0.554844	0.192955	1.758955	28.75834
		3	−0.16467	0.012416	1.478552	12.92351	0.448749	0.127834	1.69991	21.34141
		4	−0.1964	0.009247	1.512846	14.19956	0.37472	0.093584	1.610215	17.27234
3	7	0	0.002024	0.157938	2.234621	78.61409	1.004534	0.982242	2.312447	86.37732
		1	−0.1323	0.077857	1.9914	39.44096	0.67646	0.499386	2.126234	50.97791
		2	−0.21177	0.044925	1.879843	32.08649	0.503454	0.265125	1.939696	36.16978
	10	0	−0.0004	0.105689	1.831395	27.65306	0.999542	0.796568	2.021134	39.482
		1	−0.11958	0.052235	1.851664	37.89021	0.669847	0.405057	2.214512	75.30156

(*continued*)

Table 9 | Mean, Variance and coefficients of skewness and kurtosis of U_1^* and U_2^* when $\tau = 1.5$, $\theta = 0$ and $\xi = 1$ *(Continued)*

			U_1				U_2			
η	n	c	Mean	V_1	$\sqrt{\tau_1}$	τ_2	Mean	V_1	$\sqrt{\tau_1}$	τ_2
		2	−0.1925	0.029811	1.632681	18.67683	0.502932	0.196356	1.870876	30.37217
		3	−0.24453	0.019317	1.595066	16.9323	0.398287	0.118517	1.764992	22.59934
		4	−0.28135	0.014526	1.606979	17.01038	0.331043	0.086028	1.748858	22.01017

Table 10 | Average width of the Edgeworth and simulated(*) C.I.'s.

		U_1				U_2				U_3	
n	c	90%		95%		90%		95%		90%	95%
7	0	2.660745	2.67598*	2.616533	3.534921*	2.616774	2.511241*	2.59171	3.372206*	5.101516*	6.896599*
	1	2.568667	2.70841*	4.14658	3.524364*	2.489716	2.618837*	4.119881	3.469468*	11.0967*	15.75922*
	2	3.001957	2.772187*	4.207975	3.571577*	2.671747	2.645604*	4.176834	3.515287*	28.32346*	41.63974*
10	0	2.419866	2.846901*	4.12068	3.61577*	2.273196	2.673121*	2.257739	3.496869*	4.532798*	5.90981*
	1	2.131419	2.940145*	2.510081	3.721726*	2.762113	2.719455*	2.690262	3.568417*	8.824464*	11.75634*
	2	3.966812	2.945421*	4.129015	3.736519*	2.741151	2.780245*	4.085728	3.628634*	14.81699*	19.57076*
	3	4.025414	2.942563*	4.546983	3.706037*	3.27555	2.802063*	3.116921	3.62491*	23.73962*	31.9451*
	4	3.797594	2.934083*	4.467987	3.683186*	3.681019	2.873962*	4.36218	3.657831*	39.83013*	55.7535*
7	0	2.249506	2.568142*	2.238341	3.411832*	2.259165	2.51053*	2.249274	3.361044*	4.840185*	6.520606*
	1	2.587927	2.626336*	4.195851	3.481836*	2.529257	2.541276*	2.508136	3.419633*	12.75846*	17.89027*
	2	2.758169	2.684303*	4.192849	3.551572*	2.668035	2.646963*	4.187487	3.500173*	28.75793*	42.66099*
10	0	2.697834	2.719643*	2.640473	3.56129*	2.640264	2.603388*	2.608009	3.441503*	4.617257*	6.03956*
	1	4.140821	2.765767*	4.016333	3.606564*	4.056995	2.567898*	4.003325	3.40616*	10.95945*	14.47721*
	2	3.569634	2.852243*	4.22941	3.692967*	2.836481	2.693163*	4.22628	3.564463*	19.60404*	26.04864*
	3	3.683578	2.882681*	3.454311	3.678602*	3.823864	2.753427*	3.165611	3.588544*	32.77182*	43.99951*
	4	3.462393	2.855982*	4.380202	3.662456*	3.284114	2.75872*	4.360213	3.568945*	52.28107*	72.2469*

By using Table 8, the CIs for the location parameter θ as

	90% CI	95% CI
Edgeworth	(0.6257, 1.13836)	(0.27096, 1.14393)
Simulated	(0.6275, 1.23062)	(0.48281, 1.24881)

By using Table 10, the CIs for the scale parameter ξ as

	90% CI	95% CI
Edgeworth	(0.99163, 3.59568)	(1.00122, 3.65746)
Simulated	(1.02546, 3.59603)	(0.91974, 3.67228)

By using Table 10, the CIs for the location parameter θ when ξ is unknown as

	90% CI	95% CI
Simulated	(0.56691, 2.91362)	(0.48181, 3.54143)

7. CONCLUSION

In this paper, the moments and product moments of the order statistics from the TIIELL distribution are derived in explicit forms. The single and double moments are used to obtain the BLUEs of the location and scale parameters of TIIELL distribution. The variances and covariances are calculated to show the performance of the BLUEs. Next, we calculate mean, variance, coefficient of skewness and kurtosis for some linear pivotal quantities. The distributions of the pivotal quantities are calculated in terms of Edgeworth approximation based on BLUEs which in turn can be used to develop CIs. Hence, the distributions of the pivotal quantities are used to construct the interval estimation for the location and scale parameters. The accuracy of the estimated CIs is investigated in terms of the average width. Finally, one real data set has been used to obtain the MLEs of the model parameters, BLUEs of θ and ξ and CIs of θ and ξ.

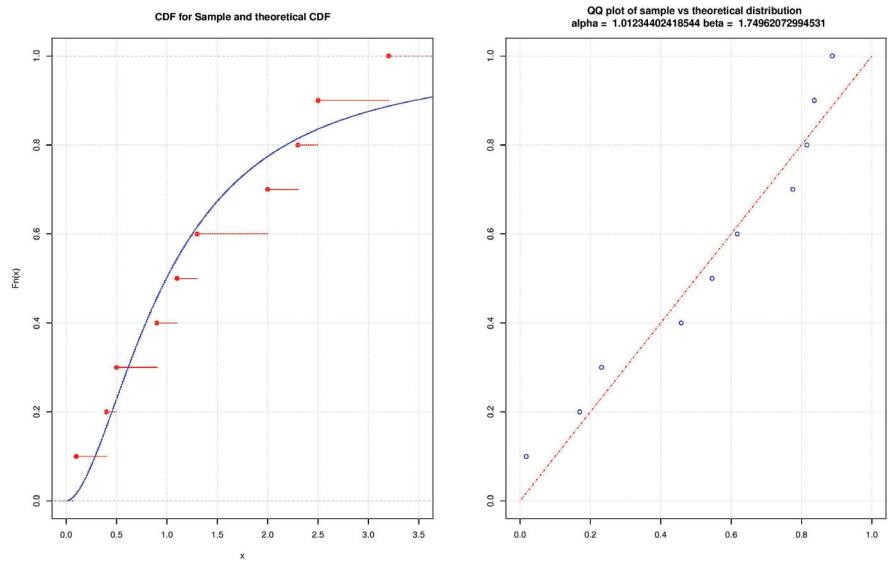

Figure 1 | Expected Cumulative Distribution Function (ECDF) and Quintile-Quintile (Q-Q) plot of the real sample based on type-II exponentiated log-logistic (TIIELL) distribution.

CONFLICTS OF INTEREST

The authors declare that there are no conflicts of interest regarding the publication of this paper.

AUTHORS' CONTRIBUTIONS

All authors contributed equally to the writing of this paper. All authors have read and agreed to the published version of the manuscript.

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