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Research Article

On Relationship between *L*-valued Approximation Spaces and *L*-valued Transformation Systems

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ABSTRACT

The objective of this paper is to establish the relationship between L-valued approximation spaces and L-valued transformation systems. We show that for each L-valued upper/lower fuzzy transformation system there exist an L-valued reflexive approximation space and vice versa. In between, we study the concept of L-valued natural transformations.

1. INTRODUCTION

The concept of rough set was originally proposed by Pawlak [1]. This theory has been developed significantly due to its importance for the study of intelligent systems with insufficient and incomplete information. In rough sets introduced by Pawlak, the key role is played by equivalence relations. In literature [2–4], several generalizations of rough sets have been made by replacing the equivalence relation by an arbitrary relation. Further, Dubois and Prade [5] introduced the concept of fuzzy rough set, in which fuzzy relations play a key role instead of crisp relations. Recently, the combinations of fuzzy sets and rough sets were investigated with different fuzzy logic operations and binary fuzzy relation in [6–14], where fuzzy implications play an important role in the extensions of fuzzy rough sets. In a recent study, Li and Yue established more general model of fuzzy rough sets based on L-valued relation. They investigated it from both constructive and axiomatic approaches, where L is a GL-quantale. In this model of fuzzy rough sets, the key components are a universal set A (as an L-set), an L- valued equivalence relation on A and an L-subset of A (cf., [15], for details).

Fuzzy transform (F-transform in short), firstly proposed by Perfilieva [16], has now been significantly developed and opened a new page in the theory of semi-linear spaces. It was shown in [16] that this transform encompassed both classical transform as well as approximation methods based on fuzzy IF-THEN rules studied in fuzzy modeling. The theory of F-transform was further elaborated and extended from real valued to lattice valued functions [16,17] and from fuzzy sets to parametrized fuzzy sets [18]. Recently in [19], it is shown that F-transform is a realization of an abstract fuzzy rough set theory, more precisely, F-transforms turn out to be fuzzy approximation operators. In another direction, the concepts of upper and lower fuzzy transformation systems were introduced recently by Mo čko ř [20] and a close connection with F-transforms was established. Specifically, it was shown that a function satisfies axioms for fuzzy upper (or lower, respectively) transformation systems if and only if it is an upper (or lower, respectively) F-transform.

In view of the facts that (i) an F-transform can be viewed as a fuzzy approximation operator and (ii) there is a bijective correspondence between an F-transform and a fuzzy transformation system, it is natural to think about the relationship between a fuzzy approximation operator and a fuzzy transformation system. In this work, we have established such relationship, where fuzzy approximation operators are considered as studied in [15].

The paper is organized as follows: In Section 2,we recall some basic properties of residuated lattice, MV-algebra and L-valued relation. The concept of L-valued approximation spaces and their properties are discussed in Section 3. In Section 4, the relationship between L-valued upper natural transformation and L-valued lower natural transformation is discussed. In Section 5,we study a relationship between L-valued transformation systems and L-valued reflexive approximation spaces. At last, we conclude our research in Section 6.

2. PRELIMINARIES

In this section, we recall some basic notions of residuated lattices, MV-algebra and L-valued relations. For details on residuated

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lattices and MV-algebra, we refer the works done in [21–24,15, 25–27]. We begin with the following:

Definition 2.1. A **residuated lattice** is an algebra $L \equiv (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ such that

i. $(L, \wedge, \vee, 0, 1)$ is a bounded lattice with the least element 0 and the greatest element 1;

ii. $(L, \otimes, 1)$ is a commutative monoid; and

iii. $\forall a, b, c \in L$; $a \otimes b \leq c$ iff $a \leq b \rightarrow c$, i.e., $(\rightarrow, \otimes)$ is an adjoint pair on L.

A residuated lattice $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ is **complete** if it is complete as a lattice.

Proposition 2.1 *Let $L \equiv (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ be a residuated lattice. Then for $a, b, c, b_i \in L$, we have*

1. $a \rightarrow 1 = 1, 1 \rightarrow a = a,$
2. $a \leq b \Rightarrow a \rightarrow b = 1,$
3. $a \leq b \Rightarrow a \otimes c \leq b \otimes c, a \rightarrow c \geq b \rightarrow c, c \rightarrow a \leq c \rightarrow b,$
4. $a \otimes 0 = 0, a \otimes 1 = a,$
5. $a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c),$
6. $a \rightarrow \vee_{i \in I} b_i \geq \vee_{i \in I}(a \rightarrow b_i),$
7. $a \otimes \vee_{i \in I} b_i = \vee_{i \in I}(a \otimes b_i),$
8. $a \rightarrow \wedge_{i \in I} b_i = \wedge_{i \in I}(a \rightarrow b_i).$

Definition 2.2 Let $L \equiv (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ be a residuated lattice. Then L is said to be **divisible** if for $a, b \in L$ and $a \leq b$, $\exists\, c \in L$ such that $a = b \otimes c$. Further, it is said to satisfy **idempotency** if $a \otimes a = a$. A **negation** in L is a unary operation $\neg$ defined by $\neg a = a \rightarrow 0$, $\forall a \in L$. L is said to satisfy the law of **double negation** if $\neg(\neg a) = a$, for all $a \in L$. An **MV-algebra** is a residuated lattice satisfies both divisibility and double negation law.

Proposition 2.2 *Let $L \equiv (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ be an MV-algebra. Then for $a, b, c \in L$, we have*

1. $a \otimes (a \rightarrow b) = a \wedge b,$
2. *if* $a \leq b$, *then* $b \wedge [(b \rightarrow a) \rightarrow c] = b \otimes (a \rightarrow c),$
3. *if* $a, c \leq b$, *then* $c \otimes (b \rightarrow a) = a \otimes (b \rightarrow c),$
4. $a \rightarrow b = \neg(a \otimes \neg b),$
5. $\neg a \rightarrow \neg b = b \rightarrow a,$
6. *if* $a \otimes a = a$, *then* $a \wedge b = a \otimes b.$

*Throughout the paper, unless otherwise stated, L is a complete residuated lattice. For a nonempty set A_0, an L-**set** A of A_0 is defined as a mapping from A_0 to L. Let $B : A_0 \rightarrow L$ be an L-set and $B(x) \leq A(x), x \in A_0$. Then the mapping B is called an L-**subset** of A. The family of all L-subsets of A is denoted by PA. For any $a \in L$, $\mathbf{a}$ is a constant L-subset of A defined by $\mathbf{a}(x) = a$ if $a \leq A(x)$ and $\mathbf{a}(x) = 0$ otherwise. For any $B, C \in PA$, the **union**, **intersection**, $\otimes$-**intersection** and $\rightarrow$-**implication** of B and C are defined as L-subsets of A by $(B \vee C)(x) = B(x) \vee C(x)$; $(B \wedge C)(x) = B(x) \wedge C(x)$; $(B \otimes C)(x) = B(x) \otimes C(x)$; $(B \rightarrow C)(x) = B(x) \rightarrow C(x)$. Further, let $A : A_0 \rightarrow L$ be an L-set. Then for $y \in A_0$, two new L-sets A_y and A_{-y0} are defined by*

$$A_y(x) = \begin{cases} A(x), & \text{if } x = y, \\ 0, & \text{if } x \neq y, \end{cases} \quad A_{-y0}(x) = \begin{cases} A(x), & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

Obviously, A_y and A_{-y0} are L-subsets of A. Furthermore, for all $B \in PA$, $core(B)$ is a set of all elements $x \in A_0$ such that $B(x) = A(x)$.

Now, we recall the following from [15].

Definition 2.3 Let $A : A_0 \rightarrow L$ and $B : B_0 \rightarrow L$ be two L-sets. An L-**valued relation** P is a mapping $P : A_0 \times B_0 \rightarrow L$ such that $P(x, y) \leq A(x) \wedge B(y), \forall x \in A_0, y \in B_0$.

Definition 2.4 Let A be an L-set and let $P : A_0 \times A_0 \rightarrow L$ be an L-valued relation on A. Then P is called

i. **reflexive** if $P(x, x) = A(x)$, for all $x \in A_0$;

ii. **transitive** if $P(x, y) \otimes (A(y) \rightarrow P(y, z)) \leq P(x, z)$, for all $x, y, z \in A_0$; and

iii. **symmetric** if $P(x, y) = P(y, x)$, for all $x, y \in A_0$.

A reflexive, transitive and symmetric L-valued relation on A is called an L-**valued equivalence relation** on A.

In above definition, if we take $A = 1_X$, then P is an L-fuzzy relation. Thus an L-fuzzy relation is a special case of an L-valued relation.

Definition 2.5 Let P be an L-valued equivalence relation on A. For $x \in A_0$, an L-subset E_x^P of A such that $E_x^P(y) = P(x, y)$, for $y \in A_0$, is called an L-**valued equivalence class** of P determined by the element x.

3. *L*-VALUED APPROXIMATION SPACES

In this section, we study the concept of L-valued approximation spaces. Further, it is shown that the L-valued lower approximation operator preserves union under certain condition. Now, we recall the following from [15]:

Definition 3.1 Let $A : A_0 \rightarrow L$ be an L-set and P be an L-valued relation on A. The pair (A, P) is called an L-**valued approximation space**. The operators $\overline{P_A}, \underline{P_A} : PA \rightarrow PA$ are respectively called the L-**valued upper and** L-**valued lower approximation operator** of (A, P) where for all $B \in PA$ and all $x \in A_0$,

$$\overline{P_A}(B)(x) = \vee_{y \in A_0} B(y) \otimes (A(y) \rightarrow P(y, x))$$

$$\underline{P_A}(B)(x) = \wedge_{y \in A_0} A(x) \otimes (P(y, x) \rightarrow B(y)).$$

The pair $(\overline{P_A}(B), \underline{P_A}(B))$ is called an L-**valued rough set** of B with respect to (A, P).

Remark 3.1. In above definition, if we take $A = 1_X$, then L-valued upper approximation operator and L-valued lower approximation operator are consistent with L-fuzzy upper approximation operator and L-fuzzy lower approximation operator, respectively, studied in [26].

Let L be an MV-algebra. Then according to Proposition 2.2, the L-valued lower rough approximation operator $\underline{P_A}$ can be expressed as follows:

$$\underline{P_A}(B)(x) = \wedge_{y \in A_0} A(x) \wedge [(A(x) \to P(y,x)) \to B(y)].$$

Now, we have the following.

Proposition 3.1 *Let (A, P) be an L-valued approximation space. Then for all $B, C \in PA$,*

$$\underline{P_A}(B \vee C) \geq \underline{P_A}(B) \vee \underline{P_A}(C).$$

Proof: *For $x \in A_0$,*

$$\begin{aligned}
\underline{P_A}(B \vee C)(x) &= \wedge_{y \in A_0} A(x) \otimes \left(P(y,x) \to (B \vee C)(y)\right) \\
&\geq \wedge_{y \in A_0} \left[A(x) \otimes \left\{\left(P(y,x) \to (B)(y)\right) \vee \right.\right. \\
&\quad \left.\left.\left(P(y,x) \to (C)(y)\right)\right\}, \text{[from Proposition 2.1(8)]}\right. \\
&\geq \wedge_{y \in A_0} \left[\left\{A(x) \otimes \left(P(y,x) \to (B)(y)\right)\right\} \vee \right. \\
&\quad \left.\left\{A(x) \otimes \left(P(y,x) \to (C)(y)\right)\right\}\right], \\
&\quad \text{[from Proposition 2.1(9)]} \\
&\geq \left\{\wedge_{y \in A_0} \left\{A(x) \otimes \left(P(y,x) \to (B)(y)\right)\right\}\right\} \vee \\
&\quad \left\{\wedge_{y \in A_0} \left\{A(x) \otimes \left(P(y,x) \to (C)(y)\right)\right\}\right\} \\
&\geq \underline{P_A}(B)(x) \vee \underline{P_A}(C)(x) \\
&\geq \left(\underline{P_A}(B) \vee \underline{P_A}(C)\right)(x).
\end{aligned}$$

$$\underline{P_A}(B \vee C) \geq (\underline{P_A}(B) \vee \underline{P_A}(C)).$$

In an L-valued approximation space (A, P), for $B, C \in PA$, $\underline{P_A}(B \vee C) \neq \underline{P_A}(B) \vee \underline{P_A}(C)$, which is shown as under (cf., [28]).

Counter-Example 3.1. Let $L = \{0, n, a, b, c, d, e, f, m, 1\}$ with $0 < n < a < c < e < m < 1$, $0 < n < b < d < f < m < 1$ and the elements $\{a, b\}, \{c, d\}, \{e, f\}$ are pairwise incomparable. Then L becomes a residuated lattice to the operations shown in Tables 1 and 2. Hasse diagram of residuated lattice L is given Figure 1.

Let $A = \{(x, 1), (y, 1), (z, 1)\}$ and B, C be two L-subsets of A such that $B = \{(x, b), (y, c), (z, d)\}$, $C = \{(x, b), (y, d), (z, d)\}$. Then $B \vee C = \{(x, b), (y, e), (z, d)\}$, since $c \vee d = e$. Now, let P be an L-valued relation on A, as given in Table 3. Then

$$\begin{aligned}
\underline{P_A}(B \vee C)(x) &= \wedge_{y \in A_0} A(x) \otimes \{P(y,x) \to (B \vee C)(y)\} \\
&= (b \to b) \wedge (e \to e) \wedge (d \to d) \\
&= 1 \wedge 1 \wedge 1 \\
&= 1.
\end{aligned}$$

Also,

$$\begin{aligned}
\underline{P_A}(B)(x) &= \wedge_{y \in A_0} A(x) \otimes \{P(y,x) \to (B)(y)\} \\
&= (b \to b) \wedge (e \to c) \wedge (d \to d) \\
&= 1 \wedge e \wedge 1 \\
&= e.
\end{aligned}$$

And

$$\begin{aligned}
\underline{P_A}(C)(x) &= \wedge_{y \in A_0} A(x) \otimes \{P(y,x) \to (C)(y)\} \\
&= (b \to b) \wedge (e \to d) \wedge (d \to d) \\
&= 1 \wedge f \wedge 1 \\
&= f.
\end{aligned}$$

But, $e \vee f = m \neq 1$. Hence $\underline{P_A}(B \vee C) \neq \underline{P_A}(B) \vee \underline{P_A}(C)$.

Following is toward the condition under which equality holds.

Table 1 → operation for lattice L.

→	0	n	a	b	c	d	e	f	m	1
0	1	1	1	1	1	1	1	1	1	1
n	m	1	1	1	1	1	1	1	1	1
a	f	f	1	f	1	f	1	f	1	1
b	e	e	e	1	1	1	1	1	1	1
c	d	d	e	f	1	f	1	f	1	1
d	c	c	c	e	e	1	1	1	1	1
e	b	b	c	d	e	f	1	f	1	1
f	a	a	a	c	c	e	e	1	1	1
m	n	n	a	b	c	d	e	f	1	1
1	0	n	a	b	c	d	e	f	m	1

Table 2 ⊗ operation for lattice L.

⊗	0	n	a	b	c	d	e	f	m	1
0	0	0	0	0	0	0	0	0	0	0
n	0	0	0	0	0	0	0	0	0	n
a	0	0	a	0	a	0	a	0	a	a
b	0	0	0	0	0	0	0	b	b	b
c	0	0	a	0	a	0	a	b	c	c
d	0	0	0	0	0	b	b	d	d	d
e	0	0	a	0	a	b	c	d	e	e
f	0	0	0	b	b	d	d	f	f	f
m	0	0	a	b	c	d	e	f	m	m
1	0	n	a	b	c	d	e	f	m	1

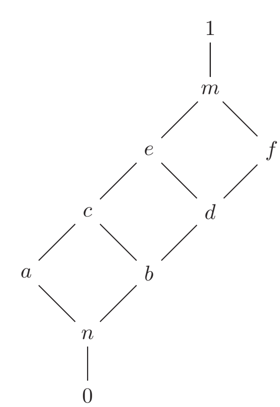

Figure 1 Hasse diagram of lattice L.

Table 3 Fuzzy binary relation on X.

P	x	y	z
x	b	o	e
y	e	b	m
z	d	f	1

Proposition 3.2 *Let (A, P) be an L-valued approximation space and $B, C \in PA$. Then $\underline{P_A}(B \vee C) = (\underline{P_A}(B) \vee \underline{P_A}(C))$, if $|E_x^P| = 1$, for every $x \in A_0$.*

Proof: If $|E_x^P| = 1$ for $x \in A_0$. Then

$$P(x,y) = \begin{cases} A(x) & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

Now, for $x \in A_0$,

$$\begin{aligned} &\underline{P_A}(B \vee C)(x) \\ &= \wedge_{y \in A_0} \{A(x) \otimes (P(y,x) \to (B \vee C)(y))\} \\ &= [\wedge_{x=y \in A_0} \{A(x) \otimes (A(x) \to (B \vee C)(y))\}] \wedge \\ &\quad [\wedge_{x \neq y \in A_0} \{A(x) \otimes (0 \to (B \vee C)(y))\}] \\ &= [\wedge_{x=y \in A_0} \{(B \vee C)(y)\}] \wedge [\wedge_{x \neq y \in A_0} \{A(x) \otimes 1\}], \\ &\quad \text{[from Proposition 2.1(1,4)]} \\ &= (B \vee C)(x) \\ &= B(x) \vee C(x). \end{aligned}$$

Again, for $x \in A_0$,

$$\begin{aligned} \underline{P_A}(B)(x) &= \wedge_{y \in A_0} A(x) \otimes (P(y,x) \to B(y)) \\ &= [\wedge_{x=y \in A_0} \{A(x) \otimes (A(x) \to B(y))\}] \wedge \\ &\quad [\wedge_{x \neq y \in A_0} \{A(x) \otimes (0 \to B(y))\}] \\ &= [\wedge_{x=y \in A_0} B(y)] \wedge [\wedge_{x \neq y \in A_0} \{A(x) \otimes 1\}], \\ &\quad \text{[from Proposition 2.1(1,4)]} \\ &= B(x). \end{aligned}$$

Similarly, we can show that, $\underline{P_A}(C)(x) = C(x)$. Thus from above $\underline{P_A}(B \vee C)(x) = \underline{P_A}(B)(x) \vee \underline{P_A}(C)(x)$. Hence $\underline{P_A}(B \vee C) = \underline{P_A}(B) \vee \underline{P_A}(C)$.

4. *L*-VALUED NATURAL TRANSFORMATIONS

In this section, we introduce the concepts of L-valued lower and upper backward natural transformations. Further, we show that there is a close connection between such transformations and maps between two L-valued approximation spaces.

Throughout the rest part of this paper, A_{10}, A_{20} are two nonempty sets and A_1, A_2 are L-sets of A_{10} and A_{20}, respectively. The following is a concept of L-valued Zadeh's backward operator.

Definition 4.1 Let A_{10} and A_{20} be two nonempty sets. Again let A_1 and A_2 be L-sets of A_{10} and A_{20}, respectively and $\phi : A_{10} \to A_{20}$ be a map. Then ***L*-valued Zadeh's backward operator** $\phi^{\leftarrow} : PA_2 \to PA_1$ is defined as follows:

$$\phi^{\leftarrow}(B_2)(x_1) = B_2(\phi(x_1)) \otimes A_1(x_1), \forall B_2 \in PA_2, \forall x_1 \in A_{10}.$$

Definition 4.2 Let (A_1, P_1) and (A_2, P_2) be two L-valued approximation spaces. A one-one map $\phi : A_{10} \to A_{20}$ is called

i. An ***L*-valued upper backward natural transformation** from (A_1, P_1) to (A_2, P_2), if $\overline{P_1}(\phi^{\leftarrow}(B_2)) \leq \phi^{\leftarrow}(\overline{P_2}(B_2)), \forall B_2 \in PA_2$, and

ii. An ***L*-valued relation preserving map** if $A_2(\phi(y_1)) \otimes P_1(y_1, x_1) \leq P_2(\phi(y_1), \phi(x_1)) \otimes A_1(x_1)$.

Now, we have the following:

Proposition 4.1 *Let L be an MV algebra, (A_1, P_1), (A_2, P_2) be two L-valued approximation spaces and $\phi : A_{10} \to A_{20}$ be a one-one map. Then ϕ is an L-valued upper backward natural transformation if and only if ϕ is L-valued relation preserving map provided L satisfies idempotency property.*

Proof: Let ϕ be an L-valued relation preserving map. Then $P_2(\phi(y_1), \phi(x_1)) \otimes A_1(x_1) \geq A_2(\phi(y_1)) \otimes P_1(y_1, x_1)$. Again, let $\phi(y_1) = y_2$. Then for $B_2 \in PA_2$,

$$\begin{aligned} A_2(\phi(y_1)) \otimes P_1(y_1, x_1) &\leq P_2(\phi(y_1), \phi(x_1)) \otimes A_1(x_1) \\ B_2(\phi(y_1)) \otimes P_1(y_1, x_1) &\leq A_1(x_1) \otimes P_2(\phi(y_1), \phi(x_1)), \text{[from Proposition 2.1(3)]} \\ \{B_2(\phi(y_1)) \otimes B_2(\phi(y_1))\} \otimes P_1(y_1, x_1) &\leq B_2(\phi(y_1)) \otimes A_1(x_1) \otimes P_2(\phi(y_1), \phi(x_1)) \\ B_2(\phi(y_1)) \otimes A_1(y_1) \otimes \{A_1(y_1) \to P_1(y_1, x_{1d})\} &\leq B_2(\phi(y_1)) \otimes A_1(x_1) \otimes \{A_2(y_2) \to P_2(\phi(y_1), \phi(x_1))\}, \\ &\quad \text{[from Proposition 2.1(3)and Proposition 2.2(1)]} \\ \vee_{y_1 \in A_{10}} [\phi^{\leftarrow}(B_2)(y_1) \otimes \{A_1(y_1) \to P_1(y_1, x_1)\}] &\leq \vee_{y_2 \in A_{20}} [B_2(y_2) \otimes \{A_2(y_2) \to P_2(y_2, \phi(x_1))\}] \otimes A_1(x_1) \\ \overline{P_1}(\phi^{\leftarrow}(B_2))(x_1) &\leq \overline{P_2}(B_2)(\phi(x_1)) \otimes A_1(x_1) \\ \overline{P_1} \circ \phi^{\leftarrow}(B_2)(x_1) &\leq \phi^{\leftarrow} \circ \overline{P_2}(B_2)(x_1) \\ \overline{P_1} \circ \phi^{\leftarrow} &\leq \phi^{\leftarrow} \circ \overline{P_2}. \end{aligned}$$

Thus ϕ is an L-valued upper backward natural transformation. Conversely, let ϕ be an L-valued upper backward natural transformation. Then

$$\begin{aligned}\overline{P_1} \circ \phi^{\leftarrow}\left(A_{2y_2}\right)(x_1) &= \vee_{z_1 \in A_{10}}\left\{\phi^{\leftarrow}\left(A_{2y_2}\right)(z_1) \otimes \{A_1(z_1) \rightarrow P_1(z_1, x_1)\}\right. \\ &= \vee_{z_1 \in A_{10}}\left[\left\{A_{2y_2}(\phi(z_1)) \otimes A_1(z_1)\right\} \otimes \{A_1(z_1) \rightarrow P_1(z_1, x_1)\}\right] \\ &= \vee_{z_1 \in A_{10}}\left[A_{2y_2}(\phi(z_1)) \otimes \{A_1(z_1) \otimes \{A_1(z_1) \rightarrow P_1(z_1, x_1)\}\right] \\ &= \vee_{z_1 \in A_{10}}\left\{A_{2y_2}(\phi(z_1)) \otimes P_1(z_1, x_1)\right\} \\ &= A_2(y_2) \otimes P_1(y_1, x_1).\end{aligned}$$

Now,

$$\begin{aligned}\phi^{\leftarrow} \circ \overline{P_2}\left(A_{2y_2}\right)(x_1) &= \overline{P_2}\left(A_{2y_2}\right)(\phi(x_1)) \otimes A_1(x_1) \\ &= \vee_{z_2 \in A_{20}}\left[A_{2y_2}(z_2) \otimes \{A_2(z_2) \rightarrow P_2(z_2, \phi(x_1))\}\right] \otimes A_1(x_1) \\ &= A_2(y_2) \otimes \{A_2(y_2) \rightarrow P_2(y_2, \phi(x_1))\} \otimes A_1(x_1) \\ &= P_2(y_2, \phi(x_1)) \otimes A_1(x_1).\end{aligned}$$

Also,

$$\begin{aligned}\phi^{\leftarrow} \circ \overline{P_2} &\geq \overline{P_1} \circ \phi^{\leftarrow} \\ \phi^{\leftarrow} \circ \overline{P_2}(A_{2y_2})(x_1) &\geq \overline{P_1} \circ \phi^{\leftarrow}(A_{2y_2})(x_1) \\ P_2(y_2, \phi(x_1)) \otimes A_1(x_1) &\geq A_2(y_2) \otimes P_1(y_1, x_1) \\ P_2(\phi(y_1), \phi(x_1)) \otimes A_1(x_1) &\geq A_2(\phi(y_1)) \otimes P_1(y_1, x_1).\end{aligned}$$

Thus ϕ is an L-valued relation preserving map.

Definition 4.3 Let (A_1, P_1) and (A_2, P_2) be two L-valued approximation spaces. A one-one map $\phi : A_{10} \rightarrow A_{20}$ is called an **L-valued lower backward natural transformation** from (A_1, P_1) to (A_2, P_2), if $\underline{P_1}(\phi^{\leftarrow}(B_2)) \geq \phi^{\leftarrow}(\underline{P_2}(B_2)), \forall\, B_2 \in PA_2$.

Before stating next, we recall the following from [15]:

Definition 4.4 Let L be an MV-algebra. Then the **pseudo complement** of $B \in PA$ is defined as follows:

$\sim B(x) = A(x) \otimes (\neg B(x)), \forall\, x \in A_0$.

Proposition 4.2 *Let (A, P) be an L-valued approximation space and let $\overline{P_A}$ and $\underline{P_A}$ be L-valued upper and L-valued lower approximation operators of (A, P). Then for all $B \in PA$,*

$\sim \overline{P_A}(\sim B) = \underline{P_A}(B)$,

$\sim \underline{P_A}(\sim B) = \overline{P_A}(B)$

Proposition 4.3 *Let L be an MV-algebra. Then*

i. *For all $B \in PA$, $\sim(\sim B) = B$,*

ii. *If $B \leq C$, then $\sim B \geq \sim C$, $\forall\, B, C \in PA$.*

Proof: (i) For all $B \in PA$ and $x \in A_0$,

$$\begin{aligned}\sim(\sim B)(x) &= A(x) \otimes (\sim B(x) \rightarrow 0) \\ &= A(x) \otimes \{(A(x) \otimes \neg B(x)) \rightarrow 0\} \\ &= A(x) \otimes \neg\{(A(x) \otimes \neg B(x))\} \\ &= A(x) \otimes (A(x) \rightarrow B(x)), \text{[from Proposition 2.2(4)]} \\ &= B(x), \text{[from Proposition 2.2(1)]}.\end{aligned}$$

Hence the proof.

(ii) Let $B \leq C$. Then $B(x) \leq C(x)$, or $\neg B(x) \geq \neg C(x)$, or $A(x) \otimes (\neg B(x)) \geq A(x) \otimes (\neg C(x))$, or that $\sim B(x) \geq \sim C(x)$, $\forall\, x \in A_0$.

Lemma 4.1 Let L be an MV-algebra and let (A_1, P_1) and (A_2, P_2) be two L-valued approximation spaces. A one-one map $\phi : A_{10} \rightarrow A_{20}$ is an L-valued upper backward natural transformation if and only if ϕ is L-valued lower backward natural transformation provided L satisfies idempotency property and $A_1(x_1) \leq A_2(\phi(x_1))$, $\forall\, x_1 \in A_{10}$.

Proof: For $B_2 \in PA_2$ and $x_1 \in A_{10}$,

$$\begin{aligned}\sim \phi^{\leftarrow}(B_2)(x_1) &= A_1(x_1) \otimes \neg\phi^{\leftarrow}(B_2)(x_1) \\ &= A_1(x_1) \otimes \neg(B_2(\phi(x_1)) \otimes A_1(x_1)) \\ &= A_1(x_1) \otimes \{A_1(x_1) \rightarrow \neg(B_2(\phi(x_1)))\} \\ &= A_1(x_1) \wedge \neg(B_2(\phi(x_1))), \text{[from Proposition 2.2(1)]} \\ &= A_1(x_1) \otimes \neg(B_2(\phi(x_1))), \text{[from Proposition 2.2(6)]}.\end{aligned}$$

Again,

$$\begin{aligned}\phi^{\leftarrow}(\sim B_2)(x_1) &= \sim B_2(\phi(x_1)) \otimes A_1(x_1) \\ &= \{A_2(\phi(x_1)) \otimes \neg B_2(\phi(x_1))\} \otimes A_1(x_1) \\ &= \{A_2(\phi(x_1)) \otimes A_1(x_1)\} \otimes \neg B_2(\phi(x_1)) \\ &= \{A_2(\phi(x_1)) \wedge A_1(x_1)\} \otimes \neg B_2(\phi(x_1)) \\ &= A_1(x_1) \otimes \neg B_2(\phi(x_1)).\end{aligned}$$

Thus $\sim \phi^{\leftarrow}(B_2)(x_1) = \phi^{\leftarrow}(\sim B_2)(x_1)$. Now,

$$\begin{aligned}\phi^{\leftarrow}(\overline{P_2}(B_2)) &\geq \overline{P_1}(\phi^{\leftarrow}(B_2)) \\ \Leftrightarrow \sim \phi^{\leftarrow}(\overline{P_2}(B_2)) &\leq \sim \overline{P_1}(\phi^{\leftarrow}(B_2)) \\ \Leftrightarrow \phi^{\leftarrow}(\sim \overline{P_2}(B_2)) &\leq \underline{P_1}(\sim \phi^{\leftarrow}(B_2)) \\ \Leftrightarrow \phi^{\leftarrow}(\underline{P_2}(\sim B_2)) &\leq \underline{P_1}(\phi^{\leftarrow}(\sim B_2)).\end{aligned}$$

Replacing $\sim B_2$ by B_2, we have $\phi^{\leftarrow}(\underline{P_2}(B_2)) \leq \underline{P_1}(\phi^{\leftarrow}(B_2))$. Hence ϕ is an L-valued lower backward natural transformation.

Finally, we have the following:

Proposition 4.4 *Let L be an MV-algebra and* (A_1, P_1), (A_2, P_2) *be two L-valued approximation spaces. A one-one map* $\phi : A_{10} \to A_{20}$ *is an L-valued lower backward natural transformation if and only if* ϕ *is L-valued relation preserving map provided L satisfies idempotency property and* $A_1(x_1) \leq A_2(\phi(x_1))$, $\forall\, x_1 \in A_{10}$.

Proof: *Follows from the Proposition 4.1 and Lemma 4.1.*

5. *L*-VALUED TRANSFORMATION SYSTEMS VERSUS *L*-VALUED APPROXIMATION SPACES

In this section, we introduce and study the concepts of L-valued upper/lower transformation systems. Interestingly, we show that there is bijection between L-valued upper/lower transformation systems and L-valued reflexive approximation spaces. We begin with the following:

Definition 5.1 Let $G : PA \to PA$ be a map. Then the system (A, G) is called an L-**valued upper transformation system** if

i. For each $B \in PA$, $B(x) \leq G(B)(x)$,

ii. For each $\{B_i : i \in I\} \in PA$, $G(\vee_{i\in I} B_i) = \vee_{i\in I} G(B_i)$,

iii. For each $a, b \in L$ with $a \leq b$ and $\vee_{x\in A_0} B(x) \leq b$, $G(\boldsymbol{a} \otimes (\boldsymbol{b} \to B)) = \boldsymbol{a} \otimes (\boldsymbol{b} \to G(B))$,

iv. Core $(G(A_y)) \neq \emptyset$.

Lemma 5.1 Let $B \in PA$. Then $B = \vee_{y\in A_0} \mathrm{B(y)} \otimes (\mathrm{A(y)} \to A_y)$, where $\mathrm{B(y)}$ and $\mathrm{A(y)}$ are constant L-subsets of A with constant values $B(y)$ and $A(y)$, respectively.

Theorem 5.1 *Let L be an MV-algebra. Then the following statements are equivalent:*

1. *(A, G) is an L-valued upper transformation system.*
2. *There exists an L-valued reflexive approximation space (A, P) such that $G = \overline{P_A}$.*

Proof: (1) ⇒(2). Let (A, G) be an L-valued upper transformation system. For $x, y \in A_0$, let $P(y, x) = G(A_y)(x)$. Then we have to show that P is an L-valued relation. Since $G(A_y) \leq A$, we have $P(y, x) \leq A(x)$. Again, as $A_y = \mathrm{A(y)} \otimes (\mathrm{A(y)} \to A_y)$, we have $G(\mathrm{A(y)} \otimes (\mathrm{A(y)} \to A_y)) = \mathrm{A(y)} \otimes (\mathrm{A(y)} \to G(A_y))$, or $G(A_y) = G(\mathrm{A(y)} \wedge A_y) = \mathrm{A(y)} \wedge G(A_y) \leq \mathrm{A(y)}$. Thus $P(y, x) = G(A_y)(x) \leq \mathrm{A(y)}(x) = A(y)$, or that $P(y, x) \leq A(x) \wedge A(y)$. Now, for $B \in PA$ and $x \in A_0$, we have

$$\begin{aligned}
\overline{P_A}(B)(x) &= \vee_{y\in A_0} B(y) \otimes (A(y) \to P(y, x)) \\
&= \vee_{y\in A_0} B(y) \otimes (A(y) \to G(A_y)(x)) \\
&= \vee_{y\in A_0} G(\mathrm{B(y)} \otimes (\mathrm{A(y)} \to A_y))(x) \\
&= G(\vee_{y\in A_0} \mathrm{B(y)} \otimes (\mathrm{A(y)} \to A_y))(x) \\
&= G(B)(x).
\end{aligned}$$

Hence $\overline{P_A} = G$.

(2) ⇒ (1). Let (A, P) be an L-valued reflexive approximation space and $\overline{P_A} : PA \to PA$ be an L-valued upper approximation operator. Then

(i) For $x \in A_0$,

$$\begin{aligned}
\overline{P_A}(B)(x) &= \vee_{y\in A_0} \{B(y) \otimes (A(y) \to P(y, x))\} \\
&= \vee_{y\in A_0} \{B(y) \otimes (P(y, y) \to P(y, x))\} \\
&= B(x) \vee [\vee_{y\neq x\in A_0} \{B(y) \otimes (P(y, y) \to P(y, x))\}] \\
&\geq B(x).
\end{aligned}$$

Hence $B \leq \overline{P_A}(B)$.

(ii) For $x \in A_0$, we have

$$\begin{aligned}
\overline{P_A}(\vee_{i\in I} B_i)(x) &= \vee_{y\in A_0} \{(\vee_{i\in I} B_i)(y) \otimes (A(y) \to P(y, x))\} \\
&= \vee_{y\in A_0} \{\vee_{i\in I} (B_i(y) \otimes (A(y) \to P(y, x)))\} \\
&= \vee_{i\in I} \{\vee_{y\in A_0} (B_i(y) \otimes (A(y) \to P(y, x)))\} \\
&= \vee_{i\in I} \overline{P_A}(B_i)(x)
\end{aligned}$$

Hence $\overline{P_A}(\vee_{i\in I} B_i) = \vee_{i\in I} \overline{P_A}(B_i)$.

(iii) For $x \in A_0$, we have

$$\begin{aligned}
\overline{P_A}(\boldsymbol{a} \otimes (\boldsymbol{b} \to B))(x) &= \vee_{y\in A_0} \{(\boldsymbol{a} \otimes (\boldsymbol{b} \to B))(y) \otimes (A(y) \to P(y, x))\} \\
&= \vee_{y\in A_0} [\{(b \to a) \otimes B(y)\} \otimes \\
&\quad (A(y) \to P(y, x))], \text{[from Proposition 2.2(3)]} \\
&= (b \to a) \otimes [\vee_{y\in A_0} \{B(y) \otimes (A(y) \to P(y, x))\}] \\
&= (b \to a) \otimes \overline{P_A}(B)(x) \\
&= (\boldsymbol{a} \otimes (\boldsymbol{b} \to \overline{P_A}(B)))(x).
\end{aligned}$$

Hence $\overline{P_A}(\boldsymbol{a} \otimes (\boldsymbol{b} \to B)) = (\boldsymbol{a} \otimes (\boldsymbol{b} \to \overline{P_A}(B)))$.

(iv) For $x \in A_0$, we have

$$\begin{aligned}
\overline{P_A}(A_y)(x) &= \vee_{z\in A_0} \{A_y(z) \otimes (A(z) \to P(z, x))\} \\
&= A(y) \otimes (A(y) \to P(y, x)) \\
&= P(y, x).
\end{aligned}$$

Since P is an L-valued reflexive relation. Therefore $core(\overline{P_A}(A_y)) \neq \emptyset$. Thus $(A, \overline{P_A})$ is an L-valued upper transformation system.

Now, we introduce the concept of an L-valued lower transformation system.

Definition 5.2 Let $H : PA \to PA$ be a map. Then the system (A, H) is called an L-**valued lower transformation system** if

i. For each $B \in PA$, $B(x) \geq H(B)(x)$,

ii. For each $\{B_i : i \in I\} \in PA$, $H(\wedge_{i\in I} B_i) = \wedge_{i\in I} H(B_i)$,

iii. For each $a \in L$, $H(A \wedge (\boldsymbol{a} \to B)) = A \wedge (\boldsymbol{a} \to H(B))$,

iv. Core$(\neg(A \to H(A_{-y0}))) \neq \emptyset$.

Lemma 5.2 Let L be an MV-algebra. Then for $B \in PA$, $B = \wedge_{z\in A_0}(A \wedge ((\mathrm{B(z)} \to 0) \to A_{-z0}))$, where B(z) is a constant L-subset of A with constant value $B(z)$.

Theorem 5.2 *Let L be an MV-algebra. Then the following statements are equivalent:*

1. *(A, H) is an L-valued lower transformation system.*
2. *There exists an L-valued reflexive approximation space (A, P) such that $H = \underline{P}_A$.*

Proof: (1) ⇒(2). Let (A, H) be an L-valued upper transformation system and $x, y \in A_0$, Then $P(y, x) = \neg(A(x) \to H(A_{-y0})(x))$. Now, for $B \in PA$ and $x \in A_0$, we have

$$\begin{aligned}
\underline{P}_A(B)(x) &= A(x) \wedge \left\{\wedge_{z\in A_0}((A(x) \to P(z,x)) \to B(z))\right\}\\
&= A(x) \wedge \left[\wedge_{z\in A_0}[\{A(x) \to (\neg(A(x) \to\right.\\
&\quad H(A_{-z0})(x)\} \to B(z)]]\\
&= A(x) \wedge \left[\wedge_{z\in A_0}[\{\{A(x) \otimes (A(x) \to\right.\\
&\quad H(A_{-z0})(x) \to 0 \to B(z), \text{[from Proposition 2.2(4)]}\\
&= A(x) \wedge \left\{\wedge_{z\in A_0}(\neg H(A_{-z0})(x) \to B(z))\right\}\\
&= A(x) \wedge \left[\wedge_{z\in A_0}\{(H(A_{-z0})(x) \to 0) \to B(z)\}\right]\\
&= A(x) \wedge \left[\wedge_{z\in A_0}\{(B(z)\right.\\
&\quad \to 0 \to H(A_{-z0})(x), \text{[from Proposition 2.2(5)]}\\
&= \wedge_{z\in A_0} H\{A \wedge ((\mathrm{B(z)} \to 0) \to A_{-z0})\}(x)\\
&= H\left(\wedge_{z\in A_0}(A \wedge ((\mathrm{B(z)} \to 0) \to A_{-z0}))\right)(x)\\
&= H(B)(x).
\end{aligned}$$

Hence $\underline{P}_A = H$.

(2) ⇒ (1). Let (A, P) be an L-valued reflexive approximation space and $\underline{P}_A : PA \to PA$ be an L-valued upper approximation operator. Then

(i) For $x \in A_0$,

$$\begin{aligned}
\underline{P}_A(B)(x) &= \wedge_{y\in A_0}\{A(x) \wedge ((A(x) \to P(y,x)) \to B(y))\}\\
&= \{A(x) \wedge ((A(x) \to P(x,x)) \to B(x))\}\wedge\\
&\quad [\wedge_{y\neq x\in A_0}\{A(x) \wedge ((A(x) \to P(y,x)) \to B(y))\}]\\
&= B(x) \wedge [\wedge_{y\neq x\in A_0}\{A(x) \wedge ((A(x) \to P(y,x)) \to B(y))\}]\\
&\leq B(x).
\end{aligned}$$

Therefore, $\underline{P}_A(B) \leq B$.

(ii) For $x \in A_0$, we have

$$\begin{aligned}
\underline{P}_A(\wedge_{i\in I}B_i)(x) &= \wedge_{y\in A_0}\{A(x) \wedge ((A(x) \to P(y,x)) \to \wedge_{i\in I}B_i(y))\}\\
&= \wedge_{y\in A_0}[A(x) \wedge \{\wedge_{i\in I}((A(x) \to\\
&\quad P(y,x)) \to B_i(y))\}], \text{[from Proposition 2.1(10)]}\\
&= \wedge_{y\in A_0}[\wedge_{i\in I}\{A(x) \wedge ((A(x) \to P(y,x)) \to B_i(y))\}]\\
&= \wedge_{i\in I}[\wedge_{y\in A_0}\{A(x) \wedge ((A(x) \to P(y,x)) \to B_i(y))\}]\\
&= \wedge_{i\in I}\underline{P}_A(B_i)(x).
\end{aligned}$$

Therefore, $\underline{P}_A(\wedge_{i\in I}B_i) = \wedge_{i\in I}\underline{P}_A(B_i)$.

(iii) For $x \in A_0$, we have

$$\begin{aligned}
\underline{P}_A(A \wedge (\boldsymbol{a} \to B))(x) &= \wedge_{y\in A_0}\left[A(x) \otimes \{P(y,x) \to (A(y) \wedge\right.\\
&\quad (a \to B(y))\}]\\
&= \wedge_{y\in A_0}\left[A(x) \otimes (P(y,x) \to A(y)) \wedge\right.\\
&\quad \{P(y,x) \to (a \to B(y))\}]\\
&= \wedge_{y\in A_0}\left[A(x) \otimes \{P(y,x) \to (a \to B(y))\}\right]\\
&= \wedge_{y\in A_0}\left[A(x) \otimes \{a \to (P(y,x) \to\right.\\
&\quad B(y), \text{[from Proposition 2.1(6)]}.
\end{aligned}$$

On other hand,

$$\begin{aligned}
\left[A \wedge \left(\alpha \to \underline{P}_A(B)\right)\right](x) &= A(x) \wedge \left[a \to \wedge_{y\in A_0}\{A(x) \wedge\right.\\
&\quad ((A(x) \to P(y,x)) \to B(y))\}]\\
&= A(x) \wedge \left[\wedge_{y\in A_0}[a \to \{A(x) \wedge\right.\\
&\quad ((A(x) \to P(y,x)) \to B(y))\}]]\\
&= \wedge_{y\in A_0}[A(x) \wedge [a \to \{(A(x) \to\\
&\quad P(y,x) \to B(y)\}]]\\
&= \wedge_{y\in A_0}\left[A(x) \wedge \left[(A(x) \to P(y,x)) \to\right.\right.\\
&\quad (a \to B(y))]], \text{[from Proposition 2.1(6)]}\\
&= \wedge_{y\in A_0}\left[A(x) \otimes \left[a \to (P(y,x) \to B(y))\right]\right].
\end{aligned}$$

Therefore, $A \wedge (\boldsymbol{a} \to \underline{P}_A(B)) = \underline{P}_A(A \wedge (\boldsymbol{a} \to B))(x)$.

(iv) For $x \in A_0$,

$$\begin{aligned}
\underline{P}_A(A_{-y0})(x) &= \wedge_{z\in A_0}[A(x) \wedge \{(A(x) \to P(z,x)) \to A_{-y0}(z)\}]\\
&= [A(x) \wedge \{(A(x) \to P(y,x)) \to 0\}] \wedge [\wedge_{z\neq y}\{A(x)\wedge\\
&\quad ((A(x) \to P(z,x)) \to A_{y0}(z))\}]\\
&= A(x) \wedge \{\neg(A(x) \to P(y,x))\}\\
&= A(x) \wedge \{A(x) \otimes \neg P(y,x)\}, \text{[from Proposition 2.2(4)]}\\
&= A(x) \otimes \neg P(y,x).
\end{aligned}$$

Again,

$$\begin{aligned}\neg(A \to \underline{P_A}(A_{-y0}))(x) &= \neg\{A(x) \to (A(x) \otimes \neg P(y,x))\}\\ &= \neg[A(x) \to \neg\{A(x) \to P(y,x)\}]\\ &= \neg\neg[A(x) \otimes (A(x) \to P(y,x))]\\ &= \neg\neg P(y,x)\\ &= P(y,x).\end{aligned}$$

Since P is reflexive, $core(\neg(A \to \underline{P_A}(A_{-y0}))) \neq \emptyset$. Hence $(A, \underline{P_A})$ is an L-valued lower transformation system.

Definition 5.3 For two L-valued upper transformation systems (A_1, G_1) and (A_2, G_2) a **homomorphism** $\phi : (A_1, G_1) \to (A_2, G_2)$ is a map $\phi : A_{10} \to A_{20}$ such that $\phi^{\leftarrow} \circ G_2 \geq G_1 \circ \phi^{\leftarrow}$.

L-valued upper transformation systems alongwith their homomorphisms form a category, say, **UFT** $^{\uparrow}$.

Definition 5.4 For two L-valued reflexive approximation spaces (A_1, P_1) and (A_2, P_2), a **homomorphism** $\phi : (A_1, P_1) \to (A_2, P_2)$ is a map $\phi : A_{10} \to A_{20}$ such that $A_1(x_1) \otimes P_2(\phi(y_1), \phi(x_1)) \geq A_2(\phi(y_1)) \otimes P_1(y_1, x_1)$.

L-valued reflexive approximation spaces alongwith their homomorphisms form a category, say, **LFAS**.

Lemma 5.3 Let $\phi : A_{10} \to A_{20}$ be a map. Then $\phi^{\leftarrow}(A_{2y_2})$ can be expressed as $\phi^{\leftarrow}(A_{2y_2}) = \mathrm{A}_2(\mathrm{y}_2) \otimes (1 \to A_{1\phi^{-1}(y_2)})$, where $\mathrm{A}_2(\mathrm{y}_2)$ is constant L-subset of A_2 with the value $A_2(y_2)$.

Theorem 5.3 *Let L be an MV-algebra which satisfies idempotency property. Then the categories* ***UFT***$^{\uparrow}$ *and* ***LFAS*** *are isomorphic.*

Proof: Let $\phi : (A_1, G_1) \to (A_2, G_2)$ be an **UFT**$^{\uparrow}$-morphism and $J : \mathbf{UFT}^{\uparrow} \to \mathbf{LFAS}$ be a functor such that $J(A_1, G_1) = (A_1, P_1)$, $J(A_2, G_2) = (A_2, P_2)$ and $J(\phi) = \phi$, where $P_1(y_1, x_1) = G_1(A_{1y_1})(x_1)$, and $P_2(y_2, x_2) = G_2(A_{2y_2})(x_2)$. For $x_1 \in A_{10}$,

$$\begin{aligned}G_1 \circ \phi^{\leftarrow}\left(A_{2y_2}\right)(x_1) &= G_1\left\{\mathrm{A}_2(\mathrm{y}_2) \otimes \left(1 \to A_{1\phi^{-1}(y_2)}\right)\right\}(x_1)\\ &= \left\{\mathrm{A}_2(\mathrm{y}_2) \otimes \left(1 \to G_1\left(A_{1\phi^{-1}(y_2)}\right)\right)\right\}(x_1)\\ &= A_2\left(y_2\right) \otimes G_1\left(A_{1\phi^{-1}(y_2)}\right)(x_1),\\ &\quad \text{[from Proposition 2.1(1)]}\\ &= A_2\left(y_2\right) \otimes P_1\left(\phi^{-1}\left(y_2\right), x_1\right).\end{aligned}$$

Again,

$$\begin{aligned}\phi^{\leftarrow} \circ G_2(A_{2y_2})(x_1) &= G_2(A_{2y_2})(\phi(x_1)) \otimes A_1(x_1)\\ &= P_2(y_2, \phi(x_1)) \otimes A_1(x_1).\end{aligned}$$

Since,

$$\phi^{\leftarrow} \circ G_2 \geq G_1 \circ \phi^{\leftarrow}$$

$$\phi^{\leftarrow} \circ G_2(A_{2y_2})(x_1) \geq G_1 \circ \phi^{\leftarrow}(A_{2y_2})(x_1)$$

$$P_2(y_2, \phi(x_1)) \otimes A_1(x_1) \geq A_2(y_2) \otimes P_1(\phi^{-1}(y_2), x_1)$$

and for $\phi^{-1}(y_2) = y_1$, or $y_2 = \phi(y_1)$, $P_2(\phi(y_1), \phi(x_1)) \otimes A_1(x_1) \geq A_2(\phi(y_1)) \otimes P_1(y_1, x_1)$. Hence ϕ is an **LFAS**-morphism.

Conversely, let J^{-1}:**LFAS** $\to$ **UFT**$^{\uparrow}$ be a functor such that $J^{-1}(A_1, P_1) = (A_1, G_1)$, $J^{-1}(A_2, P_2) = (A_2, G_2)$ and $J^{-1}(\phi) = \phi$, where $G_1 = \overline{P_1}$, $G_2 = \overline{P_2}$ and ϕ is an **LFAS**- morphism. Then $P_2(\phi(y_1), \phi(x_1)) \otimes A_1(x_1) \geq A_2(\phi(y_1)) \otimes P_1(y_1, x_1)$. Thus from Proposition 4.1, $P_2(\phi(y_1), \phi(x_1)) \otimes A_1(x_1) \geq A_2(\phi(y_1)) \otimes P_1(y_1, x_1)$, i.e., $\phi^{\leftarrow} \circ \overline{P_2} \geq \overline{P_1} \circ \phi^{\leftarrow}$, or that $\phi^{\leftarrow} \circ G_2 \geq G_1 \circ \phi^{\leftarrow}$, whereby ϕ is an **UFT**$^{\uparrow}$-morphism. Finally, let (A_1, G_1) be an L-valued upper transformation system. Then $J^{-1}J(A_1, G_1) = (A_1, \overline{P_1})$, where $\overline{P_1} = G_1$. Thus $J^{-1}J = I_1$ and $JJ^{-1} = I_2$, where I_1 and I_2 are identity functors of **UFT**$^{\uparrow}$ and **LFAS** respectively. Hence both the categories **UFT**$^{\uparrow}$ and **LFAS** are isomorphic.

6. CONCLUSION

In this paper, we have established an interesting relationship between L-valued transformation systems and L-valued reflexive approximation spaces. In view of the study done in [19], it can be seen that the relationship established in [20] between F-transforms and fuzzy transformation systems is a particular case of the results obtained in this paper. Finally, the result is expressed in terms of categories. As the construction of an L-valued preorder approximation space from an L-valued reflexive approximation space can be done just by using the concept of the transitive closure of given L-valued reflexive relation, it will be interesting to see the relationship between L-valued transformation systems and L-valued preorder approximation spaces. Further, as an L-valued preorder approximation space induce Alexandroff L-valued topology, the relationship between such topologies and L-valued transformation systems may be established.

CONFLICTS OF INTEREST

The authors declare no conflicts of interest.

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