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Research Article

Uncertain Random Optimization Models Based on System Reliability

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ABSTRACT

The reliability of a dynamic system is not constant under uncertain random environments due to the interaction of internal and external factors. The existing researches have shown that some complex systems may suffer from dependent failure processes which arising from hard failure and soft failure. In this paper, we will study the reliability of a dynamic system where the hard failure is caused by random shocks which are driven by a compound Poisson process, and soft failure occurs when total degradation processes, including uncertain degradation process and abrupt degradation shifts caused by shocks, reach a predetermined critical value. Two types of uncertain random optimization models are proposed to improve system reliability where belief reliability index is defined by chance distribution. Then the uncertain random optimization models are transformed into their equivalent deterministic forms on the basis of α-path, and the optimal solutions may be obtained with the aid of corresponding nonlinear optimization algorithms. A numerical example about a jet pipe servo valve is put forward to illustrate established models by numerical methods. The results indicate that the optimization models are effective to the reliability of engineering systems. It is our future work to consider an interdependent competing failure model where degradation processes and shocks can accelerate each other.

1. INTRODUCTION

Reliability serves as one of vital essential characteristics of the system, which represents the capability of the system to carry out required functions. System reliability will be affected by several factors which can be divided into internal factors (wear, corrosion, fatigue, etc.) and external abrupt factors (impact, stress, etc.), which also known as degradation processes and shocks. Generally, the system failure caused by degradation processes is called soft failure. At a certain moment, the system suddenly fails due to accidental shocks, which is called hard failure.

In the traditional reliability assessment, a great quantity of literatures considered that the system failure is caused by soft failure or hard failure. Since 1990s, degradation process modeling has been used to analyze the reliability and statistical characteristics of the system [1–3]. There are also extensive researches on the modeling of shocks for the system suffered from hard failure [4–6]. However, most systems may be subject to hard failure and soft failure simultaneously due to exposure to degradation processes and external shocks. A system reliability model under the competing failure of extreme shock and internal degradation was proposed by Ye *et al.* [7]. Wang *et al.* [8] presented a competing failure model where the degradation rate is increased by shocks. Jiang *et al.* [9] proposed a reliability model for the systems subject to multiple s-dependent competing failure processes with a shifting failure threshold.

Scholars used to believe that only on the basis of enough failure data can we better select the reasonable metrics. However, traditional reliability assessment based on probability theory may be hindered by failure data which are difficult to acquire within a short time. Liu [10] pioneered uncertain reliability analysis of a dynamic system based on uncertainty theory with few failure data. Uncertainty theory associated with the experts' belief degrees was put forward by Liu [11] when there is no enough data to get a frequency distribution that an event will happen. Liu [12] also proposed the definition of uncertain process which is different from random process. Zeng *et al.* [13] put up reliability indexes which can evaluate product's reliability refer to uncertainty theory. Zeng *et al.* [14] also put up a minimal cut set-based numerical evaluation method to compute the system reliability index expressed by uncertain measure.

When there is a complicated system with a mixture of uncertainty and randomness, chance theory was put forward by Liu [15]. Liu [16] also proposed an optimization method to convert an uncertain random problem which is not easy to be solved into its deterministic form. After that, Wen and Kang [17] proposed a reliability assessment method in which reliability index was defined as chance distribution. Additionally, belief reliability metrics which are concerned with uncertain random variables were defined by Zhang *et al.* [18]. Some essential formulas were put forward by Gao and Yao [19] to calculate important indexes of common discrete uncertain random systems. Liu *et al.* [20] carried out the reliability analysis for uncertain random systems with independent degradation processes and shocks.

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There are some research on the reliability optimization of uncertain random systems. Peng *et al.* [21] presented a new method to optimize the reliability of composite laminates taking into account uncertain parameters. A multi-objective optimization model to maximize the reliability of assembly line was proposed by Li *et al.* [22] under uncertain random environments. Optimization of maintenance policies which is severed as an critical role in reliability engineering has attracted much attention. Caballé *et al.* [23] proposed a preventive strategy applicable to the system based on reliability analysis. An optimal scheduling of inspection maintenance and replacement can optimize the system reliability considering the operational cost. Niwas and Garg [24] raised an approach to analyze system reliability and profit based on warranty policy. Chen and Li [25] determined an optimal replacement policy for minimizing the cost of the deteriorating system which is suffering random shocks and degradation processes. We will optimize the reliability of uncertain random systems motivated by these literature of system maintenance policy.

The system in this paper suffers from two dependent failure processes which may be modeled as uncertain degradation process and random shocks. The random shocks may also generate additional degradation shifts which may be driven by an uncertain renewal reward process. The competing failure models with extreme shock and cumulative shock are proposed respectively under uncertain random environment. This paper proposes a strategy of optimal allocation of maintenance resources to optimize the system reliability. The optimal maintenance strategy will be obtained to maximize the system reliability considering the different damage of degradation processes and shocks on system reliability. The system cost can also be saved by adjusting the proportion of maintenance resources.

The sections of this paper are listed below. Section 2 reviews some concepts involved in uncertainty theory. In Section 3, two types of uncertain random optimization system models are proposed, where reliability index is defined by chance distribution. The related algorithms of models are also discussed. In Section 4, the proposed models are verified by a numerical example. Section 5 summarizes this paper in a brief conclusion. Besides, for better comprehension of the article, some mathematical symbols used in the sequel are listed in the following notation.

Notation

H	Threshold value of soft failure
D	Threshold value of hard failure
λ	Intensity of shocks
$N_1(t)$	Total shock times by time t
$N_2(t)$	Total degradation shift times by time t
Z_i	Magnitude of the i-th shock for $i = 1, 2, \cdots, N_1(t)$
F	Probability distribution of Z_i
Y_i	Magnitude of the i-th uncertain degradation shift for $i = 1, 2, \cdots, N_2(t)$
η_i	Interarrival time between i-th and $i+1$-th degradation shift for $i = 1, 2, \cdots, N_2(t)$
X_t	Degradation process
W_t	Wiener process
Ψ	Uncertainty distribution of η_i
Υ_i	Uncertainty distribution of Yi
x_0	Initial degradation value for the uncertain process
$R_e(u)$	Reliability index of extreme shock model
$R_c(u)$	Reliability index of cumulative shock model

2. PRELIMINARY

Let Γ be a nonempty set, and $\mathcal{L}$ be a σ-algebra over Γ. Each element $\Lambda \in \mathcal{L}$ is called an event. A set function $\mathcal{M}$ defined on the σ-algebra $\mathcal{L}$ is called an uncertain measure [11] if it satisfies normality axiom, duality axiom, subadditivity axiom, and product axiom. An uncertain variable is a function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that the set $\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$ is an event for any Borel set of real numbers. The uncertainty distribution $\Phi(x) : R \to [0, 1]$ of an uncertain variable ξ is defined as $\Phi(x) = \mathcal{M}\{\xi \le x\}$ for $x \in R$. An uncertain variable ξ is called normal if it has a normal uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e-x)}{\sqrt{3}\sigma}\right)\right)^{-1}$$

denoted by $\mathcal{N}(e, \sigma)$ where e and σ are real numbers with $\sigma > 0$. An uncertain variable ξ is called lognormal if $\ln \xi$ is a normal uncertain variable $\mathcal{N}(e, \sigma)$.

Theorem 1. *(Liu [26]) Let N_t be a renewal process with i.i.d uncertain interarrival times $\{\xi_1, \xi_2, \cdots\}$. If those interarrival times have a common uncertainty distribution Φ, then N_t has an uncertainty distribution*

$$\Upsilon_t(x) = 1 - \Phi\left(\frac{t}{\| x \| + 1}\right), \forall x > 0. \quad (1)$$

Definition 1. (Yao and Chen [27]) Let α be a number with $0 < \alpha < 1$. An uncertain differential equation $dX_t = f(t, X_t)dt + g(t, X_t)dC_t$ is said to have an α-path X_t^α (C_t is a Liu process) if it solves the corresponding ordinary differential equation $dX_t^\alpha = f(t, X_t^\alpha)dt + |g(t, X_t^\alpha)|\Phi^{-1}(\alpha)dt$, where $\Phi^{-1}(\alpha)$ is the inverse standard normal uncertain distribution, i.e.,

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}.$$

Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and $(\Omega, \mathcal{A}, Pr)$ be a probability space. Then the product $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, Pr)$ is called a chance space. An uncertain random variable is a function ξ from a chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, Pr)$ to the set of real numbers such that $\{\xi \in B\}$ is an event for any Borel set B of real numbers. The chance measure of the uncertain random event $\{\xi \in B\}$ is

$$Ch\{\xi \in B\} = \int_0^1 Pr\{\omega \in \Omega | \mathcal{M}\{\gamma \in \Gamma \,|\, \xi(\gamma, \omega) \in B\} \ge r\} dr.$$

Theorem 2. *(Liu [15]) Let $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, Pr)$ be a chance space. Then*

$$Ch\{\Lambda \times A\} = \mathcal{M}\{\Lambda\} \times Pr\{A\}$$

for any $\Lambda \in \mathcal{L}$ and any $A \in \mathcal{A}$.

3. SYSTEM RELIABILITY

Suppose that the system in this paper suffers from two dependent failures which are hard failure and soft failure, where either of failure processes can lead to system failures. Randomness and uncertainty exist simultaneously when analyzing the reliability of the system.

3.1. Hard Failure

Generally, a two-dimensional random vector sequence $\{(A_i, Z_i)\}_{i=0}^{\infty}$ has been used to describe the shock model. Assume Z_i is the magnitude of the i-th shock, which represents the effect of the i-th shock on the system. $\{N_1(t), t \geq 0\}$ is the counting process generated by a sequence $\{A_{i=0}^{\infty}\}$ in which A_i represents the interarrival time between two adjacent shocks.

Poisson process is a typical mathematical model of the counting process. Intuitively, the counting process of random events may be modeled as Poisson process as long as they occur independently in disjoint intervals and only once in sufficiently small intervals. External shocks approximately satisfy the above conditions in system reliability engineering. Thus, $N_1(t)$ may be regarded as a Poisson process to model total time of shocks. The nonnegative magnitudes of the i-th shock load are expressed as Z_i which are i.i.d random variables.

3.1.1. Extreme shock model

In extreme shock model, the damage of shocks on the system is independent. Hard failure occurs when the size of any shock is beyond a specified threshold level D. The probability that no hard failure occurs by time t can be defined as

$$Pr\left\{\bigcap_{i=1}^{N_1(t)}(h(u_1, Z_i) < D)\right\} \tag{2}$$

where u_1 is the control variable, h is a real function of u_1 and Z_i.

3.1.2. Cumulative shock model

It is assumed that the damage of shocks on the system is accumulated in cumulative shock model. The system will fail if the total magnitudes of accumulated shocks exceed a critical value D. The probability that there is no hard failure by time t can be described as

$$Pr\left\{\sum_{i=1}^{N_1(t)} h(u_1, Z_i) < D\right\} \tag{3}$$

where u_1 is the control variable, h is a real function of u_1 and Z_i.

Suppose $h(u_1, Z_i)$ is a strictly monotone function with respect to Z_i where u_1 is the control variable, denoted as $h_{u_1}(Z_i)$.

3.2. Soft Failure

There is not enough failure data due to the limitation of time and cost in reliability assessment. Therefore, the traditional random process model is no longer applicable. The degradation may be evaluated by experts' experiment data based on uncertainty theory. Theoretically, it is appropriate to describe wear degradation with Liu process. The wear degradation X_t which can be modeled as

$$dX_t = f(t, u_2, X_t)dt + g(t, u_2, X_t)dC_t, \tag{4}$$

where u_2 is the control variable, X_t is the state variable ($X_0 = x_0$), f and g are both real functions of time t, control variable u_2, and state X_t, and C_t is Liu process.

It follows from Definition 1 that the solution X_t of (4) has an inverse uncertainty distribution

$$\Phi_t^{-1}(\alpha) = X_t^{\alpha} \tag{5}$$

where X_t^{α} is the α-path.

The physical structure of the system may be damaged by external shock loads which can generate degradation shifts accumulated instantaneously when random shocks arrive. Since a large amount of data is not easy to derive in reliability assessment, it is more reasonable that degradation shifts are considered as an uncertain process.

Assume that the magnitude of the i-th uncertain degradation Y_i are defined as i.i.d uncertain variables measured by experts' belief degrees. Here, the number of degradation shifts may be quantified by uncertain renewal process $N_2(t)$. The additional degradation shifts denoted as $S(t)$ may be shown by uncertain renewal reward process

$$S(t) = \sum_{i=1}^{N_2(t)} Y_i. \tag{6}$$

Thus, the entire degradation is the sum of wear degradation process and degradation shifts. Soft failure occurs when the entire degradation processes reach a predetermined critical value H. The uncertain measure that there is no soft failure may be defined as

$$\mathcal{M}\left\{X_t + \sum_{i=1}^{N_2(t)} Y_i < H\right\}. \tag{7}$$

How to define a reliability index to measure reliability of complex systems is vital important in reliability engineering. Considering above two dependent failures, the reliability index may be regarded as belief reliability distribution. It is hopeful that uncertain degradation processes will not exceed the critical value H, and random shocks will not lead to system failures. The system can operate normally when neither failure process occurs.

Definition 2. Assume that a system suffers from dependent soft failure and hard failure. Then the chance distribution of extreme shock model, i.e.,

$$Ch\left\{X_t + \sum_{i=1}^{N_2(t)} Y_i < H, \bigcap_{i=1}^{N_1(t)}(h_{u_1}(Z_i) < D)\right\} \tag{8}$$

is defined as belief reliability distribution $R_e(u)$.

Definition 3. Assume that a system suffers from dependent soft failure and hard failure. Then the chance distribution of cumulative shock model, i.e.,

$$Ch\left\{X_t + \sum_{i=1}^{N_2(t)} Y_i < H, \sum_{i=1}^{N_1(t)} h_{u_1}(Z_i) < D\right\} \tag{9}$$

is defined as belief reliability distribution $R_c(u)$.

4. UNCERTAIN RANDOM OPTIMIZATION MODEL OF RELIABILITY

We are to optimize the reliability of a system by control variables which can be shown as regular inspections, maintenance funds, etc. Assume that u is a decision vector and $R(u)$ ($R_e(u)$ or $R_c(u)$) is the objective function. Two optimization index models are presented with reference to the classification of shocks.

4.1. Optimization Model of Extreme Shock

In order to get the decision with the maximum reliability index satisfying a set of constraints, the maximum belief reliability index optimization model of extreme shock can be presented as

$$\begin{cases} \max\limits_{u_1,u_2\in U} Ch\left\{X_t + \sum\limits_{i=1}^{N_2(t)} Y_i < H, \bigcap\limits_{i=1}^{N_1(t)} (h_{u_1}(Z_i) < D)\right\} \\ \text{subject to} \\ \qquad dX_t = f(t,u_2,X_t)dt + g(t,u_2,X_t)dC_t \\ \qquad g(u_1,u_2) \le 0 \\ \qquad X_0 = x_0. \end{cases} \tag{10}$$

It is a nature approach to transform the model (10) to an equivalent problem which may be easy to be solved.

Remark 2. Compared with existing uncertain random systems in [15,16,28], the reliability index in proposed models is also defined by chance distribution. For the continuous system subject to competing failure processes, the models in [17] suffer from the independent degradation process and shocks. In contrast, the advantages of our model in this paper are mainly reflected in the following:

1. The system failure is subject to dependent competing failure processes which are uncertain wear degradation process and random shocks;
2. The dependence of the wear degradation process and shocks is that random shocks may accelerate wear degradation process by degradation shifts which are driven by an uncertain renewal reward process;
3. The proposed uncertain random constrained model may be transformed into a nonlinear constrained programming problem by uncertainty theory;
4. The system reliability is optimized by optimal allocation ratio of maintenance resources considering the different damage of degradation processes and shocks on system reliability.

Theorem 3. *Let X_t be a wear degradation process, $N_1(t)$ be a Poisson process with intensity λ, and Z_i be i.i.d shock loads with probability distribution F. Assume i.i.d interarrival times η_i and independent shock damages Y_i are uncertain variables whose uncertainty distributions are Ψ and Υ_i, respectively. Then the optimization model of extreme shock (10) is equivalent to following form:*

$$\begin{cases} \max\limits_{u_1,u_2\in U} ((1-\Psi(t)) \wedge J_1(H)) \vee (\max\limits_{m\ge 1}(1-\Psi(\frac{t}{1+m})) \\ \wedge J_2(H)) \times \exp(-\lambda t)(1 + \sum\limits_{m=1}^{\infty} \frac{(\lambda t)^m}{m!}(L(h_{u_1}^{-1}(D)))^m) \\ \textit{subject to} \\ \quad dX_t^\alpha = f(t,u_2,X_t^\alpha)dt + |g(t,u_2,X_t^\alpha)|\Phi^{-1}(\alpha)dt \\ \quad g(u_1,u_2) \le 0 \\ \quad X_0 = x_0, \end{cases} \tag{11}$$

where $L(h_{u_1}^{-1}(D))$ is

$$L(h_{u_1}^{-1}(D)) = \begin{cases} F(h_{u_1}^{-1}(D)), \\ \quad \text{if } h_{u_1}(z) \text{ is a strictly increasing} \\ \quad \text{function with respect to } z \\ 1 - F(h_{u_1}^{-1}(D)), \\ \quad \text{if } h_{u_1}(z) \text{ is a strictly decreasing} \\ \quad \text{function with respect to } z \end{cases}$$

$J_1(H)$ is the root α of the equation

$$X_t^\alpha - H = 0,$$

and $J_2(H)$ is the root α of the equation

$$X_t^\alpha + \sum_{i=1}^{k} \Upsilon_i^{-1}(\alpha) - H = 0.$$

Proof Obviously, $X_t + \sum_{i=1}^{N_2(t)} Y_i < H$ is an uncertain event, and $\bigcap_{i=1}^{N_1(t)} (h_{u_1}(Z_i) < D)$ is a random event. It follows from Theorem 2 immediately that the chance measure is the product of uncertain measure and probability measure,

$$Ch\left\{X_t + \sum_{i=1}^{N_2(t)} Y_i < H, \bigcap_{i=1}^{N_1(t)} (h_{u_1}(Z_i) < D)\right\} = \mathcal{M}\left\{X_t + \sum_{i=1}^{N_2(t)} Y_i < H\right\} \times Pr\left\{\bigcap_{i=1}^{N_1(t)} (h_{u_1}(Z_i) < D)\right\}.$$

Part I: Since the degradation shifts follow from an uncertain renewal reward process, we can get

$$
\begin{aligned}
&\mathcal{M}\left\{X_t+\sum_{i=1}^{N_2(t)}Y_i<H\right\}\\
&=\mathcal{M}\left\{\bigcup_{m=0}^{\infty}(N_2(t)=m)\cap\left(X_t+\sum_{i=1}^{m}Y_i<H\right)\right\}\\
&=\mathcal{M}\left\{\bigcup_{m=0}^{\infty}(N_2(t)\le m)\cap\left(X_t+\sum_{i=1}^{m}Y_i<H\right)\right\}\\
&=\mathcal{M}\{(N_2(t)\le 0)\cap(X_t<H)\}\\
&\vee\max_{m\ge1}\mathcal{M}\left\{(N_2(t)\le m)\cap\left(X_t+\sum_{i=1}^{m}Y_i<H\right)\right\}\\
&=(\mathcal{M}\{N_2(t)\le 0\}\wedge\mathcal{M}\{X_t<H\})\\
&\vee\left(\max_{m\ge1}\mathcal{M}\{N_2(t)\le m\}\wedge\mathcal{M}\left\{X_t+\sum_{i=1}^{m}Y_i<H\right\}\right)
\end{aligned}
$$

On the one hand, the uncertain distribution of $N_2(t)$ can be calculated by Eq. (1)

$$
\begin{aligned}
\mathcal{M}\{N_2(t)\le 0\}&=1-\Psi(t),\\
\mathcal{M}\{N_2(t)\le m\}&=1-\Psi\left(\frac{t}{1+m}\right).
\end{aligned}
$$

One the other hand, it follows from (5) that X_t has an inverse uncertainty distribution X_t^{α}. Write $J_1(H)=\mathcal{M}\{X_t<H\}=\alpha\in(0,1)$. That is, $J_1(H)$ is the root α of the equation

$$X_t^{\alpha}-H=0.$$

Referring to operational laws of uncertain variables, the inverse uncertainty distribution of $\sum_{i=1}^{m}Y_i$ can be computed by $\sum_{i=1}^{m}\Upsilon_i^{-1}(\alpha)$. Write $J_2(H)=\mathcal{M}\{X_t+\sum_{i=1}^{m}Y_i<H\}=\alpha\in(0,1)$. That is, $J_2(H)$ is the root α of equation

$$\sum_{i=1}^{m}\Upsilon_i^{-1}(\alpha)+X_t^{\alpha}-H=0.$$

We have

$$
\begin{aligned}
&\mathcal{M}\left\{X_t+\sum_{i=1}^{N_2(t)}Y_i<H\right\}\\
&=((1-\Psi(t))\wedge J_1(H))\vee\left(\max_{m\ge1}\left(1-\Psi\left(\frac{t}{1+m}\right)\right)\right.\\
&\left.\wedge J_2(H)\right).
\end{aligned}
$$

Part II: The random shock follows from the compound Poisson process, we can obtain

$$
\begin{aligned}
&Pr\left\{\bigcap_{i=1}^{N_1(t)}(h_{u_1}(Z_i)<D)\right\}\\
&=Pr\left\{\bigcup_{m=0}^{\infty}(N_1(t)=m)\cap\bigcap_{i=1}^{m}(h_{u_1}(Z_i)<D)\right\}\\
&=Pr\{N_1(t)=0\}\\
&\quad+\sum_{m=1}^{\infty}Pr\left\{(N_1(t)=m)\cap\bigcap_{i=1}^{m}(h_{u_1}(Z_i)<D)\right\}\\
&=Pr\{N_1(t)=0\}\\
&\quad+\sum_{m=1}^{\infty}Pr\{N_1(t)=m\}Pr\left\{\bigcap_{i=1}^{m}(h_{u_1}(Z_i)<D)\right\}
\end{aligned}
$$

Since $N_1(t)$ is a Poisson process with intensity λ, then

$$
\begin{aligned}
Pr\{N_1(t)=0\}&=\exp(-\lambda t)\\
Pr\{N_1(t)=m\}&=\exp(-\lambda t)\frac{(\lambda t)^m}{m!}.
\end{aligned}
$$

As mentioned above, $h_{u_1}(Z_i)$ is a monotone function with respect to Z_i. When $h_{u_1}(Z_i)$ is a strictly increasing function, we have

$$
\begin{aligned}
L(h_{u_1}^{-1}(D))&=Pr\{h_{u_1}(Z_i)\le D\}\\
&=Pr\{Z_i\le h_{u_1}^{-1}(D)\}=F(h_{u_1}^{-1}(D)).
\end{aligned}
$$

When $h_{u_1}(Z_i)$ is a strictly decreasing function, then

$$
\begin{aligned}
L(h_{u_1}^{-1}(D))&=Pr\{h_{u_1}(Z_i)\le D\}\\
&=Pr\{Z_i\ge h_{u_1}^{-1}(D)\}=1-F(h_{u_1}^{-1}(D)).
\end{aligned}
$$

And then

$$\prod_{i=1}^{m}Pr\{h_{u_1}(Z_i)<D\}=(L(h_{u_1}^{-1}(D)))^m.$$

Thus we can derive

$$Pr\left\{\bigcap_{i=1}^{N_1(t)}(h_{u_1}(Z_i)<D)\right\}=\exp(-\lambda t)\left(1+\sum_{m=1}^{\infty}\frac{(\lambda t)^m}{m!}(L(h_{u_1}^{-1}(D)))^m\right),$$

the theorem is verified.

4.2. Algorithm for Extreme Shock Optimization Model

The optimization model with extreme shock may be converted into it's equivalent problem by Theorem 3, then the problem becomes a typical nonlinear constrained programming problem. The penalty function algorithm with conjugate gradient method for nonlinear programming problems with inequality constraints is adopted in this section.

Algorithm 1: Penalty function algorithm for (11)

1. Fix a time t and set $X_0 = x_0, k = 0, l = 0, M_0 > 0, c \geq 2, \varepsilon > 0$ (a predetermined precision).
2. Construct a new function $Q(u, M_l) = R(u) + M_l G(g(u))$ where $G(g(u)) = [\max\{0, g(u)\}]^2$.
3. For initial value $u^k = (u_1^k, u_2^k)$, set $\beta = 0, \gamma = 1$, and $\alpha = (\beta + \gamma)/2$. Then solve the corresponding differential equations
$$dX_t^\alpha = f(t, u_2^k, X_t^\alpha)dt + \left|g(t, u_2^k, X_t^\alpha)\right| \Phi^{-1}(\alpha)dt$$
by Euler method.
4. If $\left(X_t^\alpha - H\right)\left(X_t^\gamma - H\right) \leq 0$, then set $\beta = \alpha$. Otherwise, set $\gamma = \alpha$.
5. If $|\gamma - \beta| > \varepsilon$, set $\alpha = (\gamma + \beta)/2$ and go to step 4. Otherwise, output α as the root $J_1(H)$.
6. Compute $J_2(H)$ similarly to the steps 3 - 5. Then set $h_k = \nabla Q(u^k, M_l), p^k = -h_k$.
7. Compute the step length λ_k by the line search $Q(u^k + \lambda_k p^k, M_l) = \max_{\lambda>0} Q(u^k + \lambda p^k, M_l)$. Then set $u^{k+1} = u^k + \lambda_k p^k$.
8. $k = k + 1, h_k = \nabla Q(u^k, M_l)$.
9. If $\| h_k \| \leq \varepsilon$, go to step 13, otherwise, perform the next step.
10. If $k = n$, go to step 12. Otherwise, perform the next step.
11. Compute $p^k = -h_k + \frac{\|h_k\|^2}{\|h_{k-1}\|^2} p^{k-1}$ and go to step 7.
12. Carry out a new round of iteration from u^k, go to step 3.
13. If $M_l G(g(u^k)) \leq \varepsilon$, then $u^* = u^k$ is the optimal solution. The algorithm ends. Otherwise, $M_{l+1} = cM_l, l = l + 1$, go to step 2.

4.3. Optimization Model of Cumulative Shock

Referring to extreme shock optimization model, we are to present a maximum reliability index optimization model with cumulative shock which can be described as

$$\begin{cases} \max\limits_{u_1,u_2\in U} Ch\left\{X_t + \sum\limits_{i=1}^{N_2(t)} Y_i < H, \sum\limits_{i=1}^{N_1(t)} h_{u_1}(Z_i) < D\right\} \\ \text{subject to} \\ \quad dX_t = f(t, u_2, X_t)dt + g(t, u_2, X_t)dC_t \\ \quad g(u_1, u_2) \leq 0 \\ \quad X_0 = x_0. \end{cases} \tag{12}$$

Theorem 4. *Let X_t be a wear degradation process, $N_1(t)$ be a Poisson process with intensity λ and Z_i be i.i.d shock loads with probability distribution F. Assume i.i.d interarrival times η_i and independent shock damages Y_i are uncertain variables whose uncertainty distributions are Ψ and Υ_i, respectively. The optimization model of cumulative shock (12) can be converted into their corresponding deterministic form*

$$\begin{cases} \max\limits_{u_1,u_2\in U} ((1-\Psi(t)) \wedge J_1(H)) \vee (\max\limits_{m\geq 1}(1-\Psi(\frac{t}{1+m})) \\ \wedge J_2(H)) \times \exp(-\lambda t)(1 + \sum\limits_{m=1}^{\infty} \frac{(\lambda t)^m}{m!} L^*_{h_{u_1}(D)}) \\ \textit{subject to} \\ \quad dX_t^\alpha = f(t, u_2, X_t^\alpha)dt + |g(t, u_2, X_t^\alpha)|\Phi^{-1}(\alpha)dt \\ \quad g(u_1, u_2) \leq 0 \\ \quad X_0 = x_0 \end{cases} \tag{13}$$

where $L^*_{h_{u_1}(D)}$ is the m-convolution

$$L^*_{h_{u_1}(D)} = L(h_{u_1}(z)) * \cdots * L(h_{u_1}(z))|_{z=D},$$

$$L(h_{u_1}^{-1}(D)) = \begin{cases} F(h_{u_1}^{-1}(D)), \\ \quad \textit{if } h_{u_1}(z) \textit{ is a strictly increasing} \\ \quad \textit{function with respect to } z \\ 1 - F(h_{u_1}^{-1}(D)), \\ \quad \textit{if } h_{u_1}(z) \textit{ is a strictly decreasing} \\ \quad \textit{function with respect to } z \end{cases}$$

$J_1(H)$ is the root α of the equation

$$X_t^\alpha - H = 0,$$

and $J_2(H)$ is the root α of the equation

$$X_t^\alpha + \sum_{i=1}^{k} \Upsilon_i^{-1}(\alpha) - H = 0.$$

Proof The proof is similar to that of Theorem 3.

4.4. Algorithm for Cumulative Shock Optimization Model

Algorithm 1 may be utilized to obtain the optimal solutions of cumulative shock optimization problem. The difference between two models (11) and (13) is that they have different terms $L(h_{u_1}^{-1}(D))$ and $L^*_{h_{u_1}(D)}$ in the objective functions. Thus, a key point applying Algorithm 1 to problem (13) is to calculate m-convolution $L^*_{h_{u_1}(D)}$ by Monte Carlo simulation (Algorithm 2).

Algorithm 2: Monte Carlo simulation for m-convolution

1. Set the initial value $k = 0, i = 1, N > 0, m > 0$ and $D > 0$.
2. Generate numbers $h_1, h_2, \cdots, h_m$ randomly with distribution $L(h_{u_1}(z))$ for each u_1.
3. Compute S_i by $S_i = h_1 + h_2 + \cdots + h_m$.
4. If $S_i < D, k = k + 1$.
5. $i = i + 1$.
6. If $i = N$, output the value k/N as $L^*_{h_{u_1}(D)}$ and stop. Otherwise, go to step 2.

5. NUMERICAL EXPERIMENT

The jet pipe servo valve is a representative two-stage electro hydraulic flow valve of EHSVs in modern industrial control systems, which can realize the servo control of mechanical equipments. In the following, we will analyze and optimize the reliability of jet pipe servo valve based on proposed models.

5.1. Proposed Model Description

The study indicates that the main cause of system failures is the fluid deterioration which would cause clamping stagnation and tribological wear between the spool and the sleeve. The clamping stagnation can cause additional wear debris, which would both affect the reliability of the system.

The clamping stagnation is considered as hard failure which is driven by a compound Poisson process with intensity λ. We assume that magnitude of the i-th clamping stagnation Z_i are i.i.d normal random variables, the magnitude of the i-th additional wear debris Y_i are i.i.d normal uncertain variables, the interarrival time between i-th and $i+1$-th wear debris η_i are i.i.d lognormal uncertain variables. Since the degradation processes of the system usually follows the law of exponential decay. It is assumed that shock rate and wear rate which are generated by maintenance funds exhibit an exponential decay function. Thus, suppose that u_1 is the proportion of maintenance funds on clamping stagnation that yields the shock rate e^{-u_1}. Then

$$h(u_1, Z_i) = e^{-u_1} Z_i.$$

The tribological wear is driven by an uncertain process

$$dX_t = ae^{-u_2}dt + be^{-u_2}dC_t,$$

where a is the drift coefficient, b is the diffusion coefficient, u_2 is the proportion of maintenance funds on tribological wear, and e^{-u_2} is the wear rate. The relevant parameters are shown in Table 1.

5.2. Traditional Model Description

The traditional reliability evaluation is modeled based on probability theory in which all the variables $\tilde{Y}_i$ involved are random variables, and the degradation process is driven by random process. The tribological wear can be described as

$$dX_t = adt + bdW_t$$

where W_t is a standard Wiener process. Referring to [29], we can get the evaluation method of system reliability. The relevant parameters are displayed in Table 1. Next, the numerical results of proposed uncertain random system reliability model would be compared with traditional reliability model.

Remark 2. In the numerical experiments, we choose the mostly common probability distribution functions which conform to the physical characteristics of the system according to the reference [30]. The uncertain distribution functions are given based on the experience of experts. Of course, other continuous distribution functions may also be applied. Since the model proposed in this paper is continuous, we have not adopted the discrete distribution function for the time being. In the future, we will carry out the reliability optimization of discrete-time systems to further improve the research of system reliability.

5.3. Results and Analysis Figure 1

In Figure 1, R_e represents the original reliability index with extreme shock of two models. The numerical result shows that the reliability index of traditional model is always more than that of the proposed model. The reliability index decreases gradually with time t, which conforms to the failure process of extreme shock model.

In Figure 2, R_c represents the original reliability index with cumulative shock of two models. Similarly, the reliability of the traditional random model is higher than that of the uncertain random model. The reliability of the system decreases with time t, which accords with the physical characteristics of system reliability.

Table 1 Parameter values.

Parameters	*Values*	*Sources*
H	5 mm	[30]
D	40 N	[30]
λ	2.5×10^{-5}	[30]
Z	$Z \sim \mathcal{N}(30, 5^2)$N	[30]
η	$\eta \sim \mathcal{LOGN}(3.4 \times 10^4, 10^2)$ h	Assumption
Y	$Y \sim \mathcal{N}(0.4, 0.1^2)$ mm	Assumption
	$Y \sim \mathcal{N}(0.5, 0.1^2)$ mm	[30]
a	10^{-3}	Assumption
b	2×10^{-3}	Assumption
x_0	0	[30]

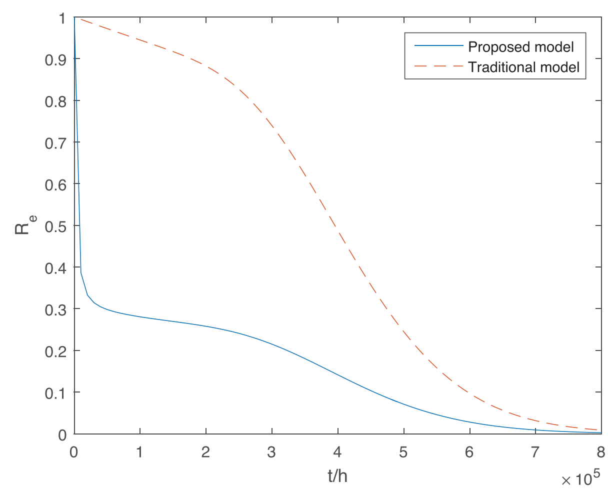

Figure 1 Original index of extreme shock model.

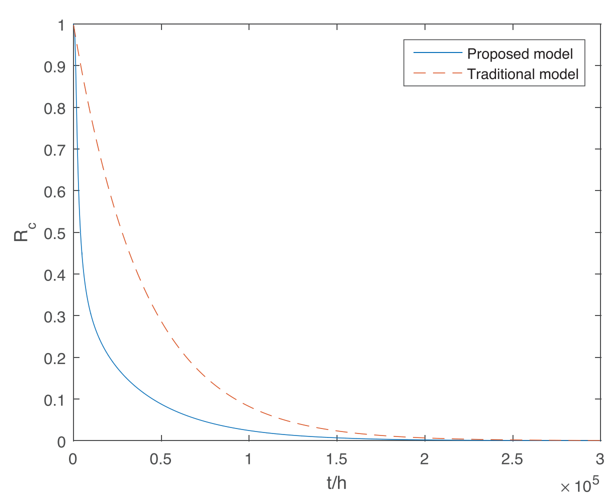

Figure 2 Original index of cumulative shock model.

The experimental data show that evaluation models proposed in this paper are relatively conservative when traditional evaluation models tend to overestimate the system reliability. The internal degradation that may not be quantified as random events due to the lack of failure data, while we may regard them as uncertain events according to experts' experience. If the degradation processes are regarded as random events, the reliability of the system will be overestimated, which will lead to the wrong configuration of maintenance strategy, causing system failure, greatly shortening the system life, and causing economic losses.

The two uncertain random optimization models have been transformed into the nonlinear constrained programming problems by Theorems 3 and 4, respectively. There are lots of software that may be used to solve such typical optimization problems. In this paper, the mathematical software MATLAB is selected to obtain the optimal solutions based on above algorithms.

Then, we select 10 sets of data from optimization model proposed to compare and illustrate the optimization decisions. In Table 2, $R_e(u^*)$ is the optimal reliability index of extreme shock optimization model with optimal control variables u^* obtained by Algorithm 1. In Table 3, $R_c(u^*)$ is the optimal reliability index of cumulative shock optimization model with optimal control variables u^* obtained by Algorithms 1 and 2. $R_e(u)$ and $R_c(u)$ are the original reliability indexes of two models without control variables, respectively.

From these 10 sets of data, we may see that the reliability of the system decreases with time t. In addition, the reliability decay rate of our optimization model is relatively slow. This means that a reasonable proportion of maintenance allocation can slow down the rate of reliability decline when considering degradation and external shocks.

Table 2 Optimal solutions and original index of extreme shock model.

$t(h)$	u^*	$R_e(u^*)$	R_e
1×10^3	(0.6355, 0.3607)	0.9963	0.9736
2×10^3	(0.6109, 0.3881)	0.9195	0.7949
3×10^3	(0.5763, 0.4226)	0.8021	0.6456
4×10^3	(0.5524, 0.4473)	0.7040	0.5552
5×10^3	(0.5209, 0.4762)	0.6349	0.4986
6×10^3	(0.5038, 0.4951)	0.5825	0.4607
7×10^3	(0.4893, 0.5106)	0.5430	0.4339
8×10^3	(0.4655, 0.5328)	0.5147	0.4139
9×10^3	(0.4312, 0.5574)	0.4932	0.3985
10×10^3	(0.4153, 0.5844)	0.4767	0.3863

Table 3 Optimal solutions and original index of cumulative shock model.

$t(h)$	u^*	$R_c(u^*)$	R_c
1×10^3	(0.7426, 0.2544)	0.9684	0.9501
2×10^3	(0.7135, 0.2863)	0.8486	0.7570
3×10^3	(0.6472, 0.3528)	0.7202	0.6000
4×10^3	(0.6157, 0.3274)	0.5977	0.5035
5×10^3	(0.5842, 0.3619)	0.5274	0.4412
6×10^3	(0.5436, 0.4246)	0.4836	0.3979
7×10^3	(0.5079, 0.4635)	0.4455	0.3657
8×10^3	(0.4318, 0.5226)	0.4194	0.3404
9×10^3	(0.4005, 0.5735)	0.3967	0.3199
10×10^3	(0.3628, 0.6247)	0.3777	0.3026

It can also be easily observed that the optimal reliability indexes of two models are always greater than their original indexes when time t varies from 10^3 to 10×10^3, which indicate that the uncertain random optimization models are effective and practical in reliability engineering.

The data also show that the maintain fund on tribological wear is increasing when time t varies from 10^3 to 10×10^3. An explanation for this phenomenon is that clamping stagnation will generate abrupt wear debris. The particle wear always exists in EHSV's working, while clamping stagnation is an abrupt failure. Soft failure caused by tribological wear gradually turns into the main failure behavior with the increase of system operation time. The wear debris increasingly to the critical value should require more funds as time goes on.

We may conclude that optimal maintenance allocation can improve the system reliability and delay the decay rate of system reliability from the above numerical experimental results. In addition, it may be seen from the optimal solutions that the clamping stagnation is the main failure process in initial time of system operation. And with the gradual accumulation of wear debris, tribological wear becomes the main failure process of the system. This shows that the damage of clamping stagnation and tribological wear on system reliability is variable in different periods. Thus, it is necessary to adjust the proportion of maintenance allocation to maximize the system reliability.

6. CONCLUSION

In this paper, we present a type of shock-degradation dependence of the uncertain random system where random shocks can accelerate wear degradation process by additional degradation shifts. The wear degradation process is driven by an uncertain process, random shocks are driven by a compound Poisson process, degradation shifts are modeled as an uncertain renew reward process. Then, two optimization reliability index models are introduced and algorithms of models are also proposed. Finally, a numerical example about a jet pipe servo valve is given to illustrate the obtained results. Furthermore, the existing research can be extended to an interdependent competing failure model where the degradation processes and shocks can accelerate each other.

CONFLICTS OF INTEREST

The authors declare they have no conflicts of interest.

AUTHORS' CONTRIBUTIONS

All authors have the same contributions to prepare the manuscript.

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