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Research Article

Information Structures in an Ordered Information System Under Granular Computing View and Their Optimal Selection Based on Uncertainty Measures

Yini Wang[1,2], Sichun Wang[1,*], Hongxiang Tang[1]

[1] *Guangxi Key Laboratory of Cross-Border E-commerce Intelligent Information Processing, Guangxi University of Finance and Economics, Guangxi 530003, P.R. China*
[2] *Panyapiwat Institute of Management, Bangkok 10310, Thailand*

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ABSTRACT

Information structures (*i*-structures) in an ordered information system (OIS) are mathematical structures of the information granules (*i*-granules) granulated from the data set of this OIS. This article investigates *i*-structures in an OIS with granular computing (GrC) view, i.e., *i*-structures in an OIS are seen as granular structures. Dependence and independence between *i*-structures are first depicted in the same OIS. Then, information distance (*i*-distance) between *i*-structures in the same OIS are proposed, and information entropy in an OIS can be expressed by *i*-distance between *i*-structures in this OIS is proved. Next, properties of *i*-structures in an OIS are shown. Moreover, group, lattice and map characters of *i*-structures in an OIS are received, and some invariant characters of *i*-structures in an OIS under homomorphisms are obtained. Finally, the optimal selection of *i*-structures in an OIS based on the proposed measures are studied. These results will contribute to build a framework of GrC in an OIS.

1. INTRODUCTION

Granular computing (GrC), presented by Zadeh [1,2], is a world view and methodology for viewing the objective world. Its main idea is to use granular thinking to solve complex problems, and transform them into the theory, method, technique and tool of the process of solving several relatively simple problems by abstracting and dividing complex problems.

An information granule (*i*-granule) exists in many fields, and its manifestation is also different for different fields. *i*-granules are elements that are similar, or indistinguishable, or have some function that binds them together by indistinguishability or similarity [3,4]. Granulation of objects is to decompose a whole into small parts and then study the decomposed parts. Each divided part is an *i*-granule. A granular structure is the family of *i*-granules [5,6].

An information system (IS) based on rough set theory was presented by Pawlak [7]. In GrC in an IS, the investigation of information structure (*i*-structure) is a meaningful subject. In an IS, an attribute subset determines an equivalence relation which divides the object set into equivalence classes that are disjoint and may be said to be *i*-granules of the IS [8,3,9]. All these *i*-granules constitute a vector that is viewed as *i*-structure induced by this attribute subset in the IS. On the light of this idea, Zhang *et al.* [10] investigated *i*-structures in fully fuzzy ISs, Liang *et al.* [4] discussed *i*-structures in ISs and Li *et al.* [11,12] considered *i*-structures in covering ISs and fuzzy relation ISs. Besides, Liang *et al.* [13] researched *i*-granules and entropy theory. Qian *et al.* [14] studied *i*-granularity in fuzzy binary GrC model. Li *et al.* [15,12] inquired into knowledge structures in a knowledge base. Yao *et al.* [16] investigated *i*-granulation and rough set approximation. These results have been shown to establish a framework of GrC in knowledge bases [17].

Given an attribute of an IS, if the domain of this attribute is a partial order set in accordance with an increasing preference, then this attribute is a criterion in this IS. If every attribute of an IS is a criterion in this IS, then this IS is viewed as an ordered IS (an OIS). So far, *i*-structures in an OIS have not been studied. This article aims to research *i*-structures in an OIS which is described by set vectors.

This article's research route is described in Figure 1.

This article is organized as follows: Section 2 reviews some elementary concepts on binary relations, OISs and homomorphisms. Section 3 introduces *i*-structures in an OIS through set vectors. Section 4 discusses relationships between *i*-structures in an OIS from dependence and separation. Section 5 researches properties of *i*-structures in an OIS. Section 6 obtains group, lattice and map

*Corresponding author. Email: sichunwang123@126.com

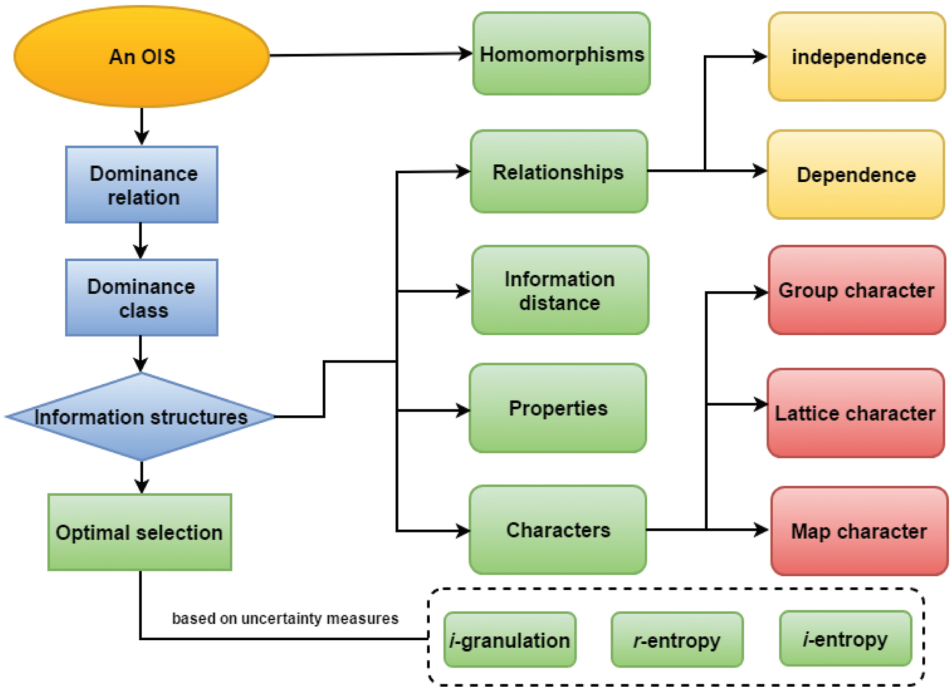

Figure 1 The work process of the article.

characters of i-structures. Section 7 studies the optimal selection of i-structures in an OIS based on the proposed measures. Section 8 makes a comparison. Section 9 concludes this article.

2. PRELIMINARIES

Throughout this article, U and V express finite sets, 2^U denotes the collection of all subsets of U and $|X|$ means the cardinality of $X \in 2^U$.

Put

$U = \{u_1, u_2, \cdots, u_n\}$,

$V = \{v_1, v_2, \cdots, v_m\}$,

$\delta = U \times U,\ \triangle = \{(u,u) : u \in U\}$,

$p(X) = \frac{|X|}{|U|}\ (X \in 2^U)$.

2.1. Binary Relations

Throughout this article, $2^{U\times U}$ expresses the collection of all binary relations on U.

Suppose that $R \in 2^{U\times U}$. Then R is called

1. reflexive, if $\forall\, x \in U \Rightarrow xRx$.
2. symmetric, if $\forall\, x, y \in U, xRy \Rightarrow yRx$.
3. transitive, if $\forall\, x, y, z \in U, xRy$ and $yRz \Rightarrow xRz$.

If R satisfies the conditions of (1), (2) and (3), it is addressed as an equivalence relation on U.

Given $R \in 2^{U\times U}$. If $R = \delta$(resp.$R = \triangle$), then R is regarded as a universal relation (respectively identity relation).

2.2. An OIS

Definition 2.1. [7] Suppose that U and A are finite object and attribute sets, respectively. Then (U, A) is regarded as an IS, if $\forall$ $a \in A$ is able to decide a function $a : U \rightarrow W_a$, where $V_a = \{a(u) : u \in U\}$.

Definition 2.2. [18] Let (U, A) be an IS. Then (U, A) is regarded as an ordered information system (OIS), if $\forall\, a \in A$ is a criterion (i.e., $U_a = \{a(u) : u \in U\}$ is a partially ordered set).

(U, A^*) is addressed as a subsystem of (U, A), if $A^* \subseteq A$.

In this article, an OIS is denoted by $S^{\succcurlyeq} = (U, A)$, the partial order on the partially ordered set U_a is denoted by $\succcurlyeq_a$.

Given $a \in A$ and $u, v \in U$. Define

$$u \succcurlyeq_a v \Leftrightarrow a(u) \succcurlyeq_a a(v).$$

Given $A^* \subseteq A$ and $u, v \in U$. Define

$$u \succcurlyeq_{A^*} v \Leftrightarrow \forall\, a \in A^*,\ u \succcurlyeq_a v.$$

Clearly, $\succcurlyeq_{A^*} = \bigcap_{a \in A^*} \succcurlyeq_a$.

Definition 2.3. [18] Let $S^{\succcurlyeq} = (U, A)$ be an OIS. Suppose $A^* \subseteq A$. Denote

$$R^{\succcurlyeq}_{A^*} = \{(u, v) \in U \times U : \forall\, a \in A^*,\ a(u) \succcurlyeq_a a(v)\}.$$

Then $R^{\succcurlyeq}_{A^*}$ is addressed as the dominance relation (d-relation) in regard to A^*.

Clearly, $R^{\succcurlyeq}_{A^*} = \succcurlyeq_{A^*}$.

Denote

$$[u]^{\succcurlyeq}_{A^*} = \{v \in U : (v, u) \in R^{\succcurlyeq}_{A^*}\}.$$

Then, $[u]^{\succcurlyeq}_{A^*}$ is viewed as a dominance class (a d-class) $u \in U$ in regard to A^*.

Clearly, $[u]^{\succcurlyeq}_{A^*} = \{v \in U : v \succcurlyeq_{A^*} u\}$.

Denote

$$U/R^{\succcurlyeq}_{A^*} = \{[u]^{\succcurlyeq}_{A^*} : u \in U\}.$$

Then $U/R^{\succcurlyeq}_{A^*}$ is deemed as the quotient set of U in regard to A^*.

Proposition 2.4. [18] *Suppose that $S^{\succcurlyeq} = (U, A)$ is an OIS and $R^{\succcurlyeq}_{A^*}$ is the dominance relation of $S^{\succcurlyeq}$. Then the following properties can be obtained:*

(1) *$R^{\succcurlyeq}_{A^*}$ satisfies reflexive and transitive, but does not satisfy symmetric;*

(2) *$A_1 \subseteq A_2 \subseteq A$, then $R^{\succcurlyeq}_{A_2} \subseteq R^{\succcurlyeq}_{A_1}$;*

(3) *$A_1 \subseteq A_2 \subseteq A$, then $\forall\, u \in U$, $[u]^{\succcurlyeq}_{A_2} \subseteq [u]^{\succcurlyeq}_{A_1}$;*

(4) *Given $A^* \subseteq A$. If $v \in [u]^{\succcurlyeq}_{A^*}$, then $[v]^{\succcurlyeq}_{A^*} \subseteq [u]^{\succcurlyeq}_{A^*}$;*

(5) *Given $A^* \subseteq A$. Then $[u]^{\succcurlyeq}_{A^*} = [v]^{\succcurlyeq}_{A^*}$ if for any $a \in A^*$, $a(u) = a(v)$;*

(6) *Given $A^* \subseteq A$. Then $U/R^{\succcurlyeq}_{A^*}$ constitute a covering of U.*

Let $S^{\succcurlyeq} = (U, A)$ be an OIS. Then $\forall\, A^* \subseteq A, X \in 2^U$ can be described as $\overline{A^*}(X)$ and $\underline{A^*}(X)$ by means of A^*, where

$$\underline{A^*}(X) = \bigcup\{Y \in U/R^{\succcurlyeq}_{A^*} : Y \subseteq X\},$$

$$\overline{A^*}(X) = \bigcap\{Y \in U/R^{\succcurlyeq}_{A^*} : Y \cap X \neq \varnothing\}.$$

It should be noted that $U/R^{\succcurlyeq}_{A^*}$ provides i-granules for depicting the concept $X \in 2^U$.

2.3. Homomorphisms between OISs

Communication between ISs is a significant subject in rough set, and could be viewed as a map between ISs. The notion of homomorphisms between ISs was first presented by Grzymala-Busse in [19,20].

Definition 2.5. Assume that (U, A) and (W, P) are OISs. Assume that $h_1 : U \to W$, $h_2 : A \to P$ and $h_3 : U^* \to W^*$ are three maps, where

$$U^* = \bigcup_{a \in A} U_a, \quad U_a = \{a(u) : u \in U\},$$

$$W^* = \bigcup_{b \in P} W_b, \quad W_b = \{b(w) : w \in W\}.$$

The triple $h = (h_1, h_2, h_3)$ is viewed as a homomorphism from (U, A) to (W, P), if $\forall\, a \in A$, $h_3 \restriction_{U_a} : U_a \to W_{h_2(a)}$ is order-preserving, and $\forall\, u \in U, a \in A$,

$$h_3(a(u)) = h_2(a)(h_1(u)).$$

Denote

$$(U, A) \sim_h (W, P)$$

3. THE CONCEPT OF *i*-STRUCTURES IN AN OIS

Suppose that $S^{\succcurlyeq} = (U, A)$ is an OIS. Given $A^* \subseteq A$. Then, the d-relation $R^{\succcurlyeq}_{A^*}$ on U is obtained. For $u \in U$, $[u]^{\succcurlyeq}_{A^*}$ is the d-class of u under $R^{\succcurlyeq}_{A^*}$. If u_1, u_2 belong to the same d-class, then u_1, u_2 will be said to be indistinguishable under $[u]^{\succcurlyeq}_{A^*}$. Each d-class can be viewed as an i-granule, the objects of which are indistinguishable. All of d-classes or i-granules constitutes a vector that is viewed as i-structure of (U, A^*).

Definition 3.1. Assume that $S^{\succcurlyeq} = (U, A)$ is an OIS. Then $\forall\, A^* \subseteq A$,

$$I(A^*) = ([u_1]^{\succcurlyeq}_{A^*}, [u_2]^{\succcurlyeq}_{A^*}, \cdots, [u_n]^{\succcurlyeq}_{A^*})$$

is addressed as the i-structure of the subsystem (U, A^*).

If $R^{\succcurlyeq}_{A^*}$ is an identity relation, then

$$I(A^*) = (\{u_1\}, \{u_2\}, \cdots, \{u_n\}).$$

Example 1. $I(\varnothing) = \tilde{U}$.

Definition 3.2. Let $S^{\succcurlyeq} = (U, A)$ be an OIS. Assume that $I(A_1)$, $I(A_2)$ are the i-structures of (U, A_1) and (U, A_2), respectively. Then $I(A_1)$ and $I(A_2)$ are viewed as the same, if $\forall\, i$, $[u_i]^{\succcurlyeq}_{A_1} = [u_i]^{\succcurlyeq}_{A_2}$. Denote $I(A_1) = I(A_2)$.

Definition 3.3. Let $S^{\succcurlyeq} = (U, A)$ be an OIS. Then

$$\mathbf{K}(U, A) = \{I(A^*) : A^* \in 2^A\}$$

is viewed as the i-structure base of (U, A).

Example 3.4. (Continued from Example 2.1 in [21]) Given an OIS $S^{\succcurlyeq} = (U, A)$ in Table 1, where $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, $A = \{a_1, a_2, a_3\}$.

Pick $A_1 = \{a_1\}$, $A_2 = \{a_2\}$, $A_3 = \{a_3\}$, $A_4 = \{a_1, a_2\}$. $A_5 = \{a_1, a_3\}$, $A_6 = \{a_2, a_3\}$.

Table 1 An ordered information system (OIS) (U, A).

	a_1	a_2	a_3
u_1	1	2	1
u_2	3	2	2
u_3	1	1	2
u_4	2	1	3
u_5	3	3	2
u_6	3	2	3

Note that

$[u_1]^{\succcurlyeq}_{A_1} = [u_3]^{\succcurlyeq}_{A_1} = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, $[u_2]^{\succcurlyeq}_{A_1} = [u_5]^{\succcurlyeq}_{A_1} = [u_6]^{\succcurlyeq}_{A_1} = \{u_2, u_5, u_6\}$, $[u_4]^{\succcurlyeq}_{A_1} = \{u_2, u_4, u_5, u_6\}$;

$[u_1]^{\succcurlyeq}_{A_2} = [u_2]^{\succcurlyeq}_{A_2} = [u_6]^{\succcurlyeq}_{A_2} = \{u_1, u_2, u_5, u_6\}$, $[u_3]^{\succcurlyeq}_{A_2} = [u_4]^{\succcurlyeq}_{A_2} = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, $[u_5]^{\succcurlyeq}_{A_2} = \{u_5\}$;

$[u_1]^{\succcurlyeq}_{A_3} = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, $[u_2]^{\succcurlyeq}_{A_3} = [u_3]^{\succcurlyeq}_{A_3} = [u_5]^{\succcurlyeq}_{A_3} = \{u_2, u_3, u_4, u_5, u_6\}$, $[u_4]^{\succcurlyeq}_{A_3} = [u_6]^{\succcurlyeq}_{A_3} = \{u_4, u_6\}$;

$[u_1]^{\succcurlyeq}_{A_4} = \{u_1, u_2, u_5, u_6\}$, $[u_2]^{\succcurlyeq}_{A_4} = [u_6]^{\succcurlyeq}_{A_4} = \{u_2, u_5, u_6\}$, $[u_3]^{\succcurlyeq}_{A_4} = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, $[u_4]^{\succcurlyeq}_{A_4} = \{u_2, u_4, u_5, u_6\}$, $[u_5]^{\succcurlyeq}_{A_4} = \{u_5\}$;

$[u_1]^{\succcurlyeq}_{A_5} = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, $[u_2]^{\succcurlyeq}_{A_5} = [u_5]^{\succcurlyeq}_{A_5} = \{u_2, u_5, u_6\}$, $[u_3]^{\succcurlyeq}_{A_5} = \{u_2, u_3, u_4, u_5, u_6\}$, $[u_4]^{\succcurlyeq}_{A_5} = \{u_4, u_6\}$, $[u_6]^{\succcurlyeq}_{A_5} = \{u_6\}$;

$[u_1]^{\succcurlyeq}_{A_6} = \{u_1, u_2, u_5, u_6\}$, $[u_2]^{\succcurlyeq}_{A_6} = \{u_2, u_5, u_6\}$, $[u_3]^{\succcurlyeq}_{A_6} = \{u_2, u_3, u_4, u_5, u_6\}$, $[u_4]^{\succcurlyeq}_{A_6} = \{u_4, u_6\}$, $[u_5]^{\succcurlyeq}_{A_6} = \{u_5\}$, $[u_6]^{\succcurlyeq}_{A_6} = \{u_6\}$;

$[u_1]^{\succcurlyeq}_{A} = \{u_1, u_2, u_5, u_6\}$, $[u_2]^{\succcurlyeq}_{A} = \{u_2, u_5, u_6\}$, $[u_3]^{\succcurlyeq}_{A} = \{u_2, u_3, u_4, u_5, u_6\}$, $[u_4]^{\succcurlyeq}_{A} = \{u_4, u_6\}$, $[u_5]^{\succcurlyeq}_{A} = \{u_5\}$, $[u_6]^{\succcurlyeq}_{A} = \{u_6\}$.

Then

$I(\emptyset) = (U,\ U,\ U,\ U,\ U,\ U) = \tilde{U}$.

$I(A_1) = (U, \{u_2, u_5, u_6\}, U, \{u_2, u_4, u_5, u_6\}, \{u_2, u_5, u_6\}, \{u_2, u_5, u_6\})$,

$I(A_2) = (\{u_1, u_2, u_5, u_6\}, \{u_1, u_2, u_5, u_6\}, U, U, \{u_5\}, \{u_1, u_2, u_5, u_6\})$,

$I(A_3) = (U, \{u_2, u_3, u_4, u_5, u_6\}, \{u_2, u_3, u_4, u_5, u_6\}, \{u_4, u_6\}, \{u_2, u_3, u_4, u_5, u_6\}, \{u_4, u_6\})$,

$I(A_4) = (\{u_1, u_2, u_5, u_6\}, \{u_2, u_5, u_6\}, U, \{u_2, u_4, u_5, u_6\}, \{u_5\}, \{u_2, u_5, u_6\})$,

$I(A_5) = (U, \{u_2, u_5, u_6\}, \{u_2, u_3, u_4, u_5, u_6\}, \{u_4, u_6\}, , \{u_2, u_5, u_6\}, \{u_6\})$,

$I(A) = I(A_6) = (\{u_1, u_2, u_5, u_6\}, \{u_2, u_5, u_6\}, \{u_2, u_3, u_4, u_5, u_6\}, \{u_4, u_6\}, \{u_5\}, \{u_6\})$.

Then

$\mathbf{K}(U, A) = \{I(\emptyset), I(A_1), I(A_2), I(A_3), I(A_4), I(A_5), I(A)\}$.

4. RELATIONSHIPS BETWEEN *i*-STRUCTURES IN AN OIS

4.1. Dependence and Independence between *i*-Structures

Definition 4.1. Let $S^{\succcurlyeq} = (U, A)$ be an OIS. Assume that $I(A_1)$, $I(A_2)$ are the *i*-structures of (U, A_1) and (U, A_2), respectively.

1. $I(A_2)$ is addressed as depend on $I(A_1)$, if $\forall\ i$, $[u_i]^{\succcurlyeq}_{A_1} \subseteq [u_i]^{\succcurlyeq}_{A_2}$, it can be written as $I(A_1) \preceq I(A_2)$; $I(A_2)$ is viewed as depend strictly on $I(A_1)$, if $I(A_1) \preceq I(A_2)$ and $I(A_1) \neq I(A_2)$, it can be written as $I(A_1) \prec I(A_2)$.
2. $I(A_2)$ is addressed as depend partially on $I(A_1)$, if $\exists\ i$, $[u_i]^{\succcurlyeq}_{A_1} \subseteq [u_i]^{\succcurlyeq}_{A_2}$, it can be written as $I(A_1) \sqsubseteq I(A_2)$; $I(A_2)$ is viewed as depend partially strictly on $I(A_1)$, if $I(A_1) \sqsubseteq I(A_2)$ and $I(A_1) \neq I(A_2)$, it can be written as $I(A_1) \sqsubset I(A_2)$.
3. $I(A_2)$ is viewed as independent of $I(A_1)$, if $\forall\ i$, $[u_i]^{\succcurlyeq}_{A_1} \nsubseteq [u_i]^{\succcurlyeq}_{A_2}$, it can be written as $I(A_1) \bowtie I(A_2)$.

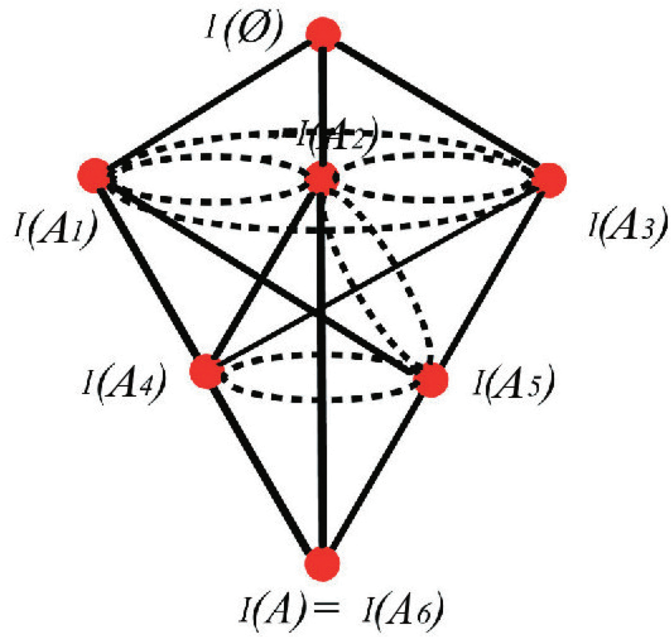

Figure 2 | Relationships among *i*-structures.

Example 4.2. (Continued from Example 3.5) It can be obtained that

$I(A) = I(A_6)$;

$I(A) \prec I(A_4) \prec I(A_1) \prec I(\emptyset)$, $I(A) \prec I(A_5) \prec I(A_3) \prec I(\emptyset)$, $I(A) \prec I(A_2) \prec I(\emptyset)$, $I(A_4) \prec I(A_2)$, $I(A_4) \prec I(A_3)$, $I(A_5) \prec I(A_1)$;

$I(A_1) \sqsubset I(A_2)$, $I(A_2) \sqsubset I(A_1)$, $I(A_1) \sqsubset I(A_3)$, $I(A_3) \sqsubset I(A_1)$, $I(A_2) \sqsubset I(A_3)$, $I(A_3) \sqsubset I(A_2)$, $I(A_2) \sqsubset I(A_5)$, $I(A_5) \sqsubset I(A_2)$, $I(A_4) \sqsubset I(A_5)$, $I(A_5) \sqsubset I(A_4)$ (see Figure 2).

4.2. Information Distance between Two *i*-Structures

$\forall\ X, Y \subseteq U$, $X \oplus Y$ is addressed as the symmetric difference X and Y, where

$$X \oplus Y = X \cup Y - X \cap Y.$$

Definition 4.3. Let $S^{\succcurlyeq} = (U, A)$ be an OIS. Assume that $I(A_1)$, $I(A_2)$ are the *i*-structures of (U, A_1) and (U, A_2), respectively. Information distance (*i*-distance) between $I(A_1)$ and $I(A_2)$ is defined as

$$d(I(A_1), I(A_2)) = \frac{1}{n^2} \sum_{i=1}^{n} \mid [u_i]^{\succcurlyeq}_{A_1} \oplus [u_i]^{\succcurlyeq}_{A_2} \mid .$$

Lemma 4.4. ([10]) Given $X, Y \subseteq U$. Then

$$X = Y \Leftrightarrow \mid X \oplus Y \mid = 0.$$

Lemma 4.5. ([10]) Given $X, Y, Z \subseteq U$. Then

$$\mid X \oplus Y \mid + \mid Y \oplus Z \mid \geqslant \mid X \oplus Z \mid .$$

Lemma 4.6. ([10]) Given $X, Y, Z \subseteq U$. If $X \subseteq Y \subseteq Z$ or $Z \subseteq Y \subseteq X$, then

$$|X \oplus Y| + |Y \oplus Z| = |X \oplus Z|.$$

Theorem 4.7. *Let $S^{\succcurlyeq} = (U, A)$ be an OIS. Then $(\mathbf{K}(U, A), d)$ is a distance space.*

Proof. Assume $A_1, A_2, A_3 \subseteq A$.

Clearly,

$d(I(A_1), I(A_2)) \geqslant 0$, $d(I(A_1), I(A_2)) = d(I(A_2), I(A_1))$.

By Lemma 4.4,

$d(I(A_1), I(A_2)) = 0 \Leftrightarrow \forall\, i, |[u_i]^{\succcurlyeq}_{A_1} \oplus [u_i]^{\succcurlyeq}_{A_2}| = 0 \Leftrightarrow \forall\, i, [u_i]^{\succcurlyeq}_{A_1} = [u_i]^{\succcurlyeq}_{A_2} \Leftrightarrow I(A_1) = I(A_2)$.

By Lemma 4.5,

$$|[u_i]^{\succcurlyeq}_{A_1} \oplus [u_i]^{\succcurlyeq}_{A_2}| + |[u_i]^{\succcurlyeq}_{A_2} \oplus [u_i]^{\succcurlyeq}_{A_3}| \geqslant |[u_i]^{\succcurlyeq}_{A_1} \oplus [u_i]^{\succcurlyeq}_{A_3}|.$$

Then $d(I(A_1), I(A_2)) + d(I(A_2), I(A_3))$

$$= \frac{1}{n^2}\sum_{i=1}^{n} |[u_i]^{\succcurlyeq}_{A_1} \oplus [u_i]^{\succcurlyeq}_{A_2}| + \frac{1}{n^2}\sum_{i=1}^{n} |[u_i]^{\succcurlyeq}_{A_2} \oplus [u_i]^{\succcurlyeq}_{A_3}|$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}(|[u_i]^{\succcurlyeq}_{A_1} \oplus [u_i]^{\succcurlyeq}_{A_2}| + |[u_i]^{\succcurlyeq}_{A_2} \oplus [u_i]^{\succcurlyeq}_{A_3}|)$$

$$\geqslant \frac{1}{n^2}\sum_{i=1}^{n} |[u_i]^{\succcurlyeq}_{A_1} \oplus [u_i]^{\succcurlyeq}_{A_3}|$$

$$= d(I(A_1), I(A_3))$$

So, $(\mathbf{K}(U, A), d)$ is a distance space.

Proposition 4.8. *Let $S^{\succcurlyeq} = (U, A)$ be an OIS. Then $\forall\, A_1, A_2 \subseteq A$,*

1. $$0 \leqslant d(I(A_1), I(A_2)) \leqslant 1 - \frac{1}{n};$$

2. *If $I(A_1) \preceq I(A_2)$ and $R^{\succcurlyeq}_{A_1^*}$ is an identity relation on U, then*
$$d(I(A_1), I(A_1^*)) \leqslant d(I(A_2), I(A_1^*));$$

3. *If $I(A_1) \preceq I(A_2)$, then*
$$d(I(A_1), I(\varnothing)) \geqslant d(I(A_2), I(\varnothing)).$$

Proof. (1) Clearly,

$\forall\, i, 1 \leqslant |[u_i]^{\succcurlyeq}_{A_1} \cup [u_i]^{\succcurlyeq}_{A_2}| \leqslant n$ and $1 \leqslant |[u_i]^{\succcurlyeq}_{A_1} \cap [u_i]^{\succcurlyeq}_{A_2}| \leqslant n$.

Then, $\forall\, i$,

$$0 \leqslant |[u_i]^{\succcurlyeq}_{A_1} \oplus [u_i]^{\succcurlyeq}_{A_2}| \leqslant n - 1.$$

Therefore

$$0 \leqslant d(I(A_1), I(A_2)) \leqslant \frac{1}{n^2}\sum_{i=1}^{n}(n-1) = \frac{n^2 - n}{n^2} = 1 - \frac{1}{n}.$$

(2) Due to $I(A_1) \preceq I(A_2)$, it can be obtained that $[u_i]^{\succcurlyeq}_{A_1} \subseteq [u_i]^{\succcurlyeq}_{A_2}, \forall\, i$.

Then, $\forall\, i$, $|[u_i]^{\succcurlyeq}_{A_1}| \leqslant |[u_i]^{\succcurlyeq}_{A_2}|$.

Thus

$$d(I(A_1), I(A_1^*)) = \frac{1}{n^2}\sum_{i=1}^{n} |[u_i]^{\succcurlyeq}_{A_1} \oplus \{u_i\}|$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}(|[u_i]^{\succcurlyeq}_{A_1} \cup \{u_i\}| - |[u_i]^{\succcurlyeq}_{A_1} \cap \{u_i\}|)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}(|[u_i]^{\succcurlyeq}_{A_1}| - 1) \leqslant \frac{1}{n^2}\sum_{i=1}^{n}(|[u_i]^{\succcurlyeq}_{A_2}| - 1)$$

$$= d(I(A_2), I(A_1^*)).$$

(3) On account of $I(A_1) \preceq I(A_2)$, then, $\forall\, i$, $|[u_i]^{\succcurlyeq}_{A_1}| \leqslant |[u_i]^{\succcurlyeq}_{A_2}|$.

So,

$$d(I(A_1), I(\varnothing)) = \frac{1}{n^2}\sum_{i=1}^{n} |[u_i]^{\succcurlyeq}_{A_1} \oplus U|$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}(|[u_i]^{\succcurlyeq}_{A_1} \cup U| - |[u_i]^{\succcurlyeq}_{A_1} \cap U|)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}(n - |[u_i]^{\succcurlyeq}_{A_1}|) \geqslant \frac{1}{n^2}\sum_{i=1}^{n}(n - |[u_i]^{\succcurlyeq}_{A_2}|)$$

$$= d(I(A_2), I(\varnothing)).$$

Proposition 4.9. *Let $S^{\succcurlyeq} = (U, A)$ be an OIS. If $R^{\succcurlyeq}_{B^*} (B^* \subseteq A)$ is an identity relation on U, then for any $B \subseteq A$,*

$$d(I(B), I(B^*)) + d(I(B), I(\varnothing)) = 1 - \frac{1}{n}.$$

Proof. It can be obtained that

$$d(I(B), I(B^*)) + d(I(B), I(\varnothing))$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} |[u_i]^{\succcurlyeq}_{B} \oplus \{u_i\}| + \frac{1}{n^2}\sum_{i=1}^{n} |[u_i]^{\succcurlyeq}_{B} \oplus U|$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}(|[u_i]^{\succcurlyeq}_{B}| - 1) + \frac{1}{n^2}\sum_{i=1}^{n}(n - |[u_i]^{\succcurlyeq}_{B}|)$$

$$= \frac{1}{n^2} n(n-1) = 1 - \frac{1}{n}.$$

Proposition 4.10. *Let $S^{\succcurlyeq} = (U, A)$ be an OIS. Put $A_1, A_2, A_3 \subseteq A$. If $I(A_1) \preceq I(A_2) \preceq I(A_3)$ or $I(A_3) \preceq I(A_2) \preceq I(A_1)$, then*

$$d(I(A_1), I(A_2)) + d(I(A_2), I(A_3)) = d(I(A_1), I(A_3)).$$

Proof. Due to $I(A_1) \preceq I(A_2) \preceq I(A_3)$ or $I(A_3) \preceq I(A_2) \preceq I(A_1)$, $\forall\, i$, it can be obtained that

$$[u_i]^{\succcurlyeq}_{A_1} \subseteq [u_i]^{\succcurlyeq}_{A_2} \subseteq [u_i]^{\succcurlyeq}_{A_3} \text{ or } [u_i]^{\succcurlyeq}_{A_3} \subseteq [u_i]^{\succcurlyeq}_{A_2} \subseteq [u_i]^{\succcurlyeq}_{A_1}.$$

By Lemma 4.6, $\forall\, i$, it can be obtained that

$|[u_i]^{\succcurlyeq}_{A_1} \oplus [u_i]^{\succcurlyeq}_{A_2}| + |[u_i]^{\succcurlyeq}_{A_2} \oplus [u_i]^{\succcurlyeq}_{A_3}| = |[u_i]^{\succcurlyeq}_{A_1} \oplus [u_i]^{\succcurlyeq}_{A_3}|$.

$d(I(A_1), I(A_2)) + d(I(A_2), I(A_3))$

$$= \frac{1}{n^2}\sum_{i=1}^{n} |[u_i]^{\succcurlyeq}_{A_1} \oplus [u_i]^{\succcurlyeq}_{A_2}| + \frac{1}{n^2}\sum_{i=1}^{n} |[u_i]^{\succcurlyeq}_{A_2} \oplus [u_i]^{\succcurlyeq}_{A_3}|$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}(\mid [u_i]^{\succcurlyeq}_{A_1} \oplus [u_i]^{\succcurlyeq}_{A_2} \mid + \mid [u_i]^{\succcurlyeq}_{A_2} \oplus [u_i]^{\succcurlyeq}_{A_3} \mid)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} \mid [u_i]^{\succcurlyeq}_{A_1} \oplus [u_i]^{\succcurlyeq}_{A_3} \mid$$

$$= d(I(A_1), I(A_3)).$$

Example 4.11. (Continued from Example 3.4)

1. By Definition 4.3, one can obtain that
   $d(I(\varnothing), I(A_1)) = \frac{11}{36}$, $d(I(\varnothing), I(A_2)) = \frac{11}{36}$,
   $d(I(\varnothing), I(A_3)) = \frac{11}{36}$, $d(I(\varnothing), I(A_4)) = \frac{15}{36}$,
   $d(I(\varnothing), I(A_5)) = \frac{16}{36}$, $d(I(\varnothing), I(A)) = \frac{20}{36}$,
   $d(I(A_1), I(A_2)) = \frac{8}{36}$, $d(I(A_1), I(A_3)) = \frac{10}{36}$,
   $d(I(A_1), I(A_4)) = \frac{4}{36}$, $d(I(A_1), I(A_5)) = \frac{5}{36}$,
   $d(I(A_1), I(A)) = \frac{9}{36}$, $d(I(A_2), I(A_3)) = \frac{18}{36}$,
   $d(I(A_2), I(A_4)) = \frac{4}{36}$, $d(I(A_2), I(A_5)) = \frac{15}{36}$,
   $d(I(A_2), I(A)) = \frac{9}{36}$, $d(I(A_3), I(A_4)) = \frac{14}{36}$,
   $d(I(A_3), I(A_5)) = \frac{5}{36}$, $d(I(A_3), I(A)) = \frac{9}{36}$,
   $d(I(A_4), I(A_5)) = \frac{9}{36}$, $d(I(A_4), I(A)) = \frac{5}{36}$,
   $d(I(A_5), I(A)) = \frac{4}{36}$ (see Figure 3).
2. Let $R^{\succcurlyeq}_{A_1^*}$ be an identity relation on U. Then

$$d(I(A_3), I(A_1^*)) = \frac{19}{36},\ \ d(I(A_5), I(A_1^*)) = \frac{14}{36}.$$

   It can be obtained that $I(A_5) \prec I(A_3)$,

$$d(I(A_5), I(A_1^*)) = \frac{14}{36} \leqslant \frac{19}{36} = d(I(A_3), I(A_1^*)),$$

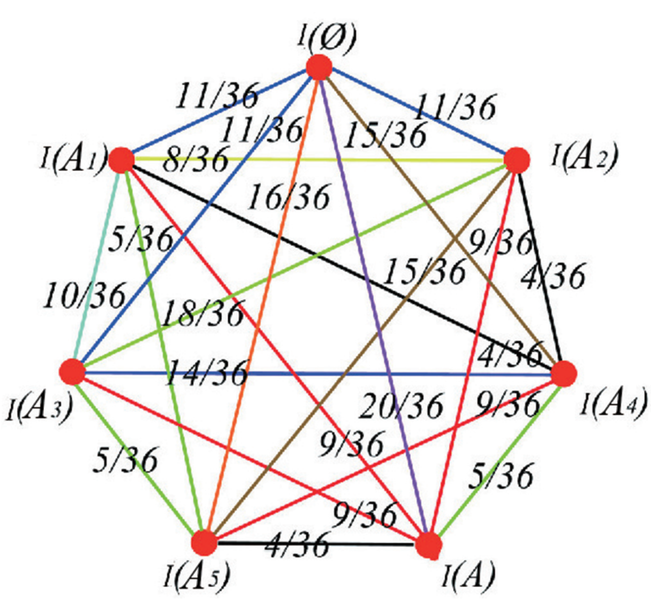

Figure 3 Information distance between i-structures.

$$d(I(A_5), I(\varnothing)) = \frac{16}{36} \geqslant \frac{11}{36} = d(I(A_3), I(\varnothing)).$$

3. On account of $R^{\succcurlyeq}_{A_1^*} = \triangle$, then $\forall\, A_1 \subseteq A$,

$$d(I(A_1), I(A_1^*)) + d(I(A_1), I(\varnothing)) = 1 - \frac{1}{6} = 1 - \frac{1}{n}.$$

4. On account of $I(A) \preceq I(A_4) \prec I(A_1) \preceq I(\varnothing)$, then
   $d(I(A), I(A_4)) + d(I(A_4), I(A_1)) = \frac{5}{36} + \frac{4}{36} = \frac{9}{36} = d(I(A), I(A_1))$,
   $d(I(A_4), I(A_1)) + d(I(A_1), I(\varnothing)) = \frac{4}{36} + \frac{11}{36} = \frac{15}{36} = d(I(A_4), I(\varnothing))$.

5. PROPERTIES OF *i*-STRUCTURES IN AN OIS

Theorem 5.1. *Let $S^{\succcurlyeq} = (U, A)$ be an OIS. Then $\forall\, A_1, A_2 \subseteq A$, the following are equivalent:*

1. $I(A_1) = I(A_2)$;
2. $R^{\succcurlyeq}_{A_1} = R^{\succcurlyeq}_{A_2}$;
3. $d(I(A_1), I(A_2)) = 0$

Proof. (1) ⇔ (2). Obviously.

(1) ⇔ (3). It can be proved by Theorem 4.7.

Theorem 5.2. *Assuming $S^{\succcurlyeq} = (U, A)$ is an OIS. Then $\forall\, A_1, A_2 \subseteq A$,*

$$I(A_1) \preceq I(A_2) \ \Leftrightarrow\ R^{\succcurlyeq}_{A_1} \subseteq R^{\succcurlyeq}_{A_2}.$$

Proof. Evidently.

Proposition 5.3. *Let $S^{\succcurlyeq} = (U, A)$ be an OIS. If $A_1 \subseteq A_2 \subseteq A$, then $I(A_2) \preceq I(A_1)$.*

Proof. On account of $A_1 \subseteq A_2$, by it can be obtained that $R^{\succcurlyeq}_{A_2} \subseteq R^{\succcurlyeq}_{A_1}$. By Theorem 5.2, $I(A_2) \preceq I(A_1)$.

Proposition 5.4. *Let $S^{\succcurlyeq} = (U, A)$ be an OIS. Given $A^* \subseteq A$. Then $I(A) \preceq I(A^*) \preceq I(\varnothing)$.*

Proof. It can be proved by Proposition 5.3.

Definition 5.5. [22] Let $S^{\succcurlyeq} = (U, A)$ be an OIS. Let $\mathbf{K}(U, A)$ be the i-structure base of (U, A). Then a map $D : \mathbf{K}(U, A) \times \mathbf{K}(U, A) \to [0, 1]$ is addressed as the inclusion degree on $\mathbf{K}(U, A)$, if

1. $0 \leqslant D(I(A_2)/I(A_1)) \leqslant 1$;
2. $I(A_1) \preceq I(A_2) \Rightarrow D(I(A_2)/I(A_1)) = 1$;
3. $I(A_1) \sqsubseteq I(A_2) \sqsubseteq I(A_3) \Rightarrow D(I(A_1)/I(A_3)) \leqslant D(I(A_1)/I(A_2))$.

Definition 5.6. Let $S^{\succcurlyeq} = (U, A)$ be an OIS, $\forall\, A_1, A_2 \subset A$, define

$$D(I(A_2)/I(A_1)) = \sum_{l=1}^{n} \frac{\left|[u_l]^{\succcurlyeq}_{A_2}\right|}{\sum_{i=1}^{n} \left|[u_i]^{\succcurlyeq}_{A_2}\right|} \chi_{[u_l]^{\succcurlyeq}_{A_2}}([u_l]^{\succcurlyeq}_{A_1}),$$

where

$$\chi_{[u_l]^{\succcurlyeq}_{A_2}}([u_l]^{\succcurlyeq}_{A_1}) = \begin{cases} 1, & if\ [u_l]^{\succcurlyeq}_{A_1} \subseteq [u_l]^{\succcurlyeq}_{A_2}, \\ 0, & if\ [u_l]^{\succcurlyeq}_{A_1} \nsubseteq [u_l]^{\succcurlyeq}_{A_2}. \end{cases}$$

Proposition 5.7. *D in Definition 5.6 is the inclusion degree.*

Proof. Obviously.

Theorem 5.8. *Let $S^{\succcurlyeq} = (U, A)$ be an OIS. Suppose $P, A_2 \subset A$. Then*

1. $I(A_1) \preceq I(A_2) \Leftrightarrow D(I(A_2)/I(A_1)) = 1.$
2. $I(A_1) \bowtie I(A_2) \Leftrightarrow D(I(A_2)/I(A_1)) = 0.$
3. $I(A_1) \sqsubseteq I(A_2) \Leftrightarrow 0 < D(I(A_2)/I(A_1)) \leqslant 1.$

Proof. Please see "Appendix."

Corollary 5.9. Let $S^{\succcurlyeq} = (U, A)$ be an OIS. Then $\forall\ A_1, A_2 \subseteq A$,

$$I(A_1) \prec I(A_2) \ \Leftrightarrow\ D(I(\mathcal{A}_2)/I(\mathcal{A}_1)) = 1, \quad d(I(\mathcal{A}_1), I(\mathcal{A}_2)) \neq 0.$$

Proof. It can be proved by Theorems 4.7 and 5.8.

6. CHARACTERS OF *i*-STRUCTURES IN AN OIS

In this section, we obtain group, lattice and map characters of *i*-structures in an OIS. These characters may be helpful for understanding the nature of *i*-structures in an OIS.

6.1. Group Characters of *i*-Structures

Theorem 6.1. *$(\mathbf{K}(U, A), \odot)$ is a commutative semigroup with the identity element $I(\varnothing)$.*

Proof. Suppose $A_1, A_2, A_3 \subseteq A$. Then

$I(A_1) \odot I(A_2)$

$= ([u_1]^{\succcurlyeq}_{A_1} \cap [u_1]^{\succcurlyeq}_{A_2}, [u_2]^{\succcurlyeq}_{A_1} \cap [u_2]^{\succcurlyeq}_{A_2}, \cdots, [u_n]^{\succcurlyeq}_{A_1} \cap [u_n]^{\succcurlyeq}_{A_2})$

$= ([u_1]^{\succcurlyeq}_{A_1 \cup A_2}, [u_2]^{\succcurlyeq}_{A_1 \cup A_2}, \cdots, [u_n]^{\succcurlyeq}_{A_1 \cup A_2})$

$= I(A_1 \cup A_2).$

Clearly, $I(A_2) \odot I(A_1) = I(A_2 \cup A_1).$

Thus

$$I(A_1) \odot I(A_2) = I(A_2) \odot I(A_1).$$

Additionally, $(I(A_1) \odot I(A_2)) \odot I(L) = I(P \cup A_2) \odot I(A^*) = I(A_1 \cup A_2 \cup A_3),$

$I(A_1) \odot (I(A_2) \odot I(A_3)) = I(A_1) \odot I(A_2 \cup A_3) = I(A_1 \cup A_2 \cup A_3).$ Then

$$(I(A_1) \odot I(A_2)) \odot I(A_3) = I(A_1) \odot (I(A_2) \odot I(A_3)).$$

Thus, $(I(U), \odot)$ is a commutative semigroup.

Clearly, $I(\varnothing)$ is the identity element.

Example 6.2. (Continued from Example 3.4) $(\mathbf{K}(U, A), \odot)$ is not a group.

By Theorem 6.1, $(\mathbf{K}(U, A), \odot)$ is a commutative semigroup with the identity element $I(\varnothing)$.

Additionally, $\forall\ P \subseteq A$,

$$I(A_1) \odot I(A) = I(A_1 \cup A) = I(A) \neq I(\varnothing).$$

Then $I(A)$ does not contain inverse elements.

Thus $(\mathbf{K}(U, A), \odot)$ is not a group.

6.2. Lattice Characters of *i*-Structures

Theorem 6.3. *Let $S^{\succcurlyeq} = (U, A)$ be an OIS. Then*

1. *$L = (\mathbf{K}(U, A), \preceq)$ is a lattice with $1_L = I(\varnothing)$ and $0_L = I(A)$;*
2. *If $A_1, A_2 \subseteq A$ and $\mathcal{A}(A_1, A_2) = \{A^* : A^* \subseteq A,\ R^{\succcurlyeq}_{A_1} \cup R^{\succcurlyeq}_{A_2} \subseteq R^{\succcurlyeq}_{A^*}\}$, then*

$$I(A_1) \wedge I(A_2) = I(A_1) \odot I(A_2) = I(A_1 \cup A_2),$$

$$I(A_1) \vee I(A_2) = \bigodot_{A^* \in \mathcal{A}(A_1, A_2)} I(A_1) = I(\cup \mathcal{A}(A_1, A_2)).$$

Proof. Please see "Appendix."

Corollary 6.4. $(\mathbf{K}(U, A), \wedge)$ is a commutative semigroup with the identity element $I(\varnothing)$.

Example 6.5. (Continued from Example 3.4) $L = (\mathbf{K}(U, A), \preceq)$ is not a distributive lattice.

By Theorem 6.3, $L = (\mathbf{K}(U, A), \preceq)$ is a lattice with $1_L = I(\varnothing)$ and $0_L = I(A)$.

It can be obtained that

$\mathcal{A}(A_1, A_2 \cup A_3) = \mathcal{A}(A_1, A_6) = \{A^* \subseteq A : R^{\succcurlyeq}_{A_1} \cup R^{\succcurlyeq}_{A_6} \subseteq R^{\succcurlyeq}_{A_1}\} = \{\varnothing, A_1\},$

$\mathcal{A}(A_1, A_2) = \{A^* \subseteq A : R^{\succcurlyeq}_{A_1} \cup R^{\succcurlyeq}_{A_2} \subseteq R^{\succcurlyeq}_{A_1}\} = \{\varnothing\},$

$\mathcal{A}(A_1, A_3) = \{A^* \subseteq A : R^{\succcurlyeq}_{A_1} \cup R^{\succcurlyeq}_{A_3} \subseteq R^{\succcurlyeq}_{A_1}\} = \{\varnothing\}.$

Then

$$\cup \mathcal{A}(A_1, A_2 \cup A_3) = \{a_1\}, \quad (\cup \mathcal{A}(A_1, A_2)) \bigcup (\cup \mathcal{A}(A_1, A_3)) = \varnothing.$$

So

$$I(\cup \mathcal{A}(A_1, A_2 \cup A_3)) \neq I((\cup \mathcal{A}(A_1, A_2)) \bigcup (\cup \mathcal{A}(A_1, A_3))).$$

By Theorem 6.3,

$I(A_1) \vee (I(A_2) \wedge I(A_3)) = I(A_1) \vee I(A_2 \cup A_3) = I((\cup \mathcal{A}(A_1, A_2 \cup A_3))),$

$(I(A_1) \vee I(A_2)) \wedge (I(A_1) \vee I(A_3)) = I((\cup A(A_1, A_2))) \bigcup I((\cup A(A_1, A_3))) = I((\cup A(A_1, A_2)) \bigcup (\cup A(A_1, A_3))).$

Thus

$I(A_1) \vee (I(A_2) \wedge I(A_3)) \neq (I(A_1) \vee I(A_2)) \wedge (I(A_1) \vee I(A_3)).$ So $(\mathbf{K}(U, A), \preceq)$ is not a distributive lattice.

Example 6.6. (Continued from Example 3.4) $(\mathbf{K}(U,A),\sqsubseteq)$ is not a partially ordered set.

It can be obtained that $I(A_1) \sqsubseteq I(A_2), I(A_2) \sqsubseteq I(A_1)$. But $I(A_1) \neq I(A_2)$. So $\sqsubseteq$ is not anti-symmetric.

Hence $(\mathbf{K}(U,A),\sqsubseteq)$ is not a partially ordered set.

6.3. Map Characters of *i*-Structures

Lemma 6.7. Assume that (U,A) and (W,P) are OISs. If $(U,A) \sim_h (W,P)$ with $h=(h_1,h_2,h_3)$, then $\forall\, A^* \subseteq A$ and $u^* \in U$,

$$h_1(u) \in [h_1(u^*)]^{\succcurlyeq}_{h_2(A^*)} \Leftrightarrow u \in g_{A^*}(u^*),$$

where $g_a = h_3 \circ a$, $g_{A^*} = \{g_a : a \in A^*\}$, $g_{A^*}(u^*) = \{u \in U : \forall\, a \in A^*,\ g_a(u) = g_a(u^*)\}$.

Proof. "$\Rightarrow$". Assume $h_1(u) \in [h_1(u^*)]^{\succcurlyeq}_{h_2(A^*)}$. Then $(h_1(u^*), h_1(u)) \in R^{\succcurlyeq}_{h_2(A^*)}$. Thus $\forall\, a \in A^*$,

$$h_2(a)(h_1(u)) = h_2(a)(h_1(u^*)).$$

On account of $(U,A) \sim_h (W,P)$, then $\forall\, a \in A^*$,

$h_2(a)(h_1(u)) = h_3(a(u)) = g_a(u)$, $h_2(a)(h_1(u^*)) = h_3(a(u^*)) = g_a(u^*)$.

So $\forall\, a \in A^*, g_a(u) = g_a(u^*)$.

Thus $u \in g_{A^*}(u^*)$.

"$\Leftarrow$". Suppose $u \in g_{A^*}(u^*)$. Then $\forall\, a \in A^*$, $g_a(u) = g_a(u^*)$, i.e., $\forall\, a \in A^*$, $h_3(a(u)) = (h_3(a(u^*))$. On account of $(U,A) \sim_h (W,P)$, it can be obtained that $\forall\, a \in A^*$, $h_2(a)(h_1(u)) = h_3(a(u))$, $h_3(a(u^*)) = h_2(a)(h_1(u^*)$. Then $\forall\, a \in A^*$,

$$h_2(a)(h_1(u)) = h_2(a)(h_1(u^*).$$

Thus $h_1(u) \in [h_1(u^*)]^{\succcurlyeq}_{h_2(A^*)}$.

Hence

$$h_1(u) \in [h_1(u^*)]^{\succcurlyeq}_{h_2(A^*)} \Leftrightarrow u \in g_{A^*}(u^*).$$

Theorem 6.8. *Let (U,A) and (W,P) be two OISs. Assume $(U,A) \sim_h (W,P)$ with $h=(h_1,h_2,h_3)$. Denote $w_j = h_1(u_{k_j})$ $(j=1,2,\cdots,m)$. Then $\forall\, A^* \in 2^A$,*

$$I(h_2(A^*)) = (h_1(g_{A^*}(u_{k_1})), h_1(g_{A^*}(u_{k_2}))\cdots, h_1(g_{A^*}(u_{k_m}))),$$

where $g_a = h_3 \circ a$, $g_{A^*} = \{g_a : a \in A^*\}$, $g_{A^*}(u_{k_j}) = \{u \in U : \forall\, a \in A^*,\ g_a(u) = g_a(u_{k_j})\}$ $(j=1,2,\cdots,m)$.

Proof. By Lemma 6.7,

$$h_1(u) \in [h_1(u_{k_j})]^{\succcurlyeq}_{h_2(A^*)} \Leftrightarrow u \in g_{A^*}(u_{k_j})\ (j=1,2,\cdots,m).$$

Then

$$h_1(g_{A^*}(u_{k_j})) = [h_1(u_{k_j})]^{\succcurlyeq}_{h_2(A^*)}\ (j=1,2,\cdots,m).$$

Thus

$I(h_2(A^*)) = ([w_1]^{\succcurlyeq}_{h_2(A^*)}, [w_2]^{\succcurlyeq}_{h_2(A^*)}, \cdots, [w_m]^{\succcurlyeq}_{h_2(A^*)})$

$= ([h_1(u_{k_1})]^{\succcurlyeq}_{h_2(A^*)}, [h_1(u_{k_2})]^{\succcurlyeq}_{h_2(A^*)}, \cdots, [h_1(u_{k_m})]^{\succcurlyeq}_{h_2(A^*)})$

$= (h_1(g_{A^*}(u_{k_1})), h_1(g_{A^*}(u_{k_2}))\cdots, h_1(g_{A^*}(u_{k_m}))).$

Proposition 6.9. *Let (U,A) and (W,P) be two OISs. Assume $(U,A) \sim_h (W,P)$ with $h=(h_1,h_2,h_3)$. Then $\forall\, A_1, A_2 \subseteq A$,*

$$R^{\succcurlyeq}_{A_1} \subseteq R^{\succcurlyeq}_{A_2} \Leftrightarrow R^{\succcurlyeq}_{h_2(A_1)} \subseteq R^{\succcurlyeq}_{h_2(A_2)}.$$

Proof. Please see "Appendix."

Corollary 6.10. Assume that (U,A) and (W,P) are OISs. Given $(U,A) \sim_h (W,P)$ with $h=(h_1,h_2,h_3)$. Then $\forall\, A_1, A_2 \subseteq A$,

$$R^{\succcurlyeq}_{A_1} = R^{\succcurlyeq}_{A_2} \Leftrightarrow R^{\succcurlyeq}_{h_2(A_1)} = R^{\succcurlyeq}_{h_2(A_2)}.$$

Proof. It can be proved by Proposition 6.9.

Theorem 6.11. *Assume that (U,A) and (W,P) are OISs. Given $(U,A) \sim_h (W,P)$ with $h=(h_1,h_2,h_3)$. Then $\forall\, A_1, A_2 \subseteq A$,*

1. $I(A_1) = I(A_2) \Leftrightarrow I(h_2(A_1)) = I(h_2(A_2))$;
2. $I(A_1) \preceq I(A_2) \Leftrightarrow I(h_2(A_1)) \preceq I(h_2(A_2))$.

Proof. (1) It can be proved by Theorem 5.1 and Corollary 6.10.

(1) This follows from Theorem 5.2 and Proposition 6.9.

Corollary 6.12. Let (U,A) and (W,P) be two OISs. Suppose $(U,A) \sim_h (W,P)$ with $h=(h_1,h_2,h_3)$. Then $\forall\, A_1, A_2 \subseteq A$,

$$I(A_1) \prec I(A_2) \Leftrightarrow I(h_2(A_1)) \prec I(h_2(A_2)).$$

Proof. It can be proved by Theorem 6.11.

Theorem 6.13. *Assume that (U,A) and (W,P) are two OISs. Suppose $(U,A) \sim_h (W,P)$ with $h=(h_1,h_2,h_3)$. Then $\forall\, A_1, A_2 \subseteq A$,*

$$I(A_1) \sqsubseteq I(A_2) \Leftrightarrow I(h_2(A_1)) \sqsubseteq I(h_2(A_2)).$$

Proof. Please see "Appendix."

Corollary 6.14. Assume that (U,A) and (W,P) are OISs, as well as $(U,A) \sim_h (W,P)$ with $h=(h_1,h_2,h_3)$. Then $\forall\, A_1, A_2 \subseteq A$,

1. $I(A_1) \sqsubset I(A_2) \Leftrightarrow I(h_2(A_1)) \sqsubset I(h_2(A_2))$;
2. $I(A_1) \bowtie I(A_2) \Leftrightarrow I(h_2(A_1)) \bowtie I(h_2(A_2))$.

Proof. (1) It can be proved by Theorems 6.11 and 6.13.

(2) This follows from Theorem 6.13.

Example 6.15. Assume that (U,A) and (W,P) are two OISs (see Tables 2 and 3).

A map $h_1 : U \to W$ is defined as follows:

u_1, u_{10}	u_2, u_8, u_{11}	u_3, u_6, u_7, u_{14}	u_4	u_5, u_9, u_{12}	u_{13}, u_{15}
w_1	w_2	w_3	w_4	w_5	w_6

Table 2 An ordered information system (OIS) (U, A).

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
u_1	2	{0, 1}	2	0.1	{0}	2	0.1	4	2	0.2
u_2	6	{0, 2}	4	0.3	{0, 1}	4	0.2	4	4	0.2
u_3	2	{1}	4	0.1	{0, 1}	4	0.2	2	4	0.1
u_4	4	{2}	6	0.2	{0, 1, 2}	6	0.3	2	6	0.1
u_5	6	{0, 1, 2}	4	0.3	{1, 2}	4	0.2	6	4	0.3
u_6	2	{0}	4	0.1	{0, 2}	4	0.2	2	4	0.1
u_7	2	{0}	4	0.1	{0, 1}	4	0.2	2	4	0.1
u_8	6	{1, 2}	4	0.3	{1, 2}	4	0.2	4	4	0.2
u_9	6	{0, 1, 2}	4	0.3	{0, 1}	4	0.2	6	4	0.3
u_{10}	2	{1, 2}	2	0.1	{1}	2	0.1	4	2	0.2
u_{11}	6	{0, 2}	4	0.3	{0, 1}	4	0.2	4	4	0.2
u_{12}	6	{0, 1, 2}	4	0.3	{1, 2}	4	0.2	6	4	0.3
u_{13}	6	{0, 2}	6	0.3	{0, 1, 2}	6	0.3	4	6	0.2
u_{14}	2	{2}	4	0.1	{0, 2}	4	0.2	2	4	0.1
u_{15}	6	{0, 1}	6	0.3	{0, 1, 2}	6	0.3	4	6	0.2

Table 3 An ordered information system (OIS) (W, P).

	b_1	b_2	b_3
w_1	1	2	1
w_2	3	2	2
w_3	1	1	2
w_4	2	1	3
w_5	3	3	2
w_6	3	2	3

A map $h_2 : A \to P$ is defined as follows:

$$\frac{a_1, a_4 \quad\quad a_2, a_8, a_{10} \quad\quad a_3, a_5, a_6, a_7, a_9}{b_1 \quad\quad\quad b_2 \quad\quad\quad\quad b_3}.$$

Put $U_{a_i} = \{a_i(u_s) : 1 \leqslant s \leqslant 15\}$, $W_{b_j} = \{b_j(w_t) : 1 \leqslant t \leqslant 6\}$. Then

$$U^* = \bigcup_{i=1}^{10} U_{a_i}, \ W^* = \bigcup_{j=1}^{3} W_{b_j}.$$

A map $h_3 : U^* \to W^*$ is defined as follows:

$$\forall\, w \in U_a,\ h_3 \restriction_{U_a} (w) = \begin{cases} \frac{1}{2} w, & a = a_1, a_3, a_6, a_8 \text{ or } a_9 \\ |w|, & a = a_2 \text{ or } a_5 \\ 10w, & a = a_4, a_7 \text{ or } a_{10} \end{cases}.$$

Obviously, $\forall\, u \in U$ and $a \in A$,

$$h_3(a(u)) = h_2(a)(h_1(u)).$$

Therefore $(U, A) \sim_h (W, P)$ with $h = (h_1, h_2, h_3)$.

Pick $Q_1 = \{b_1\}$, $Q_2 = \{b_2\}$, $Q_3 = \{b_3\}$, $Q_4 = \{b_1, b_2\}$. $Q_5 = \{b_1, b_3\}$, $Q_6 = \{b_2, b_3\}$.

It is easy to verify that

$I(Q) = I(Q_6)$;

$I(Q) \prec I(Q_4) \prec I(Q_1) \prec I(\varnothing)$, $I(Q) \prec I(Q_5) \prec I(Q_3) \prec I(\varnothing)$, $I(Q) \prec I(Q_2) \prec I(\varnothing)$, $I(Q_4) \prec I(Q_2)$, $I(Q_4) \prec I(Q_3)$, $I(Q_5) \prec I(Q_1)$.

$I(Q_1) \sqsubset I(Q_2)$, $I(Q_2) \sqsubset I(Q_1)$, $I(Q_1) \sqsubset I(Q_3)$, $I(Q_3) \sqsubset I(Q_1)$, $I(Q_2) \sqsubset I(Q_3)$, $I(Q_3) \sqsubset I(Q_2)$, $I(Q_2) \sqsubset I(Q_5)$, $I(Q_5) \sqsubset I(Q_2)$, $I(Q_4) \sqsubset I(Q_5)$, $I(Q_5) \sqsubset I(Q_4)$.

Now, dependence and independence between i-structures of the image OIS (W, P) are obtained. So these also can be obtained from the original OIS (U, A).

Pick $P^* = \{a_1, a_4\}$, $P' = \{a_9\}$. Then $h_2(P^*) = Q_1$, $h_2(P') = Q_3$.

Note that

$U/R^{\geqslant}_{P^*} = \{U, \{u_2, u_5, u_8, u_9, u_{11}, u_{12}, u_{13}, u_{15}\}, \{u_2, u_4, u_5, u_8, u_9, u_{11}, u_{12}, u_{13}, u_{15}\}\}$,

$U/R^{\geqslant}_{P'} = \{U, \{u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}\}, \{u_4, u_{13}, u_{15}\}\}$.

$W/R^{\geqslant}_{Q_1} = \{W, \{w_2, w_5, w_6\}, \{w_2, w_4, w_5, w_6\}\}$,

$W/R^{\geqslant}_{Q_3} = \{W, \{w_2, w_3, w_4, w_5, w_6\}, \{w_4, w_6\}\}$.

Clearly,

$I(P^*) = (U, \{u_2, u_5, u_8, u_9, u_{11}, u_{12}, u_{13}, u_{15}\}, U, \{u_2, u_4, u_5, u_8, u_9, u_{11}, u_{12}, u_{13}, u_{15}\}, \{u_2, u_5, u_8, u_9, u_{11}, u_{12}, u_{13}, u_{15}\}, U,\ U,\ \{u_2, u_5, u_8, u_9, u_{11}, u_{12}, u_{13}, u_{15}\}, \{u_2, u_5, u_8, u_9, u_{11}, u_{12}, u_{13}, u_{15}\}, U,\ \{u_2, u_5, u_8, u_9, u_{11}, u_{12}, u_{13}, u_{15}\}, \{u_2, u_5, u_8, u_9, u_{11}, u_{12}, u_{13}, u_{15}\}, \{u_2, u_5, u_8, u_9, u_{11}, u_{12}, u_{13}, u_{15}\}, U,\ \{u_2, u_5, u_8, u_9, u_{11}, u_{12}, u_{13}, u_{15}\})$,

$I(P') = (U, \{u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}\}, \{u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}\}, \{u_4, u_{13}, u_{15}\}, \{u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}\}, \{u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}\}, \{u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}\}, \{u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}\}, \{u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}\}, U, \{u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}\}, \{u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}\}, \{u_4, u_{13}, u_{15}\}, \{u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}\}, \{u_4, u_{13}, u_{15}\})$,

$I(Q_1) = (W, \{w_2, w_5, w_6\}, W, \{w_2, w_4, w_5, w_6\}, \{w_2, w_5, w_6\}, \{w_2, w_5, w_6\})$,

$I(Q_3) = (W, \{w_2, w_3, w_4, w_5, w_6\}, \{w_2, w_3, w_4, w_5, w_6\}, \{w_4, w_6\}, \{w_2, w_3, w_4, w_5, w_6\}, \{w_4, w_6\})$.

Then

$$d(I(P^*), I(P')) = \frac{1}{15^2} \sum_{i=1}^{15} |\, [u_i]^{\geqslant}_{P^*} \oplus [u_i]^{\geqslant}_{P'} \,| = \frac{58}{225},$$

$$d(I(h_2(P^*)), I(h_2(P'))) = d(I(Q_1), I(Q_3)) = \frac{1}{6^2} \sum_{j=1}^{6} |\, [w_j]^{\geqslant}_{Q_1} \oplus [w_j]^{\geqslant}_{Q_3} \,| = \frac{10}{36}.$$

Thus $d(I(P^*), I(P')) \neq d(I(h_2(P^*)), I(h_2(P')))$.

From this example and these discussions, we can know that a complex massive OIS can be compressed into a relatively small-scale OIS and some same data structures can be received.

7. UNCERTAINTY MEASURES OF AN OIS AND THE OPTIMAL SELECTION OF *i*-STRUCTURES BASED ON THEM

In this section, we select the optimal i-structure in an OIS based on uncertainty measures.

7.1. Uncertainty Measures of an OIS

Definition 7.1. Consider that (U, A) is an OIS. Suppose $B \subseteq A$. Then information granulation (i-granulation) of the subsystem (U, B) is defined as

$$G(B) = \frac{1}{n^2}\sum_{i=1}^{n}|[u_i]_B^{\succcurlyeq}|^2.$$

In [21], $R_B^{\succcurlyeq}$ is viewed as a knowledge and then $GK(R_B^{\succcurlyeq}) = \frac{1}{n^2}\sum_{i=1}^{n}|[u_i]_B^{\succcurlyeq}|^2$ is called granulation of the knowledge $R_B^{\succcurlyeq}$. Actually,

$$GK(R_B^{\succcurlyeq}) = G(B).$$

Theorem 7.2. *(Equivalence) Let (U, A) be an OIS, and $U/B^{\succcurlyeq} = \{[u]_B^{\succcurlyeq}|u \in U\}$, $U/C^{\succcurlyeq} = \{[u]_C^{\succcurlyeq}|u \in U\}$ be classifications of two dominance relations $B^{\succcurlyeq}$ and $C^{\succcurlyeq}$ respectively. If $|U/B^{\succcurlyeq}| = |U/C^{\succcurlyeq}|$, and it exists a bijective map h: $U/B^{\succcurlyeq} \to U/C^{\succcurlyeq}$ such that $|[u]_B^{\succcurlyeq}| = |h([u]_C^{\succcurlyeq})|$, then $G(B) = G(C)$.*

Proof. It can be achieved by Definition 7.1.

Proposition 7.3. *Assume that (U, A) is an OIS. Then for any $B \subseteq A$,*

$$\frac{1}{n} \leqslant G(B) \leqslant n.$$

Moreover, if $R_B^{\succcurlyeq}$ is an identity relation on U, then $G(B)$ achieves the minimum value $\frac{1}{n}$; if $R_B^{\succcurlyeq}$ is a universal relation on U, then $G(B)$ achieves the maximum value n.

Proof. Since for each i, $1 \leqslant |[u]_B^{\succcurlyeq}| \leqslant n$, we have $n \leqslant \sum_{i=1}^{n}|[u]_B^{\succcurlyeq}|^2 \leqslant n^3$.

By Definition 7.1,

$$\frac{1}{n} \leqslant G(B) \leqslant n.$$

If $R_B^{\succcurlyeq}$ is an identity relation on U, then $\forall\, i$, $|[u]_B^{\succcurlyeq}| = 1$. So $G(B) = \frac{1}{n}$.

If $R_B^{\succcurlyeq}$ is a universal relation on U, then $\forall\, i$, $|[u]_B^{\succcurlyeq}| = n$. So $G(B) = n$.

Theorem 7.4. *Suppose that (U, A) is an OIS. Suppose $B, C \subseteq A$.*

1. *If $I(B) \preceq I(C)$, then $G(B) \leqslant G(C)$.*
2. *If $I(B) \prec I(C)$, then $G(B) < G(C)$.*

Proof. (1) This is obvious.

(2) By Definition 7.1,

$$G(B) = \frac{1}{n^2}\sum_{i=1}^{n}|[u_i]_B^{\succcurlyeq}|^2,\ G(C) = \frac{1}{n^2}\sum_{i=1}^{n}|[u_i]_C^{\succcurlyeq}|^2.$$

Notice the $I(B) \prec I(C)$. Then $\forall\, i$, $[u_i]_B^{\succcurlyeq} \subseteq [u_i]_C^{\succcurlyeq}$ and $\exists\, j$, $[u_j]_B^{\succcurlyeq} \subsetneq [u_j]_C^{\succcurlyeq}$. Thus $\forall\, i$, $|[u_i]_B^{\succcurlyeq}| \leqslant |[u_i]_C^{\succcurlyeq}|$ and $\exists\, j$, $|[u_j]_B^{\succcurlyeq}| < |[u_j]_C^{\succcurlyeq}|$.

Consequently, $G(B) < G(C)$.

This theorem shows that when the available information turns into coarse, the i-granulation increases, and when the available information develops into finer, the i-granulation decreases. That is to say, the greater the uncertainty of the existing information, the greater the value of the i-granulation. Thus, we can conclude that the i-granulation introduced in Definition 7.1 can be used to assess the degree of OIS.

Proposition 7.5. *Assume that (U, A) is an OIS. If $B \subseteq C \subseteq A$, then $G(C) \preceq G(B)$.*

Proof. It can be proved by Theorem 7.4 and Proposition 5.3.

Definition 7.6. [21] Suppose that (U, A) is an OIS. Given $B \subseteq A$. Then rough entropy (r-entropy) of the subsystem (U, B) is defined as

$$E_r(B) = -\sum_{i=1}^{n}\frac{1}{n}log_2\frac{1}{|[u_i]_B^{\succcurlyeq}|}.$$

In [21], $R_B^{\succcurlyeq}$ is viewed as a knowledge and then $E_r(R_B^{\succcurlyeq}) = -\sum_{i=1}^{n}\frac{1}{n}log_2\frac{1}{|[u]_B^{\succcurlyeq}|}$ is called r-entropy of the knowledge $R_B^{\succcurlyeq}$. Actually,

$$E_r(R_B^{\succcurlyeq}) = E_r(B).$$

Proposition 7.7. *Assume that (U, A) is an OIS. Given $B \subseteq A$. Then*

$$0 \leqslant E_r(B) \leqslant n\log_2 n.$$

Furthermore, if $R_B^{\succcurlyeq}$ is an identity relation on U, then $E_r(B)$ achieves the minimum value 0; if $R_B^{\succcurlyeq}$ is a universal relation on U, then $E_r(B)$ achieves the maximum value $n\log_2 n$.

Proof. Since $\forall\, i$, $1 \leqslant |[u_i]_B^{\succcurlyeq}| \leqslant n$, we have $0 \leqslant -\log_2\frac{1}{|[u_i]_B^{\succcurlyeq}|} = \log_2|[u_i]_B^{\succcurlyeq}| \leqslant \log_2 n$, $0 \leqslant \frac{|[u_i]_B^{\succcurlyeq}|}{n} \leqslant 1$.

Then

$$0 \leqslant \frac{|[u_i]_B^{\succcurlyeq}|}{n}\log_2\frac{1}{|[u_i]_B^{\succcurlyeq}|} \leqslant \log_2 n.$$

By Definition 7.6,

$$0 \leqslant E_r(B) \leqslant n\log_2 n.$$

If $R_B^{\succcurlyeq}$ is an identity relation on U, then $\forall\, i$, $|[u_i]_B^{\succcurlyeq}| = 1$. So $E_r(B) = 0$.

If $R_B^{\geqslant}$ is a universal relation on U, then $\forall\, i$, $|[u_i]_B^{\geqslant}| = n$. So $E_r(B) = n\log_2 n$.

Theorem 7.8. *Let (U, A) be an OIS. Suppose $B, C \subseteq A$.*

1. *If $I(B) \preceq I(C)$, then $E_r(B) \leqslant E_r(C)$.*
2. *If $I(B) \prec I(C)$, then $E_r(B) < E_r(C)$.*

Proof. (1) This is obvious.

(2) By Definition 7.6, $E_r(B) = -\sum_{i=1}^{n} \frac{|[u_i]_B^{\geqslant}|}{n} \log_2 \frac{1}{|[u_i]_B^{\geqslant}|}$, $E_r(C) = -\sum_{i=1}^{n} \frac{|[u_i]_C^{\geqslant}|}{n} \log_2 \frac{1}{|[u_i]_C^{\geqslant}|}$.

Note that $I(B) \prec I(C)$. Then $\forall\, i$, $[u_i]_B^{\geqslant} \subseteq [u_i]_C^{\geqslant}$ and $\exists\, j$, $[u_j]_B^{\geqslant} \subsetneq [u_j]_C^{\geqslant}$. Thus $\forall\, i$, $|[u_i]_B^{\geqslant}| \leqslant |[u_i]_C^{\geqslant}|$ and $\exists\, j$, $|[u_j]_B^{\geqslant}| < |[u_j]_C^{\geqslant}|$.

Hence, $\forall\, i$,

$$-|[u_i]_B^{\geqslant}| \log_2 \frac{1}{|[u_i]_B^{\geqslant}|} = |[u_i]_B^{\geqslant}| \log_2 |[u_i]_B^{\geqslant}|$$

$$\leqslant |[u_i]_C^{\geqslant}| \log_2 |[u_i]_C^{\geqslant}| = -|[u_i]_C^{\geqslant}| \log_2 \frac{1}{|[u_i]_C^{\geqslant}|},$$

$\exists\, j$,

$$-|[u_j]_B^{\geqslant}| \log_2 \frac{1}{|[u_j]_B^{\geqslant}|} = |[u_j]_B^{\geqslant}| \log_2 |[u_j]_B^{\geqslant}|$$

$$< |[u_j]_C^{\geqslant}| \log_2 |[u_j]_C^{\geqslant}| = -|[u_j]_C^{\geqslant}| \log_2 \frac{1}{|[u_j]_C^{\geqslant}|}.$$

Therefore $E_r(B) < E_r(C)$.

This theorem shows that the greater the uncertainty of the available information, the greater the r-entropy. Therefore, we can draw the conclusion that the r-entropy proposed in Definition 7.6 can be used to evaluate the degree of determination of OIS.

Proposition 7.9. *Given that (U, A) is an OIS. If $B \subseteq C \subseteq A$, then $E_r^{\theta}(C) \preceq E_r^{\theta}(B)$.*

Proof. This follows from Theorem 7.8.

Definition 7.10. Let (U, A) be an OIS. Given $B \subseteq A$. Then information entropy (i-entropy) of (U, B) is defined by

$$E(B) = \sum_{i=1}^{n} \frac{1}{n}\left(1 - \frac{|\,[u_i]_B^{\geqslant}\,|}{n}\right).$$

In [21], $R_B^{\geqslant}$ is viewed as a knowledge and then $E(R_B^{\geqslant}) = \sum_{i=1}^{n} \frac{1}{n}\left(1 - \frac{|\,[u_i]_B^{\geqslant}\,|}{n}\right)$ is called i-entropy of the knowledge $R_B^{\geqslant}$. Actually,

$$E(R_B^{\geqslant}) = E(B).$$

Theorem 7.11. *Let (U, A) be an OIS. Then for $B \subseteq A$,*

$$E(B) = d(I(B), I(\varnothing)).$$

Proof. By,

$$E(B) = \sum_{i=1}^{n} \frac{1}{n}\left(1 - \frac{|\,[u_i]_B^{\geqslant}\,|}{n}\right).$$

But

$$d(I(B), I(\varnothing)) = \frac{1}{n^2}\sum_{i=1}^{n} |\,[u_i]_B^{\geqslant} \oplus U\,|$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} (|\,[u_i]_B^{\geqslant} \cup U\,| - |\,[u_i]_B^{\geqslant} \cap U\,|)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} (n - |\,[u_i]_B^{\geqslant}\,|)$$

$$= \sum_{i=1}^{n} \frac{1}{n}\left(1 - \frac{|\,[u_i]_B^{\geqslant}\,|}{n}\right).$$

Therefore

$$E(B) = d(I(B), I(\varnothing)).$$

Example 7.12. (Continued from Example 3.4)

$$E(A_1) = E(A_2) = \frac{11}{36},$$

$$E(A_3) = d(I(A_3), I(\varnothing)) = \frac{11}{36},$$

$$E(A_4) = d(I(A_4), I(\varnothing)) = \frac{15}{36},$$

$$E(A_5) = d(I(A_5), I(\varnothing)) = \frac{16}{36},$$

$$E(A_6) = E(A) = \frac{20}{36}.$$

Proposition 7.13. *Assume that (U, A) is an OIS. Suppose $B \subseteq A$. Then*

$$0 \leqslant E(B) \leqslant 1 - \frac{1}{n}.$$

Proof. The proof is similar to Proposition 7.7.

Theorem 7.14. *Let (U, A) be an OIS. Given $B, C \subseteq A$.*

1. *If $I(B) \preceq I(C)$, then $E(B) \leqslant E(C)$.*
2. *If $I(B) \prec I(C)$, then $E(B) < E(C)$.*

Proof. (1) This is obvious.

(2) By Definition 7.10,

$$E(B)=\sum_{i=1}^{n}\frac{1}{n}\left(1-\frac{|[u_i]_B^{\geqslant}|}{n}\right),\ E(C)=\sum_{i=1}^{n}\frac{1}{n}\left(1-\frac{|[u_i]_C^{\geqslant}|}{n}\right).$$

Note that $I(B)\prec I(C)$. Then $\forall\, i, [u_i]_B^{\geqslant}\subseteq[u_i]_C^{\geqslant}$ and $\exists\, j, [u_j]_B^{\geqslant}\subsetneq[u_j]_C^{\geqslant}$. Thus $\forall\, i, |[u_i]_B^{\geqslant}|\leqslant|[u_i]_C^{\geqslant}|$ and $\exists\, j, |[u_j]_B^{\geqslant}|<|[u_j]_C^{\geqslant}|$.

It is evident that $E(B)<E(C)$.

This theorem shows that if the structure of OIS turns into finer, the i-entropy decreases, and if the structure of OIS becomes rough, the i-entropy increases.

Proposition 7.15. *Consider that (U,A) is an OIS. If $B\subseteq C\subseteq A$, then $E(C)\leqslant E(B)$.*

Proof. It can be proved by Theorem 7.14.

7.2. Effectiveness Analysis

To evaluate the expression of the presented measures for the uncertainty of FA-spaces, effectiveness analysis is performed by using the standard deviation coefficient.

Assume that $D=\{d_1,d_2,\cdots,d_n\}$ is a dataset. Its arithmetic average is $\overline{d}=\frac{1}{n}\sum_{i=1}^{n}d_i$, its standard deviation is $\sigma(D)=\sqrt{\frac{1}{n}\sum_{i=1}^{n}(d_i-\overline{d})^2}$, and its standard deviation coefficient (for short, CV) is

$$CV(D)=\frac{\sigma(D)}{\overline{d}}.$$

We select four datasets (i.e., Energy efficiency, Airfoil Self-Noise, QSAR fish toxicity and HCV for Egyptian patients) from UCI for effectiveness analysis.

Energy efficiency may express OIS (U,A) with $|U|=768, |A|=8$. Denote $A_i=\{a_1,\cdots,a_i\}$ $(i=1,\cdots,8)$. Then for each i, $R_{A_i}^{\geqslant}$ is the dominance relation induced by A_i in Energy efficiency. Three measure sets on Energy efficiency are defined as follows:

$$D_G(En)=\{G(A_1),\cdots,G(A_8)\},$$

$$D_E(En)=\{E(A_1),\cdots,E(A_8)\},$$

$$D_{E_r}(En)=\{E_r(A_1),\cdots,E_r(A_8)\}.$$

Airfoil Self-Noise may express OIS (V,B) with $|V|=1503, |B|=6$. Denote $B_i=\{b_1,\cdots,b_i\}$ $(i=1,\cdots,6)$. Then for each i, $R_{B_i}^{\geqslant}$ is the dominance relation induced by B_i in Airfoil Self-Noise. Three measure sets on Airfoil Self-Noise are defined as follows:

$$D_G(Ai)=\{G(B_1),\cdots,G(B_6)\},$$

$$D_E(Ai)=\{E(B_1),\cdots,E(B_6)\},$$

$$D_{E_r}(Ai)=\{E_r(B_1),\cdots,E_r(B_6)\}.$$

QSAR fish toxicity may express OIS (W,C) with $|W|=908, |C|=7$. Denote $C_i=\{c_1,\cdots,c_i\}$ $(i=1,\cdots,7)$. Then for each i, $R_{C_i}^{\geqslant}$ is the dominance relation induced by C_i in QSAR fish toxicity. Three measure sets on QSAR fish toxicity are defined as follows:

$$D_G(QS)=\{G(C_1),\cdots,G(C_7)\},$$

$$D_E(QS)=\{E(C_1),\cdots,E(C_7)\},$$

$$D_{E_r}(QS)=\{E_r(C_1),\cdots,E_r(C_7)\}.$$

HCV for Egyptian patients may express OIS (X,D) with $|X|=1385, |D|=29$. Denote $D_i=\{d_1,\cdots,d_i\}$ $(i=1,\cdots,29)$. Then for each i, $R_{D_i}^{\geqslant}$ is the dominance relation induced by D_i in HCV for Egyptian patients. Three measure sets on HCV for Egyptian patients are defined as follows:

$$D_G(Ht)=\{G(D_1),\cdots,G(D_{29})\},$$

$$D_E(Ht)=\{E(D_1),\cdots,E(D_{29})\},$$

$$D_{E_r}(Ht)=\{E_r(D_1),\cdots,E_r(D_{29})\}.$$

Then CV-values of three measure sets are compared. The results are shown in Figure 4.

The dispersion degree of E is minimum when Energy efficiency is selected as the test set. Similarly, the dispersion degree of E is minimum when other data sets (i.e., Airfoil Self-Noise, QSAR fish toxicity and HCV for Egyptian patients) are selected as the test sets.

Thus, i-entropy E has much better performance for measuring uncertainty of an OIS.

7.3. The Optimal Selection of *i*-Structures in an OIS Based on *i*-Granulation

In this subsection, we select the optimize i-structure based on i-granulation.

Definition 7.16. Let (U,A) be an OIS.

1. If there exists $B^*\subseteq A$ such that $G(B^*)=max\{G(B):B\subseteq A\}$, then $I(B^*)$ is called the maximum i-structure in (U,A) based on i-granulation;
2. If there exists $B^*\subseteq A$ such that $G(B^*)=min\{G(B):B\subseteq A\}$, then $I(B^*)$ is called the minimum i-structure in (U,A) based on i-granulation.

The maximum i-structure and the minimum i-structure in (U,A) are collectively called the optimal i-structure in (U,A) based on i-granulation.

Theorem 7.17. *Let (U,A) be an OIS. Then $I(A)$ is the minimum i-structure in (U,A) based on i-granulation.*

Proof. This follows from Proposition 7.5.

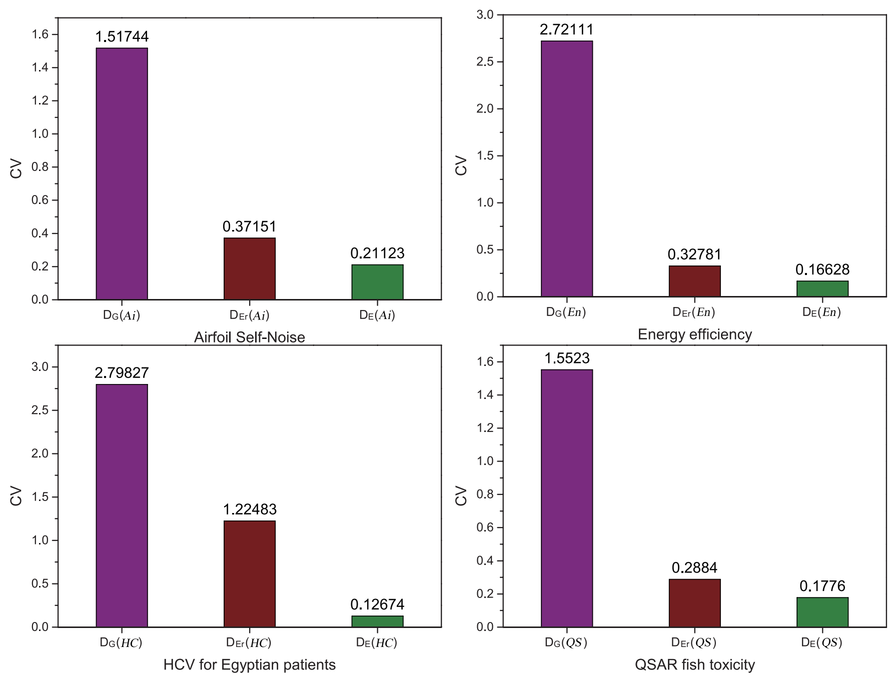

Figure 4 CV-values of three measure sets.

Example 7.18. (Continued from Example 3.4) Given an OIS (U, A) in Table 1.

Pick $A_1 = \{a_1\}, A_2 = \{a_2\}, A_3 = \{a_3\}, A_4 = \{a_1, a_2\}, A_5 = \{a_1, a_3\}, A_6 = \{a_2, a_3\}, A = \{a_1, a_2, a_3\}$.

$$G(A_1) = 3.194;\ G(A_2) = 3.361,$$

$$G(A_3) = 3.306,\ G(A_4) = 2.417;$$

$$G(A_5) = 2.333,\ G(A_6) = G(A) = 1.556.$$

Then $G(A_6) = G(A) < G(A_5) < G(A_4) < G(A_3) < G(A_1) < G(A_2)$.

Thus, $I(A_2)$ is the maximum i-structure in (U, A) based on i-granulation; $I(A_6)$ and $I(A)$ are the minimum i-structure in (U, A) based on i-granulation.

7.4. The Optimal Selection of *i*-Structures in an OIS Based on *r*-Entropy

In this subsection, we select the optimal i-structure in (U, A) based on r-entropy.

Definition 7.19. Let (U, A) be an OIS.

1. If there exists $B^* \subseteq A$ such that $E_r(B^*) = max\{E_r(B) : B \subseteq A\}$, then $I(B^*)$ is called the maximum i-structure in (U, A) based on r-entropy;
2. If there exists $B^* \subseteq A$ such that $E_r(B^*) = min\{E_r(B) : B \subseteq A\}$, then $I(B^*)$ is called the minimum i-structure in (U, A) based on r-entropy.

The maximum i-structure and the minimum i-structure in (U, A) are collectively called the optimal i-structure in (U, A) based on r-entropy.

Example 7.20. (Continued from Example 3.4) Given an OIS (U, A) in Table 1.

Pick $A_1 = \{a_1\}, A_2 = \{a_2\}, A_3 = \{a_3\}, A_4 = \{a_1, a_2\}, A_5 = \{a_1, a_3\}, A_6 = \{a_2, a_3\}, A = \{a_1, a_2, a_3\}$.

$$E_r(A_1) = 1.988;\ E_r(A_2) = 1.862,$$

$$E_r(A_3) = 1.925,\ E_r(A_4) = 1.626;$$

$$E_r(A_5) = 1.513,\ E_r(A_6) = E_r(A) = 1.151.$$

We have $E_r(A_6) = E_r(A) < E_r(A_5 < E_r(A_4) < E_r(A_2) < E_r(A_3) < E_r(A_1)$.

Thus, $I(A_1)$ is the maximum i-structure in (U, A) based on r-entropy; $I(A_6)$ and $I(A)$ are the minimum i-structure in (U, A) based on r-entropy.

7.5. The Optimal Selection of *i*-Structures in an OIS Based on *i*-Entropy

In this subsection, we select the optimal i-structure in (U, A) based on i-entropy.

Definition 7.21. Let (U, A) be an OIS.

1. If there exists $B^* \subseteq A$ such that $E(B^*) = max\{E(B) : B \subseteq A\}$, then $I(B^*)$ is called the maximum i-structure in (U, A) based on i-entropy;

2. If there exists $B^* \subseteq A$ such that $E(B^*) = min\{E(B) : B \subseteq A\}$, then $I(B^*)$ is called the minimum i-structure in (U, A) based on i-entropy.

The maximum i-structure and the minimum i-structure in (U, A) are collectively called the optimal i-structure in (U, A) based on i-entropy.

Example 7.22. (Continued from Example 3.4) Given an OIS (U, A) in Table 1.

Pick $A_1 = \{a_1\}, A_2 = \{a_2\}, A_3 = \{a_3\}, A_4 = \{a_1, a_2\}, A_5 = \{a_1, a_3\}, A_6 = \{a_2, a_3\}, A = \{a_1, a_2, a_3\}$.

$$E(A_1) = 0.306;\ E(A_2) = 0.305,$$

$$E(A_3) = 0.306,\ E(A_4) = 0.417;$$

$$E(A_5) = 0.445,\ E(A_6) = E(A) = 0.556.$$

We have $E(A_2) < E(A_1 = E(A_3) < E(A_4) < E(A_5) < E(A_6) = E(A)$.

Thus, $I(A_6)$ and $I(A)$ are the maximum i-structure in (U, A) based on i-entropy; $I(A_2)$ is the minimum i-structure in (U, A) based on i-entropy.

8. COMPARISONS

In this section, we make a comparison with literatures [11,23] so as to see the innovation of this article more clearly.

1. Three articles are based on GrC. Thus, the research ideas of three articles are the same and the obtained results are similar.
2. The research path of three articles is the same, and the details are as follows: Granular structures or i-structures in three ISs are first introduced, and then dependence, independence and difference between i-structures are discussed. Finally, mathematical characteristics of i-structures are obtained.
3. The differences of three articles are as below.
   (a) The studied ISs are different: This article considers i-structure in an OIS, literature [11] studies i-structure in a covering IS and literature [23] investigates i-structure in an incomplete interval-valued IS.
   (b) The introduced relations are different: This article introduces the dominance relation on the object set in an OIS by defining the order between two information values, literature [11] proposes the similarity relation on the universe in a covering IS by means of the neighborhood of each pointand literature [23] presents the tolerance relation on the object set in an incomplete interval-valued IS by defining the similarity degree between two information values.
   (c) The constructed i-granules are different: This article constructs an i-granule by means of the dominance class on each object in an OIS, literature [11] constructs an i-granule by means of the similarity class on each point in a covering IS and literature [23] constructs an i-granule by means of the tolerance class on each object in an incomplete interval-valued IS.
   (d) This article considers homomorphisms between OISs and then obtains map characters of i-structures. Literature [11] considers also homomorphisms between covering ISs and obtains map characters of i-structures. But these two homomorphisms are totally different. One is based on an IS and the other is based on coverings of the universe. Moreover, literature [23] does not obtain map characters of i-structures as literature [23] does not considers homomorphisms between incomplete interval-valued ISs.
   (e) This article studies the optimal selection of i-structures in an OIS based on uncertainty measures. But literature [11,23] do not study the optimal selection of i-structures in an incomplete interval-valued IS.

Table 4 Invariant characters of i-structures in an ordered information system (OIS) under homomorphisms.

Characterizations	*Invariant*
Equality between i-structures in an OIS ($=$)	√
Dependence between i-structures in an OIS ($\preceq$)	√
Strict dependence between i-structures in an OIS ($\prec$)	√
Partial dependence between i-structures in an OIS ($\sqsubseteq$)	√
Strictly partial dependence between i-structures in an OIS ($\sqsubset$)	√
Independence between i-structures in an OIS ($\bowtie$)	√
Information distance between i-structures in an OIS (**d**)	×

9. CONCLUSIONS

In this article, i-structures in an OIS have been introduced. Dependence and independence between i-structures have been depicted. Properties of i-structures are investigated by utilizing the inclusion degree. Groups, lattice and map characters of i-structures in an OIS have been obtained. Invariant characters of i-structures in an OIS under homomorphisms have been shown (see Table 4). The optimal i-structures in an OIS based on the proposed measures have been selected. These results will contribute to establishing a framework of GrC in an OIS. This vector-based framework is able to be employ to describe information's different dimension in an OIS and may contribute to knowledge discovery in an OIS. In future, we will provide some applications to deal with knowledge discovery in an OIS.

CONFLICTS OF INTEREST

The authors declare that they have no conflict of interest.

AUTHORS' CONTRIBUTIONS

Y.N. Wang designs the overall structure of this paper and improves the language, S.C. Wang writes the paper, and H.X. Tang collects the data.

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APPENDIX

Theorem 5.8. *Let $S^{\succcurlyeq} = (U, A)$ be an OIS. Suppose $P, A_2 \subset A$. Then*

1. $I(A_1) \preceq I(A_2) \Leftrightarrow D(I(A_2)/I(A_1)) = 1.$
2. $I(A_1) \bowtie I(A_2) \Leftrightarrow D(I(A_2)/I(A_1)) = 0.$
3. $I(A_1) \sqsubseteq I(A_2) \Leftrightarrow 0 < D(I(A_2)/I(A_1)) \leqslant 1.$

Proof. (1) Obviously, "$\Rightarrow$" holds.

"$\Leftarrow$". Put

$$|[u_l]^{\succcurlyeq}_{A_2}| = q_l, \quad \sum_{l=1}^{n} |[u_l]^{\succcurlyeq}_{A_2}| = q.$$

Then $q = \sum_{l=1}^{n} q_l$. Due to $D(I(A_2)/I(A_1)) = 1$, it can be obtained that

$$\sum_{l=1}^{n} q_l \chi_{[u_l]^{\succcurlyeq}_{A_2}}([u_l]^{\succcurlyeq}_{A_1}) = q = \sum_{l=1}^{n} q_l.$$

Then

$$\sum_{l=1}^{n} q_l(1 - \chi_{[u_l]^{\succcurlyeq}_{A_2}}([u_l]^{\succcurlyeq}_{A_1})) = 0.$$

Thus $\forall\, l$,

$$1 - \chi_{[u_l]^{\succcurlyeq}_{A_2}}([u_l]^{\succcurlyeq}_{A_1}) = 0.$$

It follows that $\forall\, l$, $[u_l]^{\succcurlyeq}_{A_1} \subseteq [u_l]^{\succcurlyeq}_{A_2}$.

Hence $I(A_1) \preceq I(A_2)$.

(2) "$\Rightarrow$". On account of $I(A_1) \bowtie I(A_2)$, $\forall\, l$, $[u_l]^{\succcurlyeq}_{A_1} \nsubseteq [u_l]^{\succcurlyeq}_{A_2}$, then $\forall\, l$,

$$\chi_{[u_l]^{\succcurlyeq}_{A_2}}([u_l]^{\succcurlyeq}_{A_1}) = 0.$$

Thus $D(I(A_2)/I(A_1)) = 0$.

"$\Leftarrow$". Due to $D(I(A_2)/I(A_1)) = 0$, $\forall\, l$,

$$\chi_{[u_l]^{\succcurlyeq}_{A_2}}([u_l]^{\succcurlyeq}_{A_1}) = 0.$$

Then $\forall\, l$, $[u_l]^{\succcurlyeq}_{A_1} \nsubseteq [u_l]^{\succcurlyeq}_{A_2}$. Therefore $I(A_1) \bowtie I(A_2)$.

(3) It can be obtained from (2).

Theorem 6.3. *Let $S^{\succcurlyeq} = (U, A)$ be an OIS. Then*

1. $L = (\mathbf{K}(U, A), \preceq)$ *is a lattice with* $1_L = I(\emptyset)$ *and* $0_L = I(A)$;
2. *If* $A_1, A_2 \subseteq A$ *and* $\mathcal{A}(A_1, A_2) = \{A^* : A^* \subseteq A,\ R^{\succcurlyeq}_{A_1} \cup R^{\succcurlyeq}_{A_2} \subseteq R^{\succcurlyeq}_{A^*}\}$, *then*

$$I(A_1) \wedge I(A_2) = I(A_1) \odot I(A_2) = I(A_1 \cup A_2),$$

$$I(A_1) \vee I(A_2) = \bigodot_{A^* \in \mathcal{A}(A_1, A_2)} I(A_1) = I(\cup \mathcal{A}(A_1, A_2)).$$

Proof. Clearly, $\forall\, P \subseteq A$, $I(A_1) \preceq I(A_1)$.

Assume that $I(A_1) \preceq I(A_2)$, $I(A_2) \preceq I(A_1)$. By Theorem 5.2, $R^{\succcurlyeq}_{A_1} \subseteq R^{\succcurlyeq}_{A_2}$, $R^{\succcurlyeq}_{A_2} \subseteq R^{\succcurlyeq}_{A_1}$. Then $R^{\succcurlyeq}_{A_1} = R^{\succcurlyeq}_{A_2}$. Thus $I(A_1) \preceq I(A_2)$.

Assume that $I(A_1) \preceq I(A_2)$, $I(A_2) \preceq I(L)$. By Theorem 5.2, $R^{\succcurlyeq}_{A_1} \subseteq R^{\succcurlyeq}_{A_2}$, $R^{\succcurlyeq}_{A_2} \subseteq R^{\succcurlyeq}_{L}$. Then $R^{\succcurlyeq}_{A_1} \subseteq R^{\succcurlyeq}_{L}$. By Theorem 5.2, $I(A_1) \preceq I(L)$.

Hence $(\mathbf{K}(U, A), \preceq)$ is a partially ordered set.

By the proof of Theorem 5.1, it can be obtained that

$I(A_1) \odot I(A_2) = I(A_1 \cup A_2)$,

$\bigodot_{P \in \mathcal{A}(A_1, A_2)} I(A_1) = I(\cup \mathcal{A}(A_1, A_2))$.

On account of $A_1, A_2 \subseteq A_1 \cup A_2$, by Proposition 5.3, $I(A_1 \cup A_2) \preceq I(A_1)$ and $I(A_1 \cup A_2) \preceq I(A_2)$. Then $I(A_1 \cup A_2)$ is the lower bound of $\{I(A_1), I(A_2)\}$.

Assume that $I(A_1)$ is the lower bound of $\{I(A_1), I(A_2)\}$ with $P \subseteq A$. Then $I(A_1) \preceq I(A_1)$, $I(A_1) \preceq I(A_2)$. By Theorem 5.2, $R^{\succcurlyeq}_{A_1} \subseteq R^{\succcurlyeq}_{A_1}$ and $R^{\succcurlyeq}_{A_1} \subseteq R^{\succcurlyeq}_{A_2}$. Then $R^{\succcurlyeq}_{A_1} \subseteq R^{\succcurlyeq}_{A_1} \cap R^{\succcurlyeq}_{A_2} = R^{\succcurlyeq}_{A_1 \cup A_2}$. By Theorem 5.2, $I(A_1) \preceq I(A_1 \cup A_2)$.

Thus

$$I(A_1) \wedge I(A_2) = I(A_1 \cup A_2).$$

$\forall\, A^* \in \mathcal{A}(A_1, A_2)$, since $R^{\succcurlyeq}_{A_1} \cup R^{\succcurlyeq}_{A_2} \subseteq R^{\succcurlyeq}_{A^*}$, it can be obtained that

$$R^{\succcurlyeq}_{A_1}, R^{\succcurlyeq}_{A_2} \subseteq R^{\succcurlyeq}_{A^*}.$$

Then

$$R^{\succcurlyeq}_{A_1}, R^{\succcurlyeq}_{A_2} \subseteq \bigcap_{A^* \in \mathcal{A}(A_1, A_2)} R^{\succcurlyeq}_{A^*} = R^{\succcurlyeq}_{\bigcup_{A^* \in \mathcal{A}(A_1, A_2)} A^*} = R^{\succcurlyeq}_{(\cup \mathcal{A}(A_1, A_2))}.$$

By Theorem 5.2, $I(A_1), I(A_2) \preceq I(\cup \mathcal{A}(A_1, A_2))$. Then $I(\cup \mathcal{A}(A_1, A_2))$ is the upper bound of $\{I(A_1), I(A_2)\}$.

Assume that $I(S)$ is the upper bound of $\{I(A_1), I(A_2)\}$ with $S \subseteq A$. Then $I(A_1) \preceq I(S)$, $I(A_2) \preceq I(S)$. By Theorem 5.2, $R^{\succcurlyeq}_{A_1} \subseteq R^{\succcurlyeq}_{S}$ and $R^{\succcurlyeq}_{A_2} \subseteq R^{\succcurlyeq}_{S}$. Then $S \in \mathcal{A}(A_1, A_2)$. So $S \subseteq \cup \mathcal{A}(A_1, A_2)$. By Proposition 5.3, $I(\cup \mathcal{A}(A_1, A_2)) \preceq I(S)$.

Hence, $I(A_1) \vee I(A_2) = I(\cup \mathcal{A}(A_1, A_2))$.

Thus, L is a lattice.

Clearly, $1_L = I(\emptyset)$, $0_L = I(A)$.

Proposition 6.9. *Let (U, A) and (W, P) be two OISs. Assume $(U, A) \sim_h (W, P)$ with $h = (h_1, h_2, h_3)$. Then $\forall\, A_1, A_2 \subseteq A$,*

$$R^{\succcurlyeq}_{A_1} \subseteq R^{\succcurlyeq}_{A_2} \ \Leftrightarrow\ R^{\succcurlyeq}_{h_2(A_1)} \subseteq R^{\succcurlyeq}_{h_2(A_2)}.$$

Proof. Suppose $R^{\succcurlyeq}_{A_1} \subseteq R^{\succcurlyeq}_{A_2}$. $\forall\, (w, y) \in R^{\succcurlyeq}_{h_2(A_1)}$, it can be obtained that

$$\forall\, a \in A_1,\ h_2(a)(w) \succcurlyeq_{h_2(a)} h_2(a)(y).$$

Let $y = h_1(x)$, $w = h_1(u)$.

Then $\forall\, a \in A_1$, $h_2(a)(h_1(u)) \succcurlyeq_{h_2(a)} h_2(a)(h_1(x))$.

Additionally, $(U, A) \sim_h (W, P)$. Then $\forall\, a \in A$,

$$h_2(a)(h_1(x)) = h_3(a(x)),\; h_2(a)(h_1(u)) = h_3(a(u)).$$

So $\forall\, a \in A_1$, $h_3(a(u)) \geqslant_{h_2(a)} h_3(a(x))$.

Because $h_3 \upharpoonright_{U_a}$ is order-preserving, then $\forall\, a \in A_1$, $a(u) \geqslant_a a(x)$. This means $(u, x) \in R^{\geqslant}_{A_1}$. By hypothesis, $(u, x) \in R^{\geqslant}_{A_2}$, i.e., $\forall\, a \in A_2$, $a(u) \geqslant_a a(x)$. Because $h_3 \upharpoonright_{U_a}$ is order-preserving, it can be obtained that $\forall\, a \in A_2$,

$$h_3(a(u)) \geqslant_{h_2(a)} h_3(a(x)).$$

Then $\forall\, a \in A_2$,

$h_2(a)(w) = h_2(a)(h_1(u)) = h_3(a(u)) \geqslant_{h_2(a)} h_3(a(x)) = h_2(a)(h_1(x)) = h_2(a)(y)$.

This implies $(w, y) \in R^{\geqslant}_{h_2(A_2)}$.

Thus

$$R^{\geqslant}_{h_2(A_1)} \subseteq R^{\geqslant}_{h_2(A_2)}.$$

Conversely, suppose $R^{\geqslant}_{h_2(A_1)} \subseteq R^{\geqslant}_{h_2(A_2)}$, $\forall\, (u, x) \in R^{\geqslant}_{A_1}$, it can be obtained that

$$\forall\, a \in A_1,\; a(u) \geqslant_a a(x).$$

Because $h_3 \upharpoonright_{U_a}$ is order-preserving, then $\forall\, a \in A_1$, $h_3(a(u)) \geqslant_{h_2(a)} h_3(a(x))$.

Additionally, $(U, A) \sim_h (W, P)$. Then $\forall\, a \in A$,

$h_2(a)(h_1(x)) = h_3(a(x)),\; h_3(a(x)) = h_2(a)(h_1(u))$.

Thus $\forall\, a \in A_1$, $h_2(a)(h_1(u)) \geqslant_{h_2(a)} h_2(a)(h_1(x))$. This implies $(h_1(x), h_1(u)) \in R^{\geqslant}_{h_2(A_1)}$.

On account of $R^{\geqslant}_{h_2(A_1)} \subseteq R^{\geqslant}_{h_2(A_2)}$, it can be obtained that $(h_1(x), h_1(u)) \in R^{\geqslant}_{h_2(A_2)}$, i.e., $\forall\, a \in A_2$, $h_2(a)(h_1(u)) \geqslant_{h_2(a)} h_2(a)(h_1(x))$.

Then $\forall\, a \in A_2$, $h_3(a(u)) \geqslant_{h_2(a)} h_3(a(x))$.

Because $h_3 \upharpoonright_{U_a}$ is order-preserving, it can be obtained that $\forall\, a \in A_2$, $a(u) \geqslant_a a(x)$. This implies $(u, x) \in R^{\geqslant}_{A_2}$.

Thus

$$R^{\geqslant}_{A_1} \subseteq R^{\geqslant}_{A_2}.$$

Theorem 6.13. *Assume that (U, A) and (W, P) are two OISs. Suppose $(U, A) \sim_h (W, P)$ with $h = (h_1, h_2, h_3)$. Then $\forall\, A_1, A_2 \subseteq A$,*

$$I(A_1) \sqsubseteq I(A_2) \;\Leftrightarrow\; I(h_2(A_1)) \sqsubseteq I(h_2(A_2)).$$

Proof. Suppose $I(A_1) \sqsubseteq I(A_2)$. Then $\exists\, i$, $[x_i]^{\geqslant}_{A_1} \subseteq [x_i]^{\geqslant}_{A_2}$.

Let $y_j = h_1(x_i)$. $\forall\, w \in [y_j]^{\geqslant}_{h_2(A_1)}$, it can be obtained that $(w, y_j) \in R^{\geqslant}_{h_2(A_1)}$. This implies

$$\forall\, a \in A_1,\; h_2(a)(w) \geqslant_{h_2(a)} h_2(a)(y_j).$$

Let $w = h_1(u)$. Then $\forall\, a \in A_1$, $h_2(a)(h_1(u)) \geqslant_{h_2(a)} h_2(a)(h_1(x_i))$.

Additionally, $(U, A) \sim_h (W, P)$. Then $\forall\, a \in A$,

$$h_2(a)(h_1(x_i)) = h_3(a(x_i)),\; h_2(a)(h_1(u)) = h_3(a(u)).$$

So $\forall\, a \in A_1$, $h_3(a(u)) \geqslant_{h_2(a)} h_3(a(x_i))$.

Because $h_3 \upharpoonright_{U_a}$ is order-preserving, it can be obtained that $\forall\, a \in A_1$, $a(u) \geqslant_a a(x_i)$.

This implies $(u, x_i) \in R^{\geqslant}_{A_1}$. Then $u \in [x_i]^{\geqslant}_{A_1}$.

On account of $[x_i]^{\geqslant}_{A_1} \subseteq [x_i]^{\geqslant}_{A_2}$, it can be obtained that $u \in [x_i]^{\geqslant}_{A_2}$. So $(u, x_i) \in R^{\geqslant}_{A_2}$, i.e., $\forall\, a \in A_2$, $a(u) \geqslant_a a(x_i)$.

Because $h_3 \upharpoonright_{U_a}$ is order-preserving, then $\forall\, a \in A_2$,

$$h_3(a(u)) \geqslant_{h_2(a)} h_3(a(x_i)).$$

Then $\forall\, a \in A_2$,

$h_2(a)(w) = h_2(a)(h_1(u)) = h_3(a(u)) \geqslant_{h_2(a)} h_3(a(x_i)) = h_2(a)(h_1(x_i)) = h_2(a)(y_j)$.

So $(w, y_j) \in R^{\geqslant}_{h_2(A_2)}$.

This implies $w \in [y_j]^{\geqslant}_{h_2(A_2)}$

Thus

$$I(h_2(A_1)) \sqsubseteq I(h_2(A_2)).$$

Conversely, suppose $I(h_2(A_1)) \sqsubseteq I(h_2(A_2))$. Then $\exists\, j$, $[y_j]^{\geqslant}_{h_2(A_1)} \subseteq [y_j]^{\geqslant}_{h_2(A_2)}$.

Let $h_1(x_i) = y_j$. $\forall\, u \in [x_i]^{\geqslant}_{A_1}$, it can be obtained that $(u, x_i) \in R^{\geqslant}_{A_1}$. Then

$$\forall\, a \in A_1,\; a(u) \geqslant_a a(x_i).$$

Because $h_3 \upharpoonright_{U_a}$ is order-preserving, it can be obtained that $\forall\, a \in A_1$, $h_3(a(u)) \geqslant_{h_2(a)} h_3(a(x_i))$.

Additionally, $(U, A) \sim_h (W, P)$. Then $\forall\, a \in A$,

$$h_2(a)(h_1(x_i)) = h_3(a(x_i)),\; h_3(a(u)) = h_2(a)(h_1(u)).$$

Then $\forall\, a \in A_1$, $h_2(a)(h_1(u)) \geqslant_{h_2(a)} h_2(a)(h_1(x_i))$. This implies $(h_1(u), h_1(x_i)) \in R^{\geqslant}_{h_2(A_1)}$, i.e., $h_1(u) \in [h_1(x_i)]^{\geqslant}_{h_2(A_1)} = [y_j]^{\geqslant}_{h_2(A_1)}$.

On account of $[y_j]^{\geqslant}_{h_2(A_1)} \subseteq [y_j]^{\geqslant}_{h_2(A_2)}$, it can be obtained that $h_1(u) \in [y_j]^{\geqslant}_{h_2(A_2)}$, i.e., $(h_1(u), h_1(x_i)) = (h_1(u), y_j) \in R^{\geqslant}_{h_2(A_2)}$.

Then, $\forall\, a \in A_2$, $h_2(a)(h_1(u)) \geqslant_{h_2(a)} h_2(a)(h_1(x_i))$.

So $\forall\, a \in A_2$, $h_3(a(u)) \geqslant_{h_2(a)} h_3(a(x_i))$.

Because $h_3 \upharpoonright_{U_a}$ is order-preserving, it can be obtained that $\forall\, a \in A_2$, $a(u) \geqslant_a a(x_i)$. Then $(u, x_i) \in R^{\geqslant}_{A_2}$. This implies $u \in [x_i]^{\geqslant}_{A_2}$.

Thus

$$I(A_1) \sqsubseteq I(A_2).$$