On the Application of the Open Jackson Queuing Network in Hospital

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ABSTRACT

The open Jackson queuing network is the open queuing network consisting of several service stations. On the open Jackson queueing networks, patients arrive from outside the network to get service and then leave the network. The open Jackson theory is used to determine the queue network model and the performance measures of each service station in the network. The data collecting technique used in this research is by interview and observation. The interview obtained a diagram of a queue network at the hospital, and the observation results consist of the number of arrival and departure data at each service station. The open Jackson queue network model is built based on a diagram of the service stages in hospitals. Furthermore, the average arrival rate and the average departure rate are substituted into the linear equation system of the formed queue network model. The linear equation system is solved using Moore-Penrose pseudo-inverse, to obtain the value of the transition probability between stations. The next steps are investigating steady-state conditions and counting performance measures. The analysis result at all service stations shows that the queueing network is a steady-state condition, such that queue performance measures can be calculated. The performance measures are the average of patients and the waiting time for patients in the queueing network.

Keywords: *queuing network, open Jackson, Moore-Penrose pseudo-inverse.*

1. INTRODUCTION

The queue theory is the study of standing, waiting, and being served phenomena [16]. The queue phenomena can occur when the demand for a service exceeds the available capacity [14]. One example of the queue phenomena is the queue of patients at the hospital. Patients can wait for several minutes, hours, days or months to get medical services. This situation causes the patients to lose time, energy, costs, and decrease the level of patient confidence toward the hospital. Therefore, the queue theory is needed to provide a picture or model related to the queue system problem. The hospital gives rules for patients who want to get treatment must through several stages or service stations in a queue network. On the queueing network, there are at least two service stations connected to each other [6]. Service stations in hospitals generally consist of a registration room, an initial examination room, a doctor's room, a pharmacy, etc. The open Jackson queueing network is used to model the queue problem. The queueing network model is a model where patients leaving a station arrive at another station. This situation illustrates that the input from one station is the output of one or more stations in the queueing network [18].

Queue networks are divided into two types, namely open queue networks and closed queue networks. The queue network at the hospital is classified as an open queue network. Queue networks are said to be open when customers can enter the network from outside and customers can leave the network. A breakthrough in queue network analysis was achieved by Jackson's works and he found a product-form solution. The open Jackson network has only one customer class with unlimited capacity and its queue discipline is FCFS (First Comes First Served). Every customer arrival into the queueing network follows the Poisson distribution and the service rate is Exponentially distributed [13]. This research adopted the open Jackson network to model and analyze the existing queue system. Modeling of the queueing network at the hospital was previously done by Owoloko, Edeki and Adeleke [18]. Moreover, measurements of the performance of the multi-server queue system have been carried out by Kandermir and Cavas [14].

The hospital has several service stations with different functions. If patient service is slow, it will affect the level of satisfaction and the level of patient confidence in the hospital as written by Katz-Larson [15] and Sitzia-Wood [19]. According to Katz and Larson [15], service speed is a very important competitive parameter. If the service is faster, then the level of satisfaction increases and has a positive impact on customer psychology. Another research revealed by Sitzia and Wood [19] states that patient evaluation of health services is not only influenced by actual waiting time but also influenced by felted waiting time. Therefore, researchers are interested in making further observations about the queue network system at the hospital. In this research needed some supporting literature such as books relating to the discussion of research topics. The theoretical basis for stochastic processes and Markov chains is referred to in the book written by Chee-Hock and Boon-Hee [10], Lakatos, Szeidl and Telek [17] and Bini, Latouche and Meine [5]. Basic theories regarding the network of queues are referred to from books are written by Bolch, etc. [6], Halchor-Balter [12], Boucherie and van Dijk [7], Cebulka [9], Kleinrock [16], Stewart [20].

The results of this research are hospital queue network models and queue system performance measures. Hospital queue network models can be written in the form of a system of linear equations. A system of linear equations is solved to obtain the transition probability between stations. Some of the literature used in solving the system of linear equations is referenced from books written by Goodfellow, etc. [11], Bener and Kruis [4], Anton and Rorres [2], Andrilli and Hecker [1], Ben-Israel and Greville [3], and Campbell and Meyer [8]. The queue system performance measures consist of the average patient waiting time and the average patient's number both in the queue as well as in the queue system. Furthermore, this research can be used as input for wise decision making for the hospital. Therefore, hospital service providers can serve patients well and without having to wait long. Thus, patients get a sense of comfort and satisfaction with the services provided by the hospital.

2. MATERIALS AND METHODS

2.1. Problem Description

Queue network analysis in this research is generally found in several health facilities such as eye hospitals, clinics, health centers, etc. The queue network at the health facility consists of several service rooms such as the registration room, the initial examination room by nurses, the doctor's room, the administration room, and the pharmacy room. The patient arrival rate at each service stations is notated with λ_i and the patient departure rate is denoted by μ_i $(i = 1,2,3,\dots)$. A diagram of the queue network scheme in several health facilities is given in Figure 1 as follows:

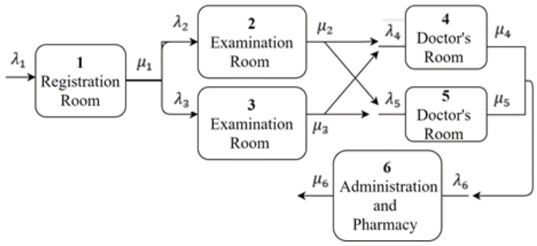

Figure 1. The schematic diagram of queue networks in several health facilities

The queue network in Figure 1 has six service stations, and each station has a single server. Moreover, the queue capacity and the call source is unlimited in number. Thus the queue network model is $(M/M/1):(FCFS/\infty/\infty)$ at each service station. The queue network is assumed there is no reneging, balking, and jockeying.

2.2. Methodology

2.2.1 Data Collecting Technique

The arrival and departure rates data of patients at each station are used to solve the system of linear equations and calculate performance measures in the queueing network. The data in this research are simulation data generated using R software. The arrival and departure rates data follows the Poisson distribution. Data collection was carried out for 3 hours, starting at 8:00 to 11:00 per 5 minutes, so there were 36 times the data collection. The following is data taken at six different service stations:

Table 1. The patient's data in five minutes

Data	Station					
	1	2	3	4	5	6
Number of arrivals	183	85	110	85	78	195
Number of departures	235	139	146	130	86	206

2.2.2 Data Analysis

In this research, several steps are done to analyze the open Jackson queuing network in the hospital, as follows:

(1) Observations were made at each service station to get the data of patient arrival rates and the data of patient departure rates per 5 minutes.

(2) Interviews with relevant parties to dig up information about the queue network system at the hospital.

(3) Forming a model of the open Jackson queuing network in hospital.

(4) Calculating the probability of transition between stations using a system of linear equations from the queue network model.

(5) Calculating the performance measure of a queue network system at a hospital using analytical formulas.

Analysis of the open Jackson queuing network in a hospital can be depicted in the flow chart Figure 2.

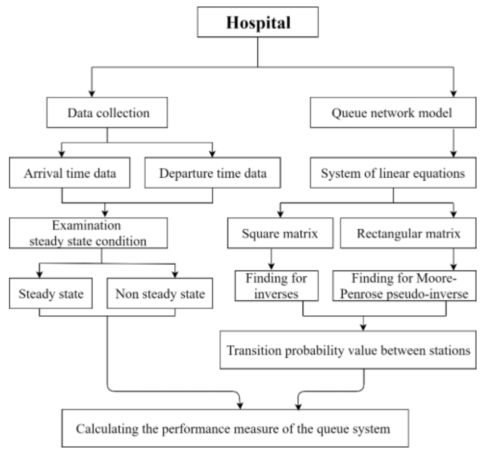

Figure 2. Diagram of the queueing network analysis process

2.3. The open Jackson queueing network

According to Zonderland [21], the open Jackson queuing network is a network consisting of N stations with a single server. The external arrival process at station i, $i = 1,2,3,\dots,N$ is Poisson distributed with rate λ_{0i} ($\lambda \geq 0 \ \forall \ i$). Each queue i has an Exponentially distributed service requirement with mean service time $E[W_i]$. Patients are routed from station i to station j with state independent routing probability p_{ij} ($0 \leq p_{ij} \leq 1$). The parameter p_{i0} denotes the fraction of patients leaving the network at queue i. The total arrival rate λ_j at station j is given by:

$$\lambda_j = \lambda_{0j} + \sum_{i=i}^{N} \lambda_i p_{ij}, \ \ j = 1,2,3,\dots,N.$$

and is composed of the arrivals to queue j from outside and inside the network. A queuing network with these characteristics is called an open Jackson network. In Jackson's theorem, product-form solutions for network types consisting of N stations with a single server are given by:

$$P(n_1,\dots,n_N) = \prod_{i=1}^{N} \rho_i^{n_i}(1-\rho_i), \ \ n_i \geq 0, \ \ i = 1,2,\dots,N,$$

where $\rho = \lambda_i/\mu_i$ and $p(n_i)$ is probability there are n patients in station i.

2.3.1. The analytic formula for a single server

The characteristic and assumption of the queue model in all service stations can be summarized into Kendall notation from $(M/M/1)\colon(FCFS/\infty/\infty)$, the arrival times and departure times are assumed that they are distributed Poisson and Exponential. Moreover, the queue system in all service stations only has one server and unlimited capacity of calling source. Figure 3 is the diagram which shows the model of $(M/M/1)\colon(FCFS/\infty/\infty)$.

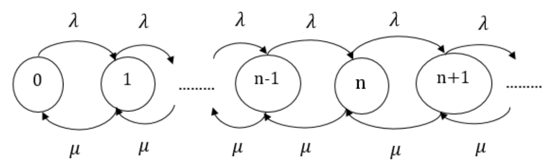

Figure 3. The state transition diagram in $M/M/1$) queue

Based on Figure 3, the system is in statistical equilibrium when the arrival rate is balanced with the departure rate at each state n, with $n = 1,2,3,\dots$. The statistical equilibrium produces the local balanced equation and can be expressed as,

$$\lambda P_0 = \mu P_1 \qquad \Rightarrow P_1 = \frac{\lambda}{\mu} P_0$$

$$(\lambda + \mu)P_1 = \lambda P_0 + \mu P_2$$

$$(\lambda + \mu)P_n = \lambda P_{n-1} + \mu P_{n+1}$$

The probability that there are n customers in the system is denoted by P_n. Therefore, the sum of all probabilities of P_n is one and can be expressed as

$$\sum_{n=0}^{\infty} P_n = 1$$

$$\sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n P_0 = 1$$

$$P_0 = \frac{1}{\sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n}$$

$$P_0 = \frac{1}{\sum_{n=0}^{\infty} (\rho)^n}$$

$$P_0 = 1 - \rho.$$

So,

$$\Rightarrow P_2 = \left(\frac{\lambda}{\mu}\right)^2 P_0$$

$$\Rightarrow P_n = \left(\frac{\lambda}{\mu}\right)^n P_0 \quad \text{untuk } n > 0.$$

$$P_n = (1 - \rho)\rho^n, \qquad \text{for } n > 0.$$

Furthermore, the performance measure of the queue system is the number average of patients and the waiting time average of patients can be expressed as

$$E[L] = \sum_{n=1}^{\infty} n\,\pi_n$$

$$E[L] = \frac{\rho}{1-\rho} \tag{1}$$

and

$$E[W] = E[W_q] + \frac{1}{\mu}$$

$$E[W] = \frac{\rho^2}{(1-\rho)\lambda} + \frac{1}{\mu}. \tag{2}$$

3. RESULT AND DISCUSSION

3.1. Result and research

The results of this research which have been conducted at six service stations are the average arrival rate and the average of departure rate of patients, which can be seen in Table 2.

Tabel 2. The utility level of each station

Station	λ	μ	ρ
1	1.0167	1.3056	0.7787
2	0.4722	0.7722	0.6114
3	0.6111	0.8111	0.7534
4	0.4722	0.7222	0.6538
5	0.4333	0.4778	0.9068
6	1.0833	1.1444	0.9466

Table 2 shows that the queue system at all service stations has reached steady-state conditions (see $\rho < 1$). Therefore, the performance measure of each station can be calculated using analytical models.

3.2. Discussion

3.2.1 The open Jackson queue network model in hospitals

From the diagram in Figure 1, the system of linear equations which captures the probabilities at which a patient enters a particular service station and leaves a service station to any other station in the system or out of the system is given below:

$$\lambda_2 = \mu_1 P_{12} \tag{3}$$
$$\lambda_3 = \mu_1 P_{13} \tag{4}$$
$$\lambda_4 = \mu_2 P_{24} + \mu_3 P_{34} \tag{5}$$
$$\lambda_5 = \mu_2 P_{25} + \mu_3 P_{35} \tag{6}$$
$$\lambda_6 = \mu_4 P_{46} + \mu_5 P_{56} \tag{7}$$
$$\mu_1 = \mu_1 P_{12} + \mu_1 P_{13} \tag{8}$$
$$\mu_2 = \mu_2 P_{24} + \mu_2 P_{25} \tag{9}$$
$$\mu_3 = \mu_3 P_{34} + \mu_3 P_{35} \tag{10}$$
$$\mu_4 = \mu_4 P_{46} \tag{11}$$
$$\mu_5 = \mu_5 P_{56} \tag{12}$$
$$\mu_6 = \mu_6 P_{60}. \tag{13}$$

Where $P_{12}, P_{13}, P_{24}, P_{25}, P_{34}, P_{35}, P_{46}, P_{46}, P_{56}, P_{60}$ are to be determined. The linear equations $(1-11)$ could be expressed as:

$$\lambda_2 = \mu_1 P_{12} + 0P_{13} + 0P_{24} + 0P_{25} + 0P_{3}4 + 0P_{35} + 0P_{46} + 0P_{56} + 0P_{60} \quad (14)$$
$$\lambda_3 = 0P_{12} + \mu_1 P_{13} + 0P_{24} + 0P_{25} + 0P_{3}4 + 0P_{35} + 0P_{46} + 0P_{56} + 0P_{60} \quad (15)$$
$$\lambda_4 = 0P_{12} + 0P_{13} + \mu_2 P_{24} + 0P_{25} + \mu_3 P_{3}4 + 0P_{35} + 0P_{46} + 0P_{56} + 0P_{60} \quad (16)$$
$$\lambda_5 = 0P_{12} + 0P_{13} + 0P_{24} + \mu_2 P_{25} + 0P_{3}4 + \mu_3 P_{35} + 0P_{46} + 0P_{56} + 0P_{60} \quad (17)$$
$$\lambda_6 = 0P_{12} + 0P_{13} + 0P_{24} + 0P_{25} + 0P_{3}4 + \mu_3 P_{35} + \mu_4 P_{46} + \mu_5 P_{56} + 0P_{60} \quad (18)$$
$$\mu_1 = \mu_1 P_{12} + \mu_1 P_{13} + 0P_{24} + 0P_{25} + 0P_{3}4 + 0P_{35} + 0P_{46} + 0P_{56} + 0P_{60} \quad (19)$$
$$\mu_2 = 0P_{12} + 0P_{13} + \mu_2 P_{24} + \mu_2 P_{25} + 0P_{3}4 + 0P_{35} + 0P_{46} + 0P_{56} + 0P_{60} \quad (20)$$
$$\mu_3 = 0P_{12} + 0P_{13} + 0P_{24} + 0P_{25} + \mu_3 P_{3}4 + \mu_3 P_{35} + 0P_{46} + 0P_{56} + 0P_{60} \quad (21)$$
$$\mu_4 = 0P_{12} + 0P_{13} + 0P_{24} + 0P_{25} + 0P_{3}4 + 0P_{35} + \mu_4 P_{46} + 0P_{56} + 0P_{60} \quad (22)$$
$$\mu_5 = 0P_{12} + 0P_{13} + 0P_{24} + 0P_{25} + 0P_{3}4 + 0P_{35} + 0P_{46} + \mu_5 P_{56} + 0P_{60} \quad (23)$$
$$\mu_6 = 0P_{12} + 0P_{13} + 0P_{24} + 0P_{25} + 0P_{3}4 + 0P_{35} + 0P_{46} + 0P_{56} + \mu_6 P_{60}. \quad (24)$$

The linear equations (14 - 24) can be represented in the following matrix:

$$\begin{bmatrix} \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 & \mu_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_2 & 0 & \mu_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu_4 & \mu_5 & 0 \\ \mu_1 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_2 & \mu_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_3 & \mu_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_6 \end{bmatrix} \begin{bmatrix} P_{12} \\ P_{13} \\ P_{24} \\ P_{25} \\ P_{34} \\ P_{35} \\ P_{46} \\ P_{56} \\ P_{60} \end{bmatrix} = \begin{bmatrix} \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \\ \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \\ \mu_6 \end{bmatrix}. \quad (25)$$

The discussion of the analysis of the open Jackson queue network consists of models that form systems of linear equations. The system of linear equations can be represented in the form of a matrix, and it is solved by finding the inverse of a rectangular matrix.

$$\begin{bmatrix} P_{12} \\ P_{13} \\ P_{24} \\ P_{25} \\ P_{34} \\ P_{35} \\ P_{46} \\ P_{56} \\ P_{60} \end{bmatrix} = \begin{bmatrix} 1.3056 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.3056 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.7722 & 0 & 0.8111 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.7722 & 0 & 0.8111 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.7222 & 0.4778 & 0 \\ 1.3056 & 1.3056 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.7722 & 0.7722 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8111 & 0.8111 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.7222 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4778 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.1444 \end{bmatrix}^{-1} \begin{bmatrix} 0.4722 \\ 0.6111 \\ 0.4722 \\ 0.4333 \\ 1.0833 \\ 1.3056 \\ 0.7722 \\ 0.8111 \\ 0.7222 \\ 0.4778 \\ 1.1444 \end{bmatrix}$$

The matrix above is singular, and hence its determinant is zero. Therefore, to solve the above matrix, we shall introduce a pseudo-inverse matrix called the Moore–Penrose pseudo-inverse. The Moore-Penrose pseudo inverse can be searched using R software with the following steps:

(1) Finding eigenvalues of $A^T A$ square matrix (A is a rectangular matrix whose inverse value will be found).

(2) Extracting the singular values and frame the $\sum$ matrix which is formed by framing a diagonal matrix with diagonal elements equal to the square root of each of these eigenvalues.

$$\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{\lambda_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \sqrt{\lambda_n} \end{bmatrix}$$

(3) Framing V as the orthogonal matrix with columns as orthonormal vectors independent spacing the domain of $A^T A$.

$$V = [V_1 \quad V_2 \ldots\ldots V_n]$$

where

$$V_n = \frac{1}{|v_n|}, n = 1,2,3,\ldots \text{ dan } |v_n|$$
$$= \text{the length of each eigenvector.}$$

(4) Framing orthogonal matrix U as the matrix with columns equal to the orthonormal eigenvectors of $A^T A$.

$$U = [U_1 \quad U_2 \ldots\ldots U_n]$$

where

$$U_n = \frac{1}{\sqrt{\lambda_n}} A\, V_n, \qquad n = 1,2,3,\ldots$$

(5) Framing the matrix $\sum^+$ as the matrix $\sum$ with its non-zero singular elements are replaced by their corresponding reciprocals and then transposed.

$$\Sigma^+ = \begin{bmatrix} \frac{1}{\sqrt{\lambda_1}} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{\sqrt{\lambda_2}} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{\sqrt{\lambda_3}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{\sqrt{\lambda_n}} \end{bmatrix}^T$$

(6) Finally, calculate the Moore-Penrose inverse by the relation $A^+ = V \sum^+ U^T$.

Based on the calculation of Moore-Penrose pseudo-inverse using the R program then obtained,

$$\begin{bmatrix} P_{12} \\ P_{13} \\ P_{24} \\ P_{25} \\ P_{34} \\ P_{35} \\ P_{46} \\ P_{56} \\ P_{60} \end{bmatrix} = \begin{bmatrix} 0.5106 & -0.2553 & 0 & 0 & 0 & 0.2553 & 0 & 0 & 0 & 0 & 0 \\ -0.2553 & 0.5106 & 0 & 0 & 0 & 0.2553 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4697 & -0.1459 & 0 & 0 & 0.4856 & -0.1619 & 0 & 0 & 0 \\ 0 & 0 & -0.1459 & 0.4697 & 0 & 0 & 0.4856 & -0.1619 & 0 & 0 & 0 \\ 0 & 0 & 0.4775 & -0.1692 & 0 & 0 & -0.1541 & 0.4623 & 0 & 0 & 0 \\ 0 & 0 & -0.1692 & 0.4775 & 0 & 0 & -0.1541 & 0.4623 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.4615 & 0 & 0 & 0 & 0.9231 & -0.4615 & 0 \\ 0 & 0 & 0 & 0 & 0.6976 & 0 & 0 & 0 & -0.6976 & 1.3953 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.8738 \end{bmatrix} \begin{bmatrix} 0.4722 \\ 0.6111 \\ 0.4722 \\ 0.4333 \\ 1.0833 \\ 1.3056 \\ 0.7722 \\ 0.8111 \\ 0.7222 \\ 0.4778 \\ 1.1444 \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} P_{12} \\ P_{13} \\ P_{24} \\ P_{25} \\ P_{34} \\ P_{35} \\ P_{46} \\ P_{56} \\ P_{60} \end{bmatrix} = \begin{bmatrix} 0.4184283 \\ 0.5248162 \\ 0.4022565 \\ 0.3783056 \\ 0.4081218 \\ 0.3829645 \\ 0.9461371 \\ 0.9185853 \\ 1.0000004 \end{bmatrix}. \tag{26}$$

The value in Equation (26) is the value of the transition probability between stations. The total value of arrival and departure probabilities from each station is equal to one. Therefore it needs to normalize the parameters. Normalization of parameters is done by taking their absolute values and rescaling them such that their sums at each service station are 1. Thus, the probability value between service stations is obtained as below

$$\begin{bmatrix} P_{12} \\ P_{13} \\ P_{24} \\ P_{25} \\ P_{34} \\ P_{35} \\ P_{46} \\ P_{56} \\ P_{60} \end{bmatrix} = \begin{bmatrix} 0.4436053 \\ 0.5563947 \\ 0.5153421 \\ 0.4846579 \\ 0.5159005 \\ 0.4840995 \\ 0.9461371 \\ 0.9185853 \\ 1.0000004 \end{bmatrix}. \tag{27}$$

Based on Equation (27), it can be seen that the probability of transition from station 1 to station 2, from station 1 to station 3 are 0.4436053 and 0.5563947, respectively. This value shows that there is a high tendency of patients leaving service station 1 to join the queue at station 3 than at station 2. Furthermore, the same meaning applies to other transition probability values.

3.2.1. The performance measure of queue network

The performance measures of the queueing network consist of the average number of patients and the waiting time average of patients in the systems. Each service station has a single server; therefore it can be calculated using the $(M/M/1):(FCFS/\infty/\infty)$ analytic formula. The average number of patients in the i-th service station system can be calculated using the formula in Equation (1), as follow

$$E[L_1] = \frac{0.7787}{1-0.7787} = 3.5188 \approx 4 \text{ patients}$$
$$E[L_2] = \frac{0.6114}{1-0.6114} = 1.5733 \approx 2 \text{ patients}$$
$$E[L_3] = \frac{0.7534}{1-0.7534} = 3.0552 \approx 3 \text{ patients}$$
$$E[L_4] = \frac{0.6538}{1-0.6538} = 1.8885 \approx 2 \text{ patients}$$
$$E[L_5] = \frac{0.9068}{1-0.9068} = 9.7296 \approx 10 \text{ patients}$$
$$E[L_6] = \frac{0.9466}{1-0.9466} = 17.7266 \approx 18 \text{ patients.}$$

The above calculation, show that station 6 is the station with the most average patients to be served, which in total is 18 patients. On the other hand, the least average patients are in stations 2 and 4, which in total 2 patients only. Therefore, the average number of patients in the queuing network is as follows,

$$E[L] = E[L_1] + E[L_2] + E[L_3] + E[L_4] + E[L_5] + E[L_6]$$
$$E[L] = 4 + 2 + 3 + 2 + 10 + 18$$
$$E[L] = 39 \text{ patients.}$$

Furthermore, the average waiting time for patients at the i-th station can be calculated using the formula in Equation (2),

$$E[W_1] = 2.695 + \frac{1}{1.3056} = 3.4609 \text{ minutes}$$
$$E[W_2] = 2.037 + \frac{1}{0.7722} = 3.332 \text{ minutes}$$
$$E[W_3] = 3.7666 + \frac{1}{0.8111} = 4.9995 \text{ minutes}$$
$$E[W_4] = 2.6148 + \frac{1}{0.7222} = 3.9994 \text{ minutes}$$
$$E[W_5] = 20.3619 + \frac{1}{0.4778} = 22.4548 \text{ minutes}$$
$$E[W_6] = 15.4897 + \frac{1}{1.1444} = 16.3635 \text{ minutes.}$$

The above calculation informs that, the patient spends the longest time at station 5 with a duration of 22,4548 minutes. On the other hand, station 2 is the station with the fastest service time with a duration of 3,332 minutes. Therefore, the average waiting time for patients in the queuing network is as follows,

$$E[W] = E[W_1] + E[W_2] + E[W_3] + E[W_4] + E[W_5] + E[W_6]$$
$$E[W] = 3.4609 + 3.332 + 4.9995 + 3.9994 + 22.4548 + 16.3635$$
$$E[W] = 54.6101 \text{ minutes.}$$

4. CONCLUSION

Based on the results of this research, the queueing network at six service stations, it can be concluded that each service station follows $(M/M/1):(FCFS/\infty/\infty)$ model. The study has obtained the value of the transition probability between stations and the performance measures of the queueing network. The transition probability from station 1 to station 2 and 3, from station 2 to station 4 and 5, from station 3 to station 4 and 5, from station 4 and 5 to station 6 are 0.4436053 and 0.5563947, 0.5153421 and 0.4846579, 0.5159005 and 0.4840995, 0.9461371 and 0.9185853 respectively. The transition probability of patients out of the queueing network from station 6 is 1.0000004. Furthermore, the average number of patients in the queueing network at all service stations is 39 patients, while the average waiting time of patients in the queueing network is 54,6101 minutes.

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