

Repetitive Pattern of L(2,1)-Labelling on Sierpinski Graphs

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ABSTRACT

An L(2,1)-labelling of a graph G is a function f which assigns labels from $\{0, 1, \dots, \lambda\}$ to the vertices of G such that vertices at distance two get different labels and adjacent vertices get labels that are at least two apart. Sierpiński graphs $S(n, k)$ generalized the Tower of Hanoi graphs that constructed by copying complete graphs recursively. By Chang-Kuo algorithm, we will show L(2,1)-labelling of Sierpinski graphs and repetitive pattern on it.

Keywords: *L(2,1)-labeling, Sierpinski graphs, repetitive pattern.*

1. INTRODUCTION

An $L(2,1)$ -labelling (also called $L(2,1)$ -labelling in [1]) of a graph G is defined as a function f which maps $f: V(G) \rightarrow \{0, 1, 2, \dots, \lambda\}$ such that

- a) If $d(u, v) = 1$ or adjacent, $|f(u) - f(v)| \geq 2$,
- b) If $d(u, v) = 2$, $|f(u) - f(v)| \geq 1$,
- c) No restriction is placed for u and v if $d(u, v) \geq 3$. It means we can assign the same label for u and v .

The difference between the largest label and the smallest label assigned by an $L(2,1)$ -labelling is denoted by $span(f)$. The λ -number $\lambda(G)$ of G is the minimum value of $span(f)$ such that G admits an $L(2,1)$ -labelling. Because the labelling number starts at 0, so in [1], it is written as

$$span(f) = max\{f(u) | u \in V(G)\}.$$

And the λ -number of G is

$$\lambda(G) = min(span(f))$$

These concepts came from the problem of assigning frequencies to radio transmitters [2] and have been formulated as the $L(2,1)$ -labelling problem by [3]. The idea of channel assignment problem is to avoid the interference that caused by the distance of the radio transmitters, [4]. There are several articles that discuss about it, like [5, 6, 7, 8, 9] that label certain graphs, and [10] made condition that all the label must be used,

that called as no-hole labelling, then [4] proposed an algorithm for $L(2,1)$ -labelling, and [11] named it Chang-Kuo algorithm.

Sierpinski graphs $S(n, k)$ were introduced in [12] that defined on the Cartesian product of vertex set $\{1, 2, \dots, k\}^n$, and two different vertices $u = (i_1, i_2, \dots, i_n)$ and $v = (j_1, j_2, \dots, j_n)$ are adjacent if $u \sim v$, where

$$u \sim v \Leftrightarrow \exists h \in \{1, 2, \dots, n\}$$

Such that

1. $i_t = j_t$ for $t = 1, \dots, h-1$;
2. $i_h \neq j_h$; and
3. $i_t = j_h$ and $j_t = i_h$ for $h+1, \dots, n$.

If $h = n$, in which case the condition (iii) is formally being empty. For simplicity, in the rest of this paper we will write $i_1 i_2 \dots i_n$ instead of (i_1, i_2, \dots, i_n) . The Sierpinski graph $S(3,3)$ and $S(2,5)$ together with the corresponding vertex labelling are shown in Figure 1.

The vertices $(1 \dots 1), (2 \dots 2), \dots, (k \dots k)$ are called the *extreme vertices* of $S(n, k)$; the other vertices will be called inner vertices of $S(n, k)$. The extreme vertices of $S(n, k)$ are of degree $k-1$ while the degree of the inner vertices is k . Note also that in $S(n, k)$ there are k extreme vertices and $|V(S(n, k))| = k^n$.

Gravier et al [13] show the procedure for constructing Sierpinski graphs $S(n, k)$ as follows. Sierpinski graph can be constructed recursively started by $S(1, k)$ which is isomorphic to complete graph K_k . To construct $S(2, k)$, make k copies of

the K_k , that will be connected to each other with an edge set in one-to-one correspondence with the edges of a K_k . By repeating this procedure, $S(n, k)$ can be constructed recursively by connecting k copies of $S(n - 1, k)$ with a set of $\frac{k(k-1)}{2}$ edges.

Since the Sierpinski graph is constructed from complete graphs (K_k is subgraph of $S_{n,k}$), the following lemma shows the labelling number of graphs G and H , for H is subgraph of G .

Lemma 1 [14] If H is subgraph of G , then $\lambda(H) \leq \lambda(G)$.

Proof: Let $\lambda(G) = k$, with corresponding labelling $f: V(G) \rightarrow \{0, 1, \dots, k\}$. Then $g: V(H) \rightarrow \{0, 1, \dots, k\}$ defined by $g(v) = f(v)$ for all $v \in V(H)$, is a labelling of H that uses no label greater than k . Thus $\lambda(H) \leq k$. The idea is we can use the same labels we use on G to label the corresponding vertices of H . ■

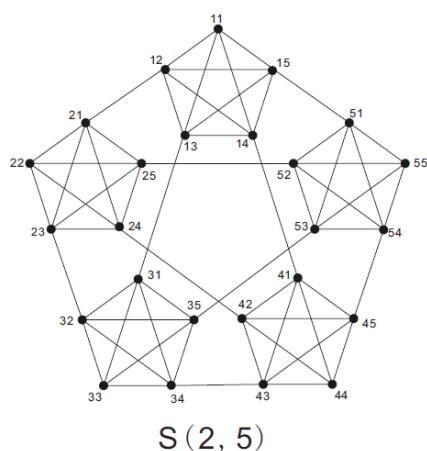
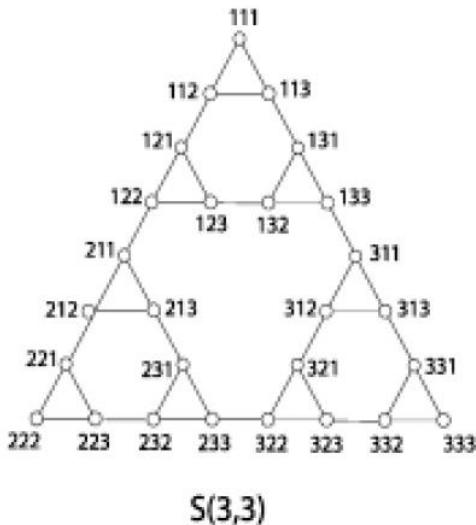


Figure 1 Sierpinski graphs $S(3,3)$ and $S(2,5)$

2. A LABELING ALGORITHM

Definition 2 [11] A subset X of $V(G)$ is called an i -stable set (or i -independent set) if the distance between any two vertices in X is greater than i . A maximal i -stable subset X of a set Y of vertices is an i -stable subset of Y such that X is not a proper subset of any other i -stable subset of Y .

Algorithm 3 ([11], Chang-Kuo Algorithm)

Input: A graph $G = (V, E)$

Output: The maximum label λ .

Idea: In each step, find a maximal 2-stable set from the unlabelled vertices that are at distance at least 2 from the vertices labeled in the previous step. Label all vertices in the 2-stable set with the index i of the current step. The index i starts from 0 and increases by 1 at each step. The maximum label k is the final value of i .

Initialization: Set $X_{-1} = \emptyset, V = V(G), i = 0$

1. Determine Y_i and X_i .

- $Y_i = \{x \in V | x \text{ is unlabelled and } d(x, y) \geq 2, \forall y \in X_{i-1}\}$
- X_i a maximal 2-stable subset of Y_i .
- If $Y_i = \emptyset$ then set $X_i = \emptyset$

2. Label these vertices in X_i (if there is any) by i .

3. $V \leftarrow V - X_i$

4. $V \neq \emptyset$ then $i \leftarrow i + 1$; go to Step 1.

5. Record the current i until all vertices are labeled. Stop.

Consider that all vertices are labeled on k -th iteration, that computed by algorithm 3 is an upper bound on $\lambda(G)$. We would like to find a bound for k in term of maximum degree $\Delta(G)$ of G , analogous to existing bounds for the chromatic number $\chi(G)$ in the term of $\Delta(G)$. Let x be a vertex with the largest label k assigned by algorithm 3. Denote

- $I_1 = \{i: 0 \leq i \leq k - 1 \text{ and } d(x, y) = 1 \text{ for some } y \in X_i\}$.
- $I_2 = \{i: 0 \leq i \leq k - 1 \text{ and } d(x, y) \leq 2 \text{ for some } y \in X_i\}$.

- $I_3 = \{i : 0 \leq i \leq k-1 \text{ and } d(x, y) \geq 3 \text{ for some } y \in X_i\}$

It is clear that $|I_2| + |I_3| = k$. For any $i \in I_3$, $x \notin Y_i$ since $X_i \cup \{x\}$ would be a 2-stable subset of Y_i , which contradicts the choice of X_i . That is, $d(x, y) = 1$ for some vertex y in X_{i-1} ; i.e., $i-1 \in I_1$, then $|I_3| \leq |I_1|$. Hence $k = |I_2| + |I_3| \leq |I_2| + |I_1|$.

L(2,1)-labelings of Related Subgraphs

In this section, we show $L(2,1)$ -labelling of some related subgraphs of Sierpinski graphs.

Lemma 4 [6] *For the complete graph K_k , $\lambda(K_k) = 2k - 2$.*

Proof: Given K_k with vertices v_1, v_2, \dots, v_k . In K_k , v_i is adjacent to v_j for $i \neq j$. An $L(2,1)$ -labelling requires that adjacent vertices be assigned with the label differ at least two, so the label we can assign is $\{0, 2, 4, \dots, 2k-2\}$. Thus, $\lambda(K_k) = 2k - 2$. ■

Lemma 5 [14] *Let P_r be a path on r vertices. Then $\lambda(P_3) = \lambda(P_4) = 3$, and for $r \geq 5$, $\lambda(P_r) = 4$.*

Proof: For P_3 , we can label the leftmost vertex 0, the middle vertex 3, and the rightmost 1, as shown in Figure 2. So, $\lambda(P_3) \leq 3$. We claim, we can't label P_3 0, 1, 2. The label 1 could not be used anywhere or else it would have to be adjacent to 0, 1, or 2, all which violates the adjacency rule. This leaves us with labels (0 and 2) that must be assigned to three vertices. By the pigeon-hole principle, two of these vertices must receive the same label, which necessarily violates the condition.

Next, for P_4 . Since P_3 is subgraph of P_4 , by lemma 1 we have $\lambda(P_3) \leq \lambda(P_4) = 3$. Figure 2 shows we can label P_4 shows we can label P_4 with no label greater than 3. Thus, $\lambda(P_4) \leq 3$.

For P_5 , we add one vertex to P_4 . We can assign the rightmost vertex with no label greater than 4. So, $\lambda(P_5) \leq 4$. We claim we can't label P_5 with just the numbers 0, 1, 2 and 3. The labels 1 and 2 cannot be assigned to any non-endpoint vertex without violating either the adjacency rule or the distance two rule. To

see this, suppose one of the non-endpoint vertices of P_5 were labeled 1. Then only the label 3 can be assigned to its neighbors without violating the adjacency rule. However, if both its neighbors receive the label 3, the distance two is violated. So this leaves us with two labels (0 and 3) that must be assigned to the three non-endpoint vertices. Then by the pigeon-hole principle, two of these vertices must receive the same label, which necessarily violates the condition. So, $\lambda(P_5) = 4$.

Now, for $P_r, r \geq 5$. Since P_5 is a subgraph of P_n , $\lambda(P_r) \geq \lambda(P_5) = 4$. Notice we can cyclically repeat the label in P_5 (2, 0, 3, 1, 4, 2, 0, ...) and still get proper labelling for any P_r . Thus, $\lambda(P_r) \leq 4$. ■

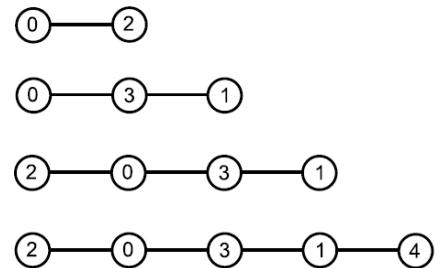


Figure 2 $L(2, 1)$ -labelling of P_2, P_3, P_4 and P_5

Lemma 6 [14] *Let C_r be a cycle of length r , then $\lambda(C_r) = 4$, for $n \geq 3$.*

Proof: By lemma 4, we have $\lambda(C_3) = 2(3) - 2 = 4$. Now consider C_4 . Figure 3 shows we can label C_4 with no label greater than 4. So, $\lambda(C_4) \leq 4$. Since C_4 is 2-regular, we cannot use labels 1 and 2 without violating the rules. This leaves us with two labels (0 and 3) that must be assigned to the four vertices. By pigeon-hole principle, two of these vertices must receive the same label, which necessarily violates the condition since any pair of vertices in C_4 are at most distance two apart. So, $\lambda(C_4) = 4$.

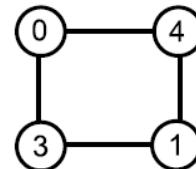


Figure 3 An $L(2, 1)$ -labelling of C_4

Now, consider $C_r, r \geq 5$. By lemma 1, $\lambda(C_r) \geq \lambda(P_5) = 4$. Now we want to show $\lambda(C_r) \leq 4$ by defining a labelling on C_r

using no label greater than 4. We have three cases. First, suppose $r \equiv 0 \pmod{3}$. Then we can label our vertices (starting at one vertex and proceeding clockwise) 0,2,4,0,2,4, ... Next, suppose $r \equiv 1 \pmod{3}$. We can label our vertices 0,2,4,0,2,4, ..., 0,2,4,0,3,1,4. If $r \equiv 2 \pmod{3}$, then we can label our vertices 0,2,4,0,2,4, ..., 0,2,4,1,3. This is illustrated in Figure 3. In each case, we repeat the labelling 0,2,4 as many times as necessary. This completes the proof. ■

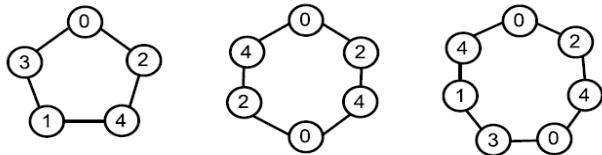


Figure 4 $L(2, 1)$ -labellings of C_5 , C_6 and C_7

Generally, we can label P_r with the label 2,0,3,1,4,2,0, ... and repeat them cyclically. If we joint the rightmost vertex and the leftmost vertex of P_r , we obtain C_r , whose label is repetitive. Since P_r and C_r have repetitive pattern of $L(2,1)$ -labelling, it allows that Sierpinski graphs have too. Next section, we will show repetitive pattern of $L(2,1)$ -labelling of Sierpinski graphs.

Proof: Sierpinski graph $S(n, 2)$ is isomorphic to path graph of length 2^n . For $n = 2$, path graph P_4 is the smallest Sierpinski graph. Lemma 5 has proven that $\lambda(S(2,2)) = 3$.

Now consider for $n \geq 3$. Sierpinski graph $S(3,2)$ is a path of length 8. If we joint the extreme vertices of $S(3,2)$ which is isomorphic to P_8 , we obtain C_8 . For $n \geq 3$, we have proven it in lemma 5 and lemma 6 that $\lambda(S(n, 2)) = 4$. ■

Lemma 4 and 5 have shown that $L(2,1)$ -labelling of path graph P_r , $r \geq 4$, which is isomorphic to Sierpinski graph $S(n, 2)$, $n \geq 2$ in lemma 7 are repeated cyclically. Next, we will show the repetitive pattern of Sierpinski graph $S(n, 3)$. To give $L(2,1)$ -labelling of Sierpinski graph $S(n, 3)$, we start from $S(2,3)$, we will use Chang-Kuo Algorithm to label it. First, we label $S(2,3)$, that shown on Figure 5.

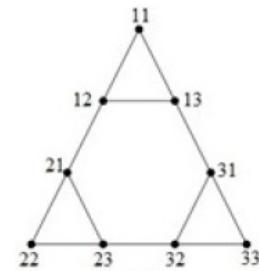


Figure 5 Sierpinski Graph $S(2,3)$

Input: Sierpinski Graph $S(2,3)$

Output: $\lambda(S(2,3))$

Initialization: $X_{-1} = \emptyset; V = \{1,2,3\}^2; i = 0$

Iteration:

1. Determine Y_0 dan X_0
 - 1) $Y_0 = \{11, 12, 13, 21, 22, 23, 31, 32, 33\}$
 - 2) $X_0 = \{12, 32\}$
2. Label 12 and 32 with 0.
3. $V = \{11, 13, 21, 22, 23, 31, 33\}$
4. Determine Y_1 dan X_1
 - 1) $Y_1 = \{22\}$
 - 2) $X_1 = \{22\}$
5. Label 22 with 1.

Repetitive Pattern of $L(2,1)$ -labelling on Sierpinski graphs

Lemma 7 [6, 7] For $n \geq 2$, and any $k \geq 3$, $\lambda(S(n, k)) = 2k$.

In this section, it is shown that Sierpinski graphs have repetitive pattern without violating lemma 7. To show the repetitive pattern, we modify as if all extreme vertices are adjacent, so we obtain one complete graph again. So, we have $k + 1$ complete graph. Since the extreme vertices are adjacent, we will consider $S(2, k)$ is the “smallest” Sierpinski graph, and $|f(u) - f(v)| \geq 2$, for u and v are extreme vertices.

Lemma 8 For $n \geq 2$, $\lambda(S(n, 2)) \leq 4$.

6. $V = \{11, 13, 21, 23, 31, 33\}$

7. Determine Y_2 dan X_2

1) $Y_2 = \{11, 13, 31, 33\}$

2) $X_2 = \{31\}$

8. Label 31 with 2

9. $V = \{11, 13, 21, 23, 33\}$

10. Determine Y_3 and X_3

1. $Y_3 = \{11, 21, 23\}$

2. $X_3 = \{11\}$

11. Label 11 with 3

12. $V = \{13, 21, 23, 33\}$

13. Determine Y_4 and X_4

a. $Y_4 = \{21, 23, 33\}$

b. $X_4 = \{21\}$

14. Label 21 with 4

15. $V = \{13, 23, 33\}$

16. Determine Y_5 and X_5

a. $Y_5 = \{13, 33\}$

b. $X_5 = \{33\}$

17. Label 33 with 5

18. $V = \{13, 23\}$

19. Determine Y_6 and X_6

a. $Y_6 = \{13, 23\}$

b. $X_6 = \{13, 23\}$

20. Label 13 and 23 with 6

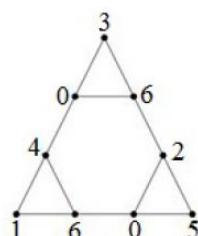


Figure 6 $L(2, 1)$ -labelling of Sierpinski graph $S(2, 3)$

From the iteration above, we get $\lambda(S(2,3)) = 6$.

Lemma 9 For $n \geq 3$, there exist repetitive pattern of $L(2,1)$ -labelling on Sierpinski graph $S(n, 3)$.

Proof: In section 1, stated that we can obtain $S(n, 3)$ by copying k -times $S(n-1, 3)$, such that the extreme vertices are adjacent. To simplify this, we can modify $S(2, 3)$ as shown on Figure 7.

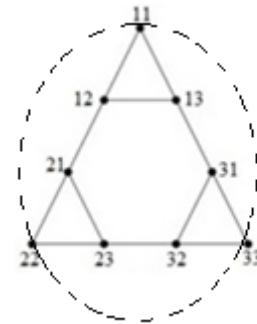


Figure 7 Modified Sierpinski graph $S(2, 3)$

Let u and v be extreme vertex of $S(2, 3)$, and $u \neq v$. Since we modify the graph such that u and v are adjacent, then

$$|f(u) - f(v)| \geq 2.$$

and let w be inner vertex of $S(2, 3)$, and $d(w, u) = 1$ and $d(w, v) = 2$, then

$$|f(w) - f(u)| \geq 2; |f(w) - f(v)| \geq 1.$$

Since u is extreme vertex, always found that $f(u) \neq f(w)$. But, there exist another inner vertex x such that $d(w, x) = 3$, like 12 and 32, 13 and 23, 21 and 31, so the same label may be assigned. On Figure 6, shown that $f(11) = 3, f(22) = 1, f(55) = 5$. Now, we assign the label to inner vertices.

- a. Since $d(12, 11) = 1$ and $d(12, 22) = 2$, then $f(12) = 0$ or $f(12) \geq 5$. Since 0 to is the minimum label we can assign to 12, then $f(12) = 0$.
- b. Since $d(13, 11) = d(13, 12) = 1$ and $d(13, 33) = 2$, then $f(13) \geq 6$. Since 6 is the minimum label we can assign to 13, then $f(13) = 6$.

- c. Since $d(21,12) = d(21,22) = 1$ and $d(21,11) = d(21,13) = 2$, then $f(21) \geq 4$. Since 4 is the minimum label we can assign to 21, then $f(21) = 4$.
- d. Since $d(23,21) = d(23,22) = 1$ and $d(23,12) = d(23,33) = 2$, then $f(21) \geq 6$. Since 6 is the minimum label we can assign to 23, then $f(23) = 6$.
- e. Since $d(32,23) = d(32,33) = 1$ and $d(32,13) = d(32,22) = 2$, then $f(21) = 0, f(21) = 2$, or $f(21) \geq 8$. Since 0 is the minimum label we can assign to 32, then $f(32) = 0$.
- f. Since $d(31,13) = d(31,32) = d(31,33) = 1$ and $d(31,11) = d(31,23) = d(31,12) = 2$, then $f(31) = 2$ or $f(31) \geq 8$. Since 2 is the minimum label we can assign to 31, then $f(32) = 2$.

Since $S(2,3)$ is subgraph of $S(n,3), n \geq 3$, (as shown on Figure 1), and label which assign to extreme vertices differ by 2, and there exist label of inner vertices which differ to another extreme vertices, then $S(3,3)$ has repetitive pattern. For $n \geq 3$, we can repeat the same pattern with $S(2,3)$ to its copies, as shown on Figure 8, since $S(n,3)$ is obtained by copying $S(2,3)$ as 3^{n-2} times. ■

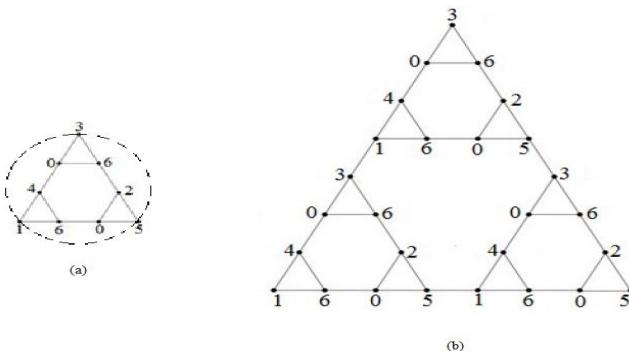


Figure 8 (a) $L(2, 1)$ -labelling of modified $S(2, 3)$, (b) $L(2, 1)$ -labelling of Sierpinski graph $S(3, 3)$

We can apply the same way to assign $L(2,1)$ -labelling to Sierpinski graph $S(2,4)$ and we obtain as shown on Figure 9.

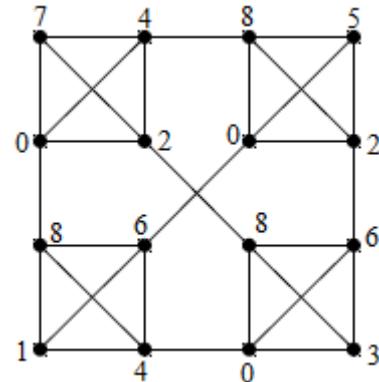


Figure 9 $L(2, 1)$ -labelling of Sierpinski graph $S(2, 4)$

3. REMARKS

Conjecture 10 *Sierpinski graph $S(n, k)$ has repetitive pattern of $L(2,1)$ -labelling, for $n, k \geq 2$.*

4. CONCLUSION

We have shown that Sierpinski graphs $S(n, k)$ have repetitive pattern of $L(2,1)$ -labelling for $n \geq 2$, and $k = 2,3,4$. If we ‘modify’ Sierpinski graph as if the extreme vertices are adjacent each other, it is guaranteed that $S(n, k)$ has repetitive pattern for higher n and k , since $S(n, k)$ is self-similar, but lemma 7 may be violated.

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