

## Research Article

# Flexible Bootstrap for Fuzzy Data Based on the Canonical Representation

Przemysław Grzegorzewski<sup>1,2,\*</sup>, Olgierd Hryniewicz<sup>2</sup>, Maciej Romaniuk<sup>2,3</sup>

<sup>1</sup>Faculty of Mathematics and Information Science, Warsaw University of Technology, Koszykowa 75, 00-662 Warsaw, Poland

<sup>2</sup>Systems Research Institute, Polish Academy of Sciences, Newelska 6, 01-447 Warsaw, Poland

<sup>3</sup>Warsaw School of Information Technology, Newelska 6, 01-447 Warsaw, Poland

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## ABSTRACT

Several new resampling methods for generating bootstrap samples of fuzzy numbers are proposed. To avoid undesired repetitions in the secondary samples we do not draw randomly directly observations from the primary samples but construct them allowing for some modifications in their membership functions, however only such which do not disturb the canonical representation of the initial fuzzy numbers. We consider both two-parameter and three-parameter canonical representations, as well as the triangular and trapezoidal outputs in the secondary samples. Numerical experiments concerning some statistical tests based on fuzzy samples show that the suggested methods may appear helpful in statistical reasoning with imprecise data.

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## 1. INTRODUCTION

The bootstrap, introduced by Efron [1], is a widespread statistical technique for assessing uncertainty and solving various complex problems. For instance, it appears extremely useful for estimating standard errors, for computing confidence intervals or testing hypotheses in all those cases where no theoretical results are available and using normal approximation or any other parametric approach remains questionable.

Such a situation is typical in statistical inference based on imprecise data modelled with fuzzy numbers. Indeed, in most cases when the analyzed phenomena is described by fuzzy random variables (random fuzzy numbers) the underlying distribution remains unknown. There, the bootstrap is as not only helpful but sometimes appears as the onliest tool to conduct any statistical reasoning. In particular, the bootstrap was successfully applied in statistical tests for fuzzy data by Colubi *et al.* [2], Gil *et al.* [3], González-Rodríguez *et al.* [4,5], Grzegorzewski and Ramos-Guajardo [6], Montenegro *et al.* [7] and Ramos-Guajardo and Lubiano [8]. Some examples on the bootstrap application for solving the real-life problems include, e.g., fuzzy questionnaires rating (Ref. [9]), also in cheese manufacturing (Ref. [10]) or fuzzy control charts (Ref. [11]).

The main idea of the classical bootstrap consists in drawing random samples with replacement from the initial sample of the experiment

outcomes and then construct the empirical distribution through averaging the bootstrap samples. Such procedure, although simple yet efficient, has an important disadvantage: in the generated secondary samples we obtain only the values belonging to the initial (primary) sample and hence, nearly every bootstrap sample contains repeated values. If the primary sample size is small, all bootstrap samples consist of only few distinct values. Such a case seems to be strange especially if the unknown original distribution is continuous. To overcome this inconvenience some modifications and improvements of the classical bootstrap were proposed, like the balanced bootstrap by Davison *et al.* [12] or Graham *et al.* [13], as well as various kinds of the so-called smoothed bootstrap discussed by Silverman and Young [14], Hall *et al.* [15] or De Angelis and Young [16].

Excessive repetitions in bootstrap samples is also a problem in fuzzy modelling. To increase the diversity of simulated results some new approaches were proposed. Romaniuk and Hryniewicz [17,18] introduced two resampling methods in which new triangular fuzzy numbers were generated from the primary sample by adding some incremental spreads for  $\alpha$ -cuts. The resampling algorithms for interval-valued fuzzy numbers were considered in Ref. [19].

In this paper, we propose another modification of the bootstrap approach which prevents from undesired repetitions in secondary fuzzy sample generation. Our key idea is to generate such triangular or trapezoidal fuzzy numbers which have the same canonical

\*Corresponding author. Email: [p.grzegorzewski@mini.pw.edu.pl](mailto:p.grzegorzewski@mini.pw.edu.pl)

representations as fuzzy observations in the primary sample. As it is known, the canonical representation of a fuzzy number, proposed by Delgado *et al.* [20], summarizes information on basic characteristics of a fuzzy number. Obviously, two fuzzy numbers with identical canonical representation may have membership functions which differ a little bit. Thus by generating fuzzy numbers with the fixed canonical representation, we may obtain secondary samples with desired characteristics but without replicating observations from the initial sample. This is the reason why the aforementioned bootstrap idea was called *flexible*.

Flexible bootstrap was suggested for the first time in Ref. [21], where the algorithm for generating fuzzy numbers with fixed two-parameter canonical representation, i.e., the value and ambiguity, was considered. In this paper, we examine extensively this kind of the flexible bootstrap for triangular and trapezoidal fuzzy numbers. Moreover, we introduce another flexible bootstrap algorithm dedicated to trapezoidal fuzzy numbers which preserves three-parameter canonical representation comprising the value, ambiguity and the fuzziness of a fuzzy number.

This paper is organized as follows: In Section 2 we recall basic definitions and concepts related to fuzzy numbers and their representation. Section 3 contains a short introduction to random fuzzy numbers. In Section 4 we develop the bootstrap procedures generating triangular and trapezoidal fuzzy numbers which preserve the value and the ambiguity of the initial sample. Then, in Section 5, we propose an algorithm to generate secondary samples of trapezoidal fuzzy numbers that preserve the value, the ambiguity and the fuzziness of observations belonging to the initial sample. In all cases, we provide the resampling algorithms in a form ready for the practical use. In Section 6 we present the results of the real-life case study in which we compare the proposed flexible bootstrap with the classical bootstrap algorithm. Section 7 contains a broad simulation study performed to evaluate some properties of the suggested methods. In particular, we compare the empirical size and the power of two one-sample tests and the two-sample test equipped with the standard bootstrap procedure and our flexible bootstrap algorithms. We also discuss the standard error of estimators based on the proposed bootstrap algorithms.

## 2. FUZZY NUMBERS AND THEIR CHARACTERISTICS

As the most often type of data used in classical statistical inference are real-valued observations, i.e., real numbers or vectors of reals, the same happens in fuzzy environment where the central role is played by fuzzy numbers. Indeed, although various types of fuzzy sets are considered for modelling imprecision, just fuzzy numbers are actually used most often.

A **fuzzy number**  $A$  is a fuzzy set in  $\mathbb{R}$  which is normal, fuzzy-convex, which has upper semicontinuous membership function  $A(x)$  and bounded support. A family of all fuzzy numbers will be denoted further on by  $\mathbb{F}(\mathbb{R})$ .

An  $\alpha$ -cut of a fuzzy number  $A$ , where  $\alpha \in [0, 1]$ , is defined by

$$A(\alpha) = \begin{cases} \{x \in \mathbb{R} : A(x) \geq \alpha\} & \text{if } \alpha \in (0, 1], \\ cl\{x \in \mathbb{R} : A(x) > 0\} & \text{if } \alpha = 0, \end{cases}$$

where  $cl$  stands for the closure operator. One can see easily that each  $\alpha$ -cut  $A(\alpha)$  of a fuzzy number  $A$  is a closed interval  $A(\alpha) = [A_L(\alpha), A_U(\alpha)]$ .

Membership functions of fuzzy numbers may assume different shapes. However, the most often used fuzzy numbers are the **trapezoidal fuzzy numbers** with membership functions of the form

$$A(x) = \begin{cases} \frac{x - a_1}{a_2 - a_1} & \text{if } a_1 < x \leq a_2, \\ 1 & \text{if } a_2 \leq x \leq a_3, \\ \frac{a_4 - x}{a_4 - a_3} & \text{if } a_3 \leq x < a_4, \\ 0 & \text{otherwise,} \end{cases} \quad (1)$$

where  $a_1, a_2, a_3, a_4 \in \mathbb{R}$  such that  $a_1 \leq a_2 \leq a_3 \leq a_4$ . Then a trapezoidal fuzzy number is denoted by  $[a_1, a_2, a_3, a_4]$ . If  $a_2 = a_3$  then  $A$  is said to be a **triangular fuzzy number**.

Instead of declaring two points  $a_1$  and  $a_4$  describing the support of  $A$  and next two points  $a_2$  and  $a_3$  for its core, it is sometimes more convenient to use another parametrization which emphasizes the location and the spread of the arms. Let us define the following parameters:

$$\begin{aligned} c &:= \frac{a_2 + a_3}{2}, \\ s &:= \frac{a_3 - a_2}{2}, \\ l &:= a_2 - a_1, \\ r &:= a_4 - a_3. \end{aligned}$$

Here  $c$  indicates the center of the core while  $s$  equals to the half of the core, while  $l$  and  $r$  stand for the spread of the left and the right arm of the membership function  $A(x)$ , respectively. Obviously,  $c \in \mathbb{R}$ , while  $s, l, r \geq 0$ . Using this notation a trapezoidal fuzzy number  $A$  would be denoted as  $A(c, s; l, r)$ . Similarly,  $A(c; l, r)$  stands for a triangular fuzzy number, since then  $s = 0$ .

One may ask why to restrict our attention to triangular or trapezoidal fuzzy numbers only. The reason is straightforward: it is due to their simplicity. Trapezoidal or triangular fuzzy numbers are easy to handle and have a natural interpretation. Moreover, even if the original data set consists of fuzzy numbers which are neither triangular nor trapezoidal, one may easily approximate them by such fuzzy numbers. In particular, an approximation algorithm which preserves the value and the ambiguity of the original fuzzy number is given in Ref. [22], while the broad collection of approximation methods satisfying various requirements can be found in Ref. [23].

To simplify the representation of fuzzy numbers Delgado *et al.* [20] suggested two parameters — the value and the ambiguity — which characterize two basic features of a fuzzy number and hence were called together as the **canonical representation of a fuzzy number**.

The first characteristic called the **value** and defined as follows:

$$\text{Val}(A) = \int_0^1 \alpha(A_U(\alpha) + A_L(\alpha))d\alpha, \quad (2)$$

indicates a general location of a fuzzy number  $A$ , whereas the second parameter, i.e., the **ambiguity**

$$\text{Amb}(A) = \int_0^1 \alpha(A_U(\alpha) - A_L(\alpha))d\alpha, \tag{3}$$

is a measure of the global spread (or vagueness) of a fuzzy number  $A$ .

Some straightforward calculations show that the value and the ambiguity of a trapezoidal fuzzy number  $A(c, s; l, r)$  are given as follows:

$$\text{Val}(A) = c + \frac{r-l}{6}, \tag{4}$$

$$\text{Amb}(A) = s + \frac{r+l}{6}. \tag{5}$$

Obviously, if  $A(c; l, r)$  is a triangular fuzzy number then its value is still given by (4), while its ambiguity reduces to

$$\text{Amb}(A) = \frac{r+l}{6}. \tag{6}$$

As advocated by Delgado *et al.*, the value and the ambiguity represent basic features of a fuzzy number and therefore two fuzzy numbers with the same ambiguity and the value might be considered as similar (sometimes they are even treated as “almost equal,” see Ref. [20]).

However, since the two-parameter canonical representation of a fuzzy number does not contain the whole information of that fuzzy number we may try to enrich this representation by adding some supplemental parameters. Besides the value and the ambiguity Delgado *et al.* [20] suggested to consider another characteristic of a fuzzy number, called its **fuzziness**. The fuzziness tries to quantify the difference between a fuzzy number and its complement and is defined as follows:

$$\begin{aligned} \text{Fuzz}(A) = & \int_0^{1/2} (A_U(\alpha) - A_L(\alpha)) d\alpha \\ & + \int_{1/2}^1 (A_L(\alpha) - A_U(\alpha)) d\alpha. \end{aligned} \tag{7}$$

One can find that the fuzziness of a trapezoidal fuzzy number  $A(c, s; l, r)$  equals

$$\text{Fuzz}(A) = \frac{r+l}{4}. \tag{8}$$

Hence, if  $A(c; l, r)$  is a triangular fuzzy number then its fuzziness is also given by (8).

On the other hand, for the trapezoidal fuzzy number  $A(c, s; l, r)$ , because of (5) we obtain immediately a relation between its fuzziness and ambiguity, i.e.,

$$\text{Amb}(A) = s + \frac{2}{3} \text{Fuzz}(A). \tag{9}$$

One can define various metrics in  $\mathbb{F}(\mathbb{R})$  but perhaps the most often used in statistical context is the one proposed by Gil *et al.* [24] and Trutschnig *et al.* [25]. Let  $\lambda$  denote a normalized measure associated

with a continuous distribution with support in  $[0, 1]$  and let  $\theta > 0$ . Then for any  $A, B \in \mathbb{F}(\mathbb{R})$  we define a metric  $D_\theta^\lambda$  as follows:

$$\begin{aligned} D_\theta^\lambda(A, B) = & \left( \int_0^1 [(\text{mid } A(\alpha) - \text{mid } B(\alpha))^2 \right. \\ & \left. + \theta \cdot (\text{spr } A(\alpha) - \text{spr } B(\alpha))^2] d\lambda(\alpha) \right)^{1/2}, \end{aligned} \tag{10}$$

where  $\text{mid } A(\alpha) = (\inf A(\alpha) + \sup A(\alpha)) / 2$  and  $\text{spr } A(\alpha) = (\sup A(\alpha) - \inf A(\alpha)) / 2$  denote the midpoint and the radius of the  $\alpha$ -cut  $A(\alpha)$ , respectively.

For more details on fuzzy numbers, their types and characteristics we refer the reader to Ref. [23].

### 3. RANDOM FUZZY NUMBERS

To analyze fuzzy data and to conduct a statistical inference we need a formal model for the random mechanism generating fuzzy number-valued data. Such model should integrate randomness, associated with data generation mechanism and fuzziness, connected with the intrinsic nature of the data, i.e., their imprecision.

To cope with this problem Puri and Ralescu [26] introduced the notion of a **fuzzy random variable**, also called a **random fuzzy number**.

**Definition 1.** Given a probability space  $(\Omega, \mathcal{A}, P)$ , a mapping  $X : \Omega \rightarrow \mathbb{F}(\mathbb{R})$  is said to be a *fuzzy random variable* if for all  $\alpha \in [0, 1]$  the  $\alpha$ -level function is a compact random interval.

Alternatively,  $X$  is a random fuzzy variable if and only if  $X$  is a Borel measurable function w.r.t. the Borel  $\sigma$ -field generated by the topology induced by the metric  $D_\theta^\lambda$ .

Puri and Ralescu [26] defined also the Aumann-type mean of a fuzzy random variable  $X$  as the fuzzy number  $\mathcal{E}(X) \in \mathbb{F}(\mathbb{R})$  such that for each  $\alpha \in [0, 1]$  the  $\alpha$ -cut  $(\mathcal{E}(X))(\alpha)$  is equal to the Aumann integral of  $X(\alpha)$  or, in other words,

$$(\mathcal{E}(X))(\alpha) = [\mathbb{E}(X_L(\alpha)), \mathbb{E}(X_U(\alpha))]. \tag{11}$$

So defined  $\mathcal{E}(X)$  preserves the main properties from the real-valued expected value (e.g., equivariance under translation and product by a scalar, additivity, etc.).

To characterize dispersion of a fuzzy random variable  $X$  we can also define the  $D_\theta^\lambda$ -Fréchet-type variance  $\mathcal{V}(X)$ , which is a nonnegative real number such that

$$\mathcal{V}(X) = \mathbb{E} \left( [D_\theta^\lambda(X, \mathcal{E}(X))]^2 \right).$$

Although random fuzzy numbers preserve some properties known from the real-valued inference, one should be aware of the problems typical of statistical reasoning with fuzzy data. In particular, due to the nonlinearities of the space of fuzzy numbers it is advisable to avoid subtraction of fuzzy numbers wherever it is possible (the same hold for the division). Some difficulties in fuzzy data analysis may be caused by the lack of universally accepted total ranking between fuzzy numbers. Another source of possible crucial problems that appear in conjunction of randomness and fuzziness is the absence of suitable models for the distribution of fuzzy random variables.

Moreover, there are not yet Central Limit Theorems for fuzzy random variables which can be applied directly in statistical inference.

The aforementioned arguments could strongly discourage statistical inference with fuzzy data if not the two brilliant ideas that indicate the way-out in this situation. Firstly, in the late 1970s Efron [1] developed the bootstrap method to approximate the distribution of inferential statistics when the population distribution is unknown. Next, in the very early 1990s Giné and Zinn [27] developed a bootstrapped approximation to the CLT for generalized random elements, which opens the door for the bootstrap application in statistical inference with fuzzy data.

#### 4. VALUE-AMBIGUITY (VA) BOOTSTRAP ALGORITHM

To avoid undesired repetitions which often appear in bootstrap Romaniuk and Hryniewicz [17–19] proposed a resampling method which enrich secondary samples with fuzzy observations imitating those from the primary sample but containing some incremental spreads on their  $\alpha$ -cuts. Then Grzegorzewski et al. [21,28] suggested another approach for generating bootstrap samples which may differ from the primary one but preserve the two-parameter canonical representation of each fuzzy observation, i.e., its value and ambiguity or the expected value and the width. We briefly discuss this method below, before introducing in Section 5 a new flexible bootstrap algorithm which preserves three-parameter canonical representation comprising the value, ambiguity and the fuzziness of a fuzzy number.

##### 4.1. VA Algorithm for Triangular Observations

Let  $x_1, \dots, x_n \in \mathbb{F}(\mathbb{R})$  denote a realization of the experiment modeled by a fuzzy random sample  $X_1, \dots, X_n$ , where each  $X_i$  is a random fuzzy number (defined as in Ref. [26]). Further on this sample will be called the primary (initial) sample.

Let us compute the value and the ambiguity of each observation in this primary sample following Eqs. (2) and (3). This way we obtain a set of pairs

$$\{(\text{Val}(x_1), \text{Amb}(x_1)), \dots, (\text{Val}(x_n), \text{Amb}(x_n))\}. \quad (12)$$

The main idea of the proposed bootstrap technique is to create a secondary sample by generating randomly fuzzy observations from the set (12). Obviously, although the value and the ambiguity characterize nicely a fuzzy number, they do not identify it completely. Thus by imposing some restrictions on the generated fuzzy numbers we have still some room for the choice of their membership function. Let us retrace in detail how it works.

In this section we assume that the desired secondary sample  $x_1^*, \dots, x_n^*$  will consist of triangular fuzzy numbers, i.e.,  $x_i^* = x_i^*(c_i^*; l_i^*, r_i^*)$  for each  $i = 1, \dots, n$ .

Let  $(\text{Val}^*, \text{Amb}^*)$  denote the output of a random drawing from (12). By (4) and (6) we obtain

$$\begin{cases} r - l = 6\text{Val}^* - 6c, \\ r + l = 6\text{Amb}^*. \end{cases}$$

Moreover, by the definition,  $r, l \geq 0$ . Hence after some transformations we obtain

$$\begin{cases} l = 3(\text{Amb}^* - \text{Val}^* + c), \\ r = 3(\text{Amb}^* + \text{Val}^* - c), \end{cases} \quad (13)$$

which (keeping in mind that  $r, l \geq 0$ ) yields in

$$\text{Val}^* - \text{Amb}^* \leq c \leq \text{Val}^* + \text{Amb}^*. \quad (14)$$

We can gather all these considerations in the following bootstrap algorithm.

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##### Algorithm 1

1. Given a fuzzy sample  $x_1, \dots, x_n \in \mathbb{F}(\mathbb{R})$  compute the value  $\text{Val}(x_i)$  and the ambiguity  $\text{Amb}(x_i)$  for each observation  $i = 1, \dots, n$ .
  2. Let  $j := 1$ .
  3. Draw randomly (with equal probabilities) a pair  $(\text{Val}_j^*, \text{Amb}_j^*)$  from
 
$$\{(\text{Val}(x_1), \text{Amb}(x_1)), \dots, (\text{Val}(x_n), \text{Amb}(x_n))\}.$$
  4. Generate a random value  $c_j^*$  from the uniform distribution on the interval
 
$$[\text{Val}_j^* - \text{Amb}_j^*, \text{Val}_j^* + \text{Amb}_j^*].$$
  5. Compute  $l_j^* := 3[\text{Amb}_j^* - \text{Val}_j^* + c_j^*]$ .
  6. Compute  $r_j^* := 3[\text{Amb}_j^* + \text{Val}_j^* - c_j^*]$ .
  7. Let  $j := j + 1$ .
  8. If  $j < n$  go to step 3.
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Thus, following Algorithm 1 we receive the secondary bootstrap sample  $x_1^*, \dots, x_n^*$  of triangular fuzzy numbers, where  $x_i^* = x_i^*(c_i^*; l_i^*, r_i^*)$  for  $i = 1, \dots, n$ .

##### 4.2. VA Algorithm for Trapezoidal Observations

As before, let  $x_1, \dots, x_n \in \mathbb{F}(\mathbb{R})$  be the primary sample which is a realization of the experiment modelled by a fuzzy random sample  $X_1, \dots, X_n$ . Similarly, as in Section 4.1, we compute the value and the ambiguity of each observation which yields in the set of pairs (12), where the corresponding values and ambiguities are calculated from (2) and (3), respectively. However, now we assume that the desired secondary sample  $x_1^*, \dots, x_n^*$  will consist of trapezoidal fuzzy numbers, i.e.,  $x_i^* = x_i^*(c_i^*, s_i^*; l_i^*, r_i^*)$  for each  $i = 1, \dots, n$ . Thus, after drawing randomly one pair from the set (12) and denoting it by  $(\text{Val}^*, \text{Amb}^*)$ , we follow Eqs. (4) and (5), so we have

$$\begin{cases} r - l = 6\text{Val}^* - 6c, \\ r + l = 6\text{Amb}^* - 6s, \end{cases}$$

where  $s, r, l \geq 0$ , which is equivalent to

$$\begin{cases} l = 3(\text{Amb}^* - \text{Val}^* + c - s), \\ r = 3(\text{Amb}^* + \text{Val}^* - c - s). \end{cases} \quad (15)$$

Since  $r, l \geq 0$ , we obtain the following inequalities

$$\text{Val}^* - \text{Amb}^* + s \leq c \leq \text{Val}^* + \text{Amb}^* - s, \quad (16)$$

where  $s \geq 0$ . Since the upper bound of (16) cannot be smaller than its lower bound we obtain additionally the following requirement

$$0 \leq s \leq \text{Amb}^*. \quad (17)$$

Summing up the aforementioned considerations as well as restrictions specified in (15), (16) and (17), we obtain the following bootstrap algorithm for generating a secondary sample of trapezoidal fuzzy numbers.

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**Algorithm 2**

1. Given a fuzzy sample  $x_1, \dots, x_n \in \mathbb{F}(\mathbb{R})$  compute the value  $\text{Val}(x_i)$  and ambiguity  $\text{Amb}(x_i)$  for each observation  $i = 1, \dots, n$ .
2. Let  $j := 1$ .
3. Draw randomly (with equal probabilities) a pair  $(\text{Val}_j^*, \text{Amb}_j^*)$  from

$$\{(\text{Val}(x_1), \text{Amb}(x_1)), \dots, (\text{Val}(x_n), \text{Amb}(x_n))\}.$$

4. Generate a random value  $s_j^*$  from the uniform distribution on the interval  $[0, \text{Amb}_j^*]$ .
5. Generate a random value  $c_j^*$  from the uniform distribution on the interval

$$[\text{Val}_j^* - \text{Amb}_j^* + s_j^*, \text{Val}_j^* + \text{Amb}_j^* - s_j^*].$$

6. Compute  $l_j^* := 3 \left[ \text{Amb}_j^* - \text{Val}_j^* + c_j^* - s_j^* \right]$ .
  7. Compute  $r_j^* := 3 \left[ \text{Amb}_j^* + \text{Val}_j^* - c_j^* - s_j^* \right]$ .
  8. Let  $j := j + 1$ .
  9. If  $j < n$  go to step 3.
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Therefore, Algorithm 2 provides a bootstrap secondary sample  $x_1^*, \dots, x_n^*$  of fuzzy trapezoidal numbers, where  $x_i^* = x_i^*(c_i^*, s_i^*; l_i^*, r_i^*)$  for  $i = 1, \dots, n$ .

## 5. VALUE-AMBIGUITY-FUZZINESS (VAF) BOOTSTRAP ALGORITHM

Both algorithms proposed in Sections 4.1 and 4.2 show how to generate secondary bootstrap samples which preserve the value and ambiguity of observations from the initial sample. While keeping track of those algorithms one may notice that by fixing two-parameter canonical representation of a fuzzy number we have something like “one degree of freedom” in generating a triangular fuzzy number and “two degrees of freedom” in trapezoidal fuzzy number generation. Indeed, besides drawing randomly a pair  $(\text{Val}^*, \text{Amb}^*)$  from the set (12) we have to generate some constants: one real number  $c^*$ , which satisfies (14), to obtain a triangular fuzzy number or two reals  $c^*$  and  $s^*$ , so (16) and (17) holds, for getting a trapezoidal fuzzy number. As a consequence, a trapezoidal fuzzy number obtained from Algorithm 2 may differ more from the original trapezoidal observation having the same value and ambiguity than its triangular counterpart.

One way to reduce the number of undesired “degrees of freedom” in generating bootstrap samples which consist of trapezoidal fuzzy

numbers is to consider a more extended canonical representation of a fuzzy number, i.e., to fix another characteristic besides its value and ambiguity. A possible candidate is the fuzziness described in Section 2.

Indeed, by computing not only the value and the ambiguity of each observation from the primary sample  $x_1, \dots, x_n$  but also its fuzziness (7), we obtain the following set of triples

$$\{(\text{Val}(x_1), \text{Amb}(x_1), \text{Fuzz}(x_1)), \dots, (\text{Val}(x_n), \text{Amb}(x_n), \text{Fuzz}(x_n))\}. \quad (18)$$

Next, assuming that the desired secondary sample  $x_1^*, \dots, x_n^*$  will consist of trapezoidal fuzzy numbers, i.e.  $x_i^* = x_i^*(c_i^*, s_i^*; l_i^*, r_i^*)$  for each  $i = 1, \dots, n$ , we draw randomly one triple from the set (21) and denoting it by  $(\text{Val}^*, \text{Amb}^*, \text{Fuzz}^*)$ , we follow the consecutive transformations as in Section 4.2 which lead us to Eq. (16), where besides the given value  $\text{Val}^*$  and the ambiguity  $\text{Amb}^*$  one can find two unknown constants  $c$  and  $s$ . However, let us recall that there exist a one-to-one correspondence between  $s$  and the fuzziness of a fuzzy number, provided we restrict our attention to trapezoidal fuzzy numbers only. Indeed, by (9) we have

$$s^* = \text{Amb}^* - \frac{2}{3} \text{Fuzz}^*, \quad (19)$$

and hence substituting it into (16) we obtain the following requirement for  $c$

$$\text{Val}^* - \frac{2}{3} \text{Fuzz}^* \leq c \leq \text{Val}^* + \frac{2}{3} \text{Fuzz}^*. \quad (20)$$

Summing up the aforementioned considerations as well as the restrictions specified in (15), (16) and (17), we obtain the following bootstrap algorithm for generating a secondary sample of trapezoidal fuzzy numbers.

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**Algorithm 3**

1. Given a fuzzy sample  $x_1, \dots, x_n \in \mathbb{F}(\mathbb{R})$  compute the value  $\text{Val}(x_i)$ , the ambiguity  $\text{Amb}(x_i)$  and the fuzziness  $\text{Fuzz}(x_i)$  for each observation  $i = 1, \dots, n$ .
2. Let  $j := 1$ .
3. Draw randomly (with equal probabilities) a triple  $(\text{Val}_j^*, \text{Amb}_j^*, \text{Fuzz}_j^*)$  from

$$\{(\text{Val}(x_1), \text{Amb}(x_1), \text{Fuzz}(x_1)), \dots, (\text{Val}(x_n), \text{Amb}(x_n), \text{Fuzz}(x_n))\}.$$

4. Compute  $s_j^* := \text{Amb}_j^* - \frac{2}{3} \text{Fuzz}_j^*$
5. Generate a random value  $c_j^*$  from the uniform distribution on the interval

$$[\text{Val}_j^* - \frac{2}{3} \text{Fuzz}_j^*, \text{Val}_j^* + \frac{2}{3} \text{Fuzz}_j^*].$$

6. Compute  $l_j^* := 3 \left[ \text{Amb}_j^* - \text{Val}_j^* + c_j^* - s_j^* \right]$ .
  7. Compute  $r_j^* := 3 \left[ \text{Amb}_j^* + \text{Val}_j^* - c_j^* - s_j^* \right]$ .
  8. Let  $j := j + 1$ .
  9. If  $j < n$  go to step 3.
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Therefore, Algorithm 3 provides the bootstrap secondary sample  $x_1^*, \dots, x_n^*$  of fuzzy trapezoidal numbers, where  $x_i^* = x_i^*(c_i^*, s_i^*; l_i^*, r_i^*)$

for  $i = 1, \dots, n$ . Please note, that by considering the fuzziness of the observations belonging to the primary sample we have reduced the number of “degrees of freedom” from two to one, required for generating randomly the center of the core, i.e.,  $c_i^*$ .

**Example 1.** Let us assume that  $x = (6, 1; 2, 3)$  is a randomly chosen observation from our primary sample. For this fuzzy number we have  $\text{Val}(x) = 6\frac{1}{6}$ ,  $\text{Amb}(x) = 1\frac{5}{6}$  and  $\text{Fuzz}(x) = 1\frac{3}{4}$ . Then, according to step 4 of Algorithm 3, we do not generate  $s^*$  randomly (as in Algorithm 2) but just calculate it from the direct formula that gives us  $s^* = \frac{2}{3}$ .

Then, according to step 5 of Algorithm 3, we generate randomly  $c^*$  from the uniform distribution on the interval  $[5, 7]$ . Suppose, that we have obtained  $c^* = 6\frac{2}{3}$  as the output of the last random generation. Hence, by (15) we compute  $l^* = 5$  and  $r^* = 2$  and finally we obtain the trapezoidal fuzzy number  $x^* = (6\frac{2}{3}, \frac{2}{3}; 5, 2)$  as the result of our flexible bootstrap approach. Although  $x^* \neq x$ , but  $\text{Val}(x^*) = 6\frac{1}{6} = \text{Val}(x)$ ,  $\text{Amb}(x^*) = 1\frac{5}{6} = \text{Amb}(x)$  and  $\text{Fuzz}(x^*) = 1\frac{3}{4} = \text{Fuzz}(x)$ , which means that the fundamental characteristics of the primary and secondary observations have been preserved.

## 6. CASE STUDY

Resampling techniques are commonly used to solve practical problems [9–11], therefore we compare the introduced algorithms also in the case of a real-life application. We utilize some data [10] related to the quality of the Gamonedo cheese, which is a kind of blue cheese produced in Asturias, Spain. Some of its testers expressed their subjective opinions using trapezoidal fuzzy numbers and we compare the overall impressions of three experts about this cheese [8,10].

To compare the means of two independent samples, i.e., to verify the hypothesis

$$H_0 : \mathcal{E}(X) = \mathcal{E}(Y) \quad \text{vs.} \quad H_1 : \mathcal{E}(X) \neq \mathcal{E}(Y), \quad (21)$$

where  $\mathcal{E}(X), \mathcal{E}(Y)$  correspond to the Aumman type means of the fuzzy random variables from the first and the second population, respectively, we apply the statistical test, which was discussed by Colubi [30] and Lubiano *et al.* [9]. From now on, this test will be denoted as the C2-test. The classical approach, the VA- and the VAF-method are used to compare the estimated  $p$  values if the number of the bootstrap replications  $b$  is set to 100, 200 or 1000.

We compare pairwise the mean opinions of the three previously mentioned experts (Expert A, B and C) to check if they have “similar overall opinions.” In the case of two pairs (Expert A vs. Expert B and Expert B vs. Expert C), all the resampling approaches for the C2-test give the same estimated  $p$  value about 0.0000 (even for  $b = 100$ ). Because the calculated sample means for the respective three samples are equal to

$$\begin{aligned} \bar{X}_A &= [57.65, 63.175, 69.175, 73.48], \\ \bar{X}_B &= [47.34, 51.21, 59.87, 66.84], \\ \bar{X}_C &= [57.24, 62.38, 67.95, 73.52], \end{aligned}$$

these results are not a surprise. The estimated  $p$  values for the pair Expert A vs. Expert C can be found in Table 1. Assuming the typical significant level, i.e.,  $\alpha = 0.05$ , the null hypothesis (21) is not

rejected in this case, so both of the experts have the same “mean opinion” about the cheese. But it seems that the estimated  $p$  values for the value-ambiguity (VA) and value-ambiguity-fuzziness (VAF) method decrease in a stable way and the classical approach gives the most restrictive results (i.e., the respective  $p$  values are the lowest).

## 7. SIMULATION STUDY

### 7.1. One-Sample Tests

Hypotheses testing with fuzzy data attract attention of many researches (see, e.g., Refs. [3,5–8,31]) and most of the cited papers utilize bootstrap methods to determine a null distribution under study. This is the reason that we also examine our new bootstrap methods suggested in this contribution with three tests for the expected value. The first one is a bootstrapped version of the Körner test [31], the second one is developed by Montenegro *et al.* [5,7] and the third one (related also to the second test) is discussed by Colubi [30] and Lubiano *et al.* [9]. Further on, they will be denoted as the  $K$ -test, the  $M$ -test and the  $C$ -test, respectively. All these tests were designed to verify the following problem

$$H_0 : \mathcal{E}(X) = v \quad \text{vs.} \quad H_1 : \mathcal{E}(X) \neq v, \quad (22)$$

where  $\mathcal{E}(X)$  corresponds to the Aumman type mean of the fuzzy random variable  $X(\alpha)$  defined by (11), while  $v \in \mathbb{F}(\mathbb{R})$  is a fixed fuzzy number considered as the true population fuzzy mean.

We consider a few types of fuzzy numbers (summarized in Table 2) to create primary fuzzy random sample  $X_1, \dots, X_n$ . The first type  $\mathbb{F}_{N, \chi^2, \chi^2}^\Delta$  is given by triangular fuzzy numbers  $X_i(c_i; l_i, r_i)$  with centers (i.e., parameter  $c_i$ ) randomly generated from the standard normal distribution  $N(0, 1)$  and spreads (i.e.,  $l_i$  and  $r_i$ ) also randomly generated from the chi-square distribution with one degree of freedom  $\chi^2(1)$ . All random variables mentioned above are independent (similarly as in the next cases). A sample of the second type  $\mathbb{F}_{N, E, E}^\Delta$  is generated with the centers  $c_i$  from  $N(0, 1)$  and asymmetrical spreads, i.e.,  $l_i$  drawn from the exponential distribution  $\text{Exp}(2)$  with parameter 2 and  $r_i$  drawn from  $\text{Exp}(4)$ .

Similarly, the centers for  $\mathbb{F}_{N, U, U}^\Delta$  are from  $N(0, 1)$ , but both spreads are independently drawn from the uniform distribution  $U([0, 0.4])$ .

**Table 1** | Empirical  $p$  values for the C2-test concerning the quality of the Gamonedo cheese (Expert A vs. C).

$b$	100	200	1000
Boot.	0.8	0.835	0.84
VA	0.96	0.955	0.941
VAF	0.87	0.86	0.848

**Table 2** | Description of types of the simulated fuzzy numbers.

Type	c	s	l	r
$\mathbb{F}_{N, \chi^2, \chi^2}^\Delta$	$N(0,1)$	–	$\chi^2(1)$	$\chi^2(1)$
$\mathbb{F}_{N, E, E}^\Delta$	$N(0,1)$	–	$\text{Exp}(2)$	$\text{Exp}(4)$
$\mathbb{F}_{N, U, U}^\Delta$	$N(0,1)$	–	$U(0,0.4)$	$U(0,0.4)$
$\mathbb{F}_{N, U, U}^T$	$N(0,1)$	$U(0,0.1)$	$U(0,0.1)$	$U(0,0.3)$
$\mathbb{F}_{\Gamma, U, E, E}^T$	$\Gamma(2, 2)$	$U(0,0.2)$	$\text{Exp}(4)$	$\text{Exp}(2)$

Trapezoidal fuzzy numbers  $X_i(c_i, s_i; l_i, r_i)$  for  $\mathbb{F}_{N,U,U}^T$  are generated using  $N(0, 1)$  (in the case of the centers  $c_i$ ), and the uniform distributions (for the radii  $s_i$  and both the spreads  $l_i, r_i$ ). And trapezoidal fuzzy numbers  $\mathbb{F}_{\Gamma,U,E,E}^T$  are drawn using the gamma distribution  $\Gamma(2, 2)$  with the scale parameter 2 and the shape parameter (for the centers), the uniform distribution (the radii) and the exponential distributions (the spreads). Similar types of fuzzy numbers were considered in various numerical studies, see Refs. [2,18,19,31].

For each type of fuzzy numbers, initial random samples of size  $n$  are used as input sets for the considered tests. To compare various starting parameters we consider  $n = 5, 10, 30, 100$  corresponding to small and medium sample sizes. Then, in the case of triangular fuzzy numbers, the  $K$ -test and the  $M$ -test are performed with the help of the classical bootstrap and the proposed VA method. Outputs for trapezoidal fuzzy numbers are compared using the classical bootstrap, the VA method and the VAF method. We consider several numbers  $b$  of the generated bootstrap samples (namely,  $b = 100, 200, 1000$ ), which consist of  $n$  elements, to investigate a possible influence of  $b$  on the final results. In each experiment the whole resampling procedure is iterated 100000 times (see, e.g., Refs. [3,5,7,8,18] for more details concerning the similar comparisons).

As the essential benchmark, we use the empirical size of the test  $\hat{\alpha}$  (i.e., the estimated percentage of the true null hypothesis rejections) and its relation to the nominal significance level  $\alpha$ . In our experiments we set the standard value  $\alpha = 0.05$ .

Results given in Tables 3–8 show that both the classical approach (denoted by “Boot.”) and the suggested VA resampling procedure are very close to each other. It seems that the empirical sizes of the test  $\hat{\alpha}$  converge to the same limits as the initial sample  $n$  and the number of the bootstrap replications  $b$  increase. To ease noting the existing differences, those  $\hat{\alpha}$ 's, which are closer to the nominal significance level  $\alpha = 0.05$ , are given in boldface. For example, if  $n = 5, b = 100$  and  $\hat{\alpha} = 0.16024$ , as in Table 3, then the true null hypothesis was rejected in 16% of the cases (on average) for the small primary sample (with 5 elements) and 100 replications of the bootstrap sample.

Going into details, when discussing the  $K$ -test (Tables 3–5), it seems that the VA method dominates the classical bootstrap for  $\mathbb{F}_{N,E,E}^{\Delta}$  fuzzy numbers. In the case of  $\mathbb{F}_{N,\chi^2,\chi^2}^{\Delta}$  fuzzy numbers, the advantage of the VA method appears only for small sample sizes. For  $\mathbb{F}_{N,U,U}^{\Delta}$  fuzzy numbers situation is not completely clear, but it seems

**Table 3** | Empirical  $K$ -test size  $\hat{\alpha}$  for  $\mathbb{F}_{N,\chi^2,\chi^2}^{\Delta}$ .

$n$	5	10	30	100
$b$	100			
Boot.	0.16024	<b>0.10113</b>	<b>0.07006</b>	<b>0.06331</b>
VA	<b>0.15954</b>	0.10476	0.07616	0.06814
$b$	200			
Boot.	0.15438	<b>0.09686</b>	<b>0.0658</b>	<b>0.05795</b>
VA	<b>0.15236</b>	0.09833	0.0711	0.06207
$b$	1000			
Boot.	0.14834	<b>0.0911</b>	<b>0.06297</b>	<b>0.05449</b>
VA	<b>0.14523</b>	0.09421	0.06519	0.0585

that the behavior of the VA method improves and prevail the classical approach as the number of the bootstrap repetitions  $b$  increases.

In the case of the  $M$ -test (see Tables 6–8), the general comparison of the classical bootstrap and the VA method shows that there is

**Table 4** | Empirical  $K$ -test size  $\hat{\alpha}$  for  $\mathbb{F}_{N,E,E}^{\Delta}$ .

$n$	5	10	30	100
$b$	100			
Boot.	0.17437	0.1078	0.07297	0.0633
VA	<b>0.168</b>	<b>0.10542</b>	<b>0.07108</b>	<b>0.05378</b>
$b$	200			
Boot.	0.16529	0.10371	0.06852	0.05739
VA	<b>0.16395</b>	<b>0.10009</b>	<b>0.06591</b>	<b>0.04933</b>
$b$	1000			
Boot.	0.16155	0.09762	0.06543	0.05515
VA	<b>0.15952</b>	<b>0.09628</b>	<b>0.06162</b>	<b>0.04674</b>

**Table 5** | Empirical  $K$ -test size  $\hat{\alpha}$  for  $\mathbb{F}_{N,U,U}^{\Delta}$ .

$n$	5	10	30	100
$b$	100			
Boot.	0.17553	0.10833	<b>0.07303</b>	<b>0.06285</b>
VA	<b>0.17183</b>	<b>0.10803</b>	0.07445	0.06319
$b$	200			
Boot.	<b>0.16669</b>	0.10309	<b>0.06878</b>	0.05775
VA	0.16806	<b>0.1021</b>	0.06895	<b>0.05746</b>
$b$	1000			
Boot.	<b>0.16255</b>	<b>0.0977</b>	0.06584	0.0556
VA	0.16352	0.09843	<b>0.06432</b>	<b>0.05432</b>

**Table 6** | Empirical  $M$ -test size  $\hat{\alpha}$  for  $\mathbb{F}_{N,\chi^2,\chi^2}^{\Delta}$ .

$n$	5	10	30	100
$b$	100			
Boot.	0.03375	<b>0.04906</b>	0.0562	0.05892
VA	<b>0.0339</b>	0.04347	<b>0.0489</b>	<b>0.05107</b>
$b$	200			
Boot.	<b>0.02988</b>	<b>0.04449</b>	<b>0.05196</b>	0.05473
VA	0.02915	0.03915	0.04493	<b>0.04597</b>
$b$	1000			
Boot.	<b>0.02748</b>	<b>0.04047</b>	<b>0.04916</b>	<b>0.05064</b>
VA	0.02603	0.03567	0.04078	0.0423

**Table 7** | Empirical  $M$ -test size  $\hat{\alpha}$  for  $\mathbb{F}_{N,E,E}^{\Delta}$ .

$n$	5	10	30	100
$b$	100			
Boot.	0.03634	<b>0.05033</b>	0.05777	0.05862
VA	<b>0.03724</b>	0.04912	<b>0.05488</b>	<b>0.05569</b>
$b$	200			
Boot.	0.03095	<b>0.04593</b>	0.05286	0.05344
VA	<b>0.03248</b>	0.04367	<b>0.04989</b>	<b>0.04911</b>
$b$	1000			
Boot.	0.02871	<b>0.04107</b>	<b>0.05009</b>	<b>0.05087</b>
VA	<b>0.02997</b>	0.04058	0.04564	0.0469

also no apparent winner: for small sample sizes sometimes the VA method is better, otherwise the classical approach dominates. When we compare the situation under the different number of bootstrap repetitions no obvious conclusion imposes.

In the case of trapezoidal fuzzy numbers, the  $C$ -test is applied to compare the empirical sizes for this test using the classical bootstrap, the VA method and the VAF method with the same values of  $n, b$  as in the previous analyses. As an essential part of the  $C$ -test, the mid-spread distance (or Bertoluzza *et al.*'s distance [32]) with the weighting parameter  $\theta = 1/3$  is used, which leads to weighing uniformly the squared distances between the convex linear combinations of the end points of the different level intervals [9].

In general, as we can see from Tables 9 and 10, there are some significant differences, especially for small  $n$ , when the VA method or the VAF method is preferred in comparison with the classical bootstrap. Moreover, both of the introduced methods seem to behave in a more stable way, i.e., the simulated values of  $\hat{\alpha}$  are generally closer to the true value for all applied  $n, b$ .

Next step of the numerical experiment was the power study of the considered tests equipped with the classical bootstrap, the VA method or the VAF method. Because the null and the alternative hypotheses (22) in these tests concern the strict equality or inequality for a fuzzy number, the respective procedure has to be applied. To examine the power of the tests, we estimated the number of rejections under increasing shift  $\epsilon \in \mathbb{R}$  of realizations of the initial fuzzy sample, when  $\epsilon = 0.25, 0.5, 0.75$ . We observed the behavior of the procedures under study using the same values of parameters  $b$  and

**Table 8** Empirical  $M$ -test size  $\hat{\alpha}$  for  $\mathbb{F}_{N,U,U}^A$ .

$n$	5	10	30	100
$b$	100			
Boot.	0.03246	0.04915	<b>0.05742</b>	<b>0.05851</b>
VA	<b>0.03352</b>	<b>0.05026</b>	0.0588	0.05903
$b$	200			
Boot.	0.02663	0.04431	<b>0.05325</b>	0.05327
VA	<b>0.02832</b>	<b>0.04433</b>	0.05351	<b>0.05314</b>
$b$	1000			
Boot.	0.02381	0.04006	<b>0.04954</b>	0.05113
VA	<b>0.02612</b>	<b>0.04026</b>	0.04874	<b>0.05046</b>

**Table 9** Empirical  $C$ -test size  $\hat{\alpha}$  for  $\mathbb{F}_{N,U,U,U}^T$ .

$n$	5	10	30	100
$b$	100			
Boot.	0.03072	<b>0.04896</b>	0.05818	0.05847
VA	0.03137	0.04862	<b>0.0574</b>	0.05772
VAF	<b>0.03217</b>	0.04862	0.05778	<b>0.05716</b>
$b$	200			
Boot.	0.02581	<b>0.04435</b>	0.05217	0.05519
VA	<b>0.02613</b>	0.04367	0.05331	0.05394
VAF	0.02512	0.04385	<b>0.05125</b>	<b>0.05383</b>
$b$	1000			
Boot.	0.0232	0.03982	0.04966	0.05156
VA	<b>0.02414</b>	0.03989	0.04856	0.04929
VAF	0.02379	<b>0.04037</b>	<b>0.04978</b>	<b>0.0494</b>

$n$  as before. Because of lack of space we present to the reader the experimental results corresponding to  $\mathbb{F}_{N,U,U}^A$  (see Tables 11 and 12),  $\mathbb{F}_{N,U,U,U}^T$  (see Table 13) and  $\mathbb{F}_{\Gamma,U,E,E}^T$  (see Table 14) only.

For  $\mathbb{F}_{N,U,U}^A$  and both the  $K$ -test and the  $M$ -test, the differences in the simulated power between the VA method and the classical bootstrap are relatively small. Both of the resampling procedures behave

**Table 10** Empirical  $C$ -test size  $\hat{\alpha}$  for  $\mathbb{F}_{\Gamma,U,E,E}^T$ .

$n$	5	10	30	100
$b$	100			
Boot.	<b>0.05272</b>	0.0676	0.06291	0.05947
VA	0.05397	0.06602	<b>0.05956</b>	<b>0.0554</b>
VAF	0.05353	<b>0.06561</b>	0.06036	0.05624
$b$	200			
Boot.	0.04773	0.06349	0.05781	0.05507
VA	0.04755	0.06168	<b>0.05507</b>	0.05049
VAF	<b>0.04802</b>	<b>0.06073</b>	0.05518	<b>0.05028</b>
$b$	1000			
Boot.	0.04456	0.05935	0.05424	0.05162
VA	0.04511	<b>0.05763</b>	0.05193	0.04718
VAF	<b>0.04545</b>	0.05777	<b>0.05118</b>	<b>0.0483</b>

**Table 11**  $K$ -test power analysis for  $\mathbb{F}_{N,U,U}^A$ .

$n$	5	10	30	100
$\epsilon, b$	0.25, 100			
Boot.	<b>0.22496</b>	<b>0.199</b>	0.3177	0.72024
VA	0.22174	0.19778	<b>0.31772</b>	<b>0.72126</b>
$\epsilon, b$	0.25, 200			
Boot.	0.21777	<b>0.19247</b>	<b>0.31134</b>	<b>0.71411</b>
VA	<b>0.21791</b>	0.19106	0.30949	0.71335
$\epsilon, b$	0.25, 1000			
Boot.	0.21218	<b>0.18781</b>	0.30239	0.71188
VA	<b>0.21284</b>	0.18705	<b>0.30398</b>	<b>0.71282</b>
$\epsilon, b$	0.5, 100			
Boot.	<b>0.36195</b>	<b>0.43961</b>	<b>0.79944</b>	<b>0.9987</b>
VA	0.35853	0.43703	0.79904	0.99864
$\epsilon, b$	0.5, 200			
Boot.	0.35157	0.42882	<b>0.79486</b>	0.99867
VA	<b>0.35244</b>	<b>0.42901</b>	0.79472	<b>0.99869</b>
$\epsilon, b$	0.5, 1000			
Boot.	0.34748	<b>0.42479</b>	0.79131	0.99885
VA	<b>0.34767</b>	0.42217	<b>0.79273</b>	<b>0.99893</b>
$\epsilon, b$	0.75, 100			
Boot.	<b>0.54304</b>	<b>0.71307</b>	0.98386	1
VA	0.54079	0.7111	<b>0.98425</b>	1
$\epsilon, b$	0.75, 200			
Boot.	0.53448	0.70511	<b>0.98386</b>	1
VA	<b>0.53467</b>	<b>0.70805</b>	0.9835	1
$\epsilon, b$	0.75, 1000			
Boot.	<b>0.52919</b>	<b>0.70456</b>	<b>0.98447</b>	1
VA	0.52863	0.7022	0.98363	1

**Table 12** | M-test power analysis for  $\mathbb{F}_{N,U,U}^{\Delta}$ .

$n$	5	10	30	100
$\epsilon, b$	0.25, 100			
Boot.	0.04708	0.1045	<b>0.27674</b>	0.70835
VA	<b>0.04763</b>	<b>0.10472</b>	0.27601	<b>0.70968</b>
$\epsilon, b$	0.25, 200			
Boot.	0.03834	<b>0.09749</b>	<b>0.26958</b>	0.70123
VA	<b>0.04177</b>	0.09739	0.26873	<b>0.70248</b>
$\epsilon, b$	0.25, 1000			
Boot.	0.03571	<b>0.09049</b>	0.26048	0.69996
VA	<b>0.03792</b>	0.0899	<b>0.26096</b>	<b>0.7012</b>
$\epsilon, b$	0.5, 100			
Boot.	0.08898	<b>0.2769</b>	<b>0.76116</b>	<b>0.99863</b>
VA	<b>0.08967</b>	0.27502	0.76064	0.9985
$\epsilon, b$	0.5, 200			
Boot.	0.07711	0.26214	<b>0.75629</b>	0.99851
VA	<b>0.08138</b>	<b>0.26332</b>	0.75503	<b>0.99855</b>
$\epsilon, b$	0.5, 1000			
Boot.	0.07073	<b>0.25315</b>	0.75066	<b>0.99871</b>
VA	<b>0.07326</b>	0.25213	<b>0.75301</b>	0.9988
$\epsilon, b$	0.75, 100			
Boot.	0.15996	<b>0.52679</b>	0.97706	1
VA	<b>0.16177</b>	0.52473	<b>0.97759</b>	1
$\epsilon, b$	0.75, 200			
Boot.	0.14084	0.51102	<b>0.97743</b>	1
VA	<b>0.14723</b>	<b>0.51362</b>	0.97705	1
$\epsilon, b$	0.75, 1000			
Boot.	0.13156	<b>0.50551</b>	<b>0.97771</b>	1
VA	<b>0.13547</b>	0.50407	0.97696	1

in a rather consistent way, i.e., the percentage of rejections generally grows as  $n$  or  $b$  increases. To emphasize, better results (if they exist) are given in boldface.

The differences between the bootstrap procedures are hardly discernible especially for the  $K$ -test, where it is difficult to designate the real winner. It can be seen also in Figure 1, where the percentages of rejections are shown for some range of smaller values of the shift (i.e.,  $\epsilon \in [0.05, 0.25]$ ) and  $n = 5, b = 100$ . The situation is slightly more clear in the case of the  $M$ -test. As it is seen in Table 12 and in Figure 2, the VA method gives the higher percentages of rejections especially for the smallest sample size  $n$ , i.e.,  $n = 5$ .

The situation is similar for the considered trapezoidal fuzzy numbers and the  $C$ -test (see Tables 13 and 14). The new resampling algorithms are favored for the smallest sample size (i.e.,  $n = 5$ ), which can be also validated with Figures 3 and 4.

### 7.2. Two-Sample Tests

In the same manner as in Section 7.1, we conduct a power study for the  $C_2$ -test. Because of the form of the hypotheses (21), the special approach has to be also applied. The number of rejections is estimated under increasing shift  $\epsilon \in \mathbb{R}$  of realizations of the second initial fuzzy sample, i.e.,  $Y$ , if both of the samples  $X, Y$  are generated using the same type of fuzzy numbers.

**Table 13** | C-test power analysis for  $\mathbb{F}_{N,U,U}^T$ .

$n$	5	10	30	100
$\epsilon, b$	0.25, 100			
Boot.	0.04398	<b>0.10497</b>	<b>0.28146</b>	<b>0.71087</b>
VA	0.04477	0.1037	0.2732	0.7032
VAF	<b>0.04481</b>	0.10392	0.2759	0.70426
$\epsilon, b$	0.25, 200			
Boot.	0.03699	0.09689	0.26573	<b>0.69984</b>
VA	0.03781	<b>0.09746</b>	0.26485	0.69834
VAF	<b>0.03788</b>	0.09595	<b>0.26643</b>	0.69762
$\epsilon, b$	0.25, 1000			
Boot.	0.03418	0.09134	<b>0.26152</b>	<b>0.69554</b>
VA	<b>0.03542</b>	0.09125	0.25929	0.69551
VAF	0.03504	<b>0.09157</b>	0.25921	0.69341
$\epsilon, b$	0.5, 100			
Boot.	0.08472	<b>0.27716</b>	<b>0.76122</b>	<b>0.99878</b>
VA	<b>0.08738</b>	0.27453	0.75812	0.9985
VAF	0.08635	0.27485	0.75776	0.99858
$\epsilon, b$	0.5, 200			
Boot.	0.07281	<b>0.26274</b>	0.75268	0.99856
VA	<b>0.07545</b>	0.26151	<b>0.75434</b>	0.99845
VAF	0.07443	0.26224	0.75257	<b>0.99858</b>
$\epsilon, b$	0.5, 1000			
Boot.	0.06929	<b>0.25442</b>	0.75005	0.99861
VA	<b>0.06979</b>	0.25425	0.75031	<b>0.99864</b>
VAF	0.0695	0.25362	<b>0.75084</b>	0.99852
$\epsilon, b$	0.75, 100			
Boot.	0.15426	<b>0.52872</b>	0.97674	1
VA	<b>0.15825</b>	0.52797	<b>0.97722</b>	1
VAF	0.15539	0.52695	0.97671	1
$\epsilon, b$	0.75, 200			
Boot.	0.13543	<b>0.51624</b>	0.97627	1
VA	<b>0.13981</b>	0.51466	0.97642	1
VAF	0.13768	0.51286	<b>0.97652</b>	1
$\epsilon, b$	0.75, 1000			
Boot.	0.12831	<b>0.50498</b>	<b>0.97689</b>	1
VA	<b>0.13004</b>	0.50411	0.9761	1
VAF	0.12874	0.5044	0.97615	1

To shorten the paper, only the results for  $\mathbb{F}_{N,U,U}^T$  (see Figure 5) and  $\mathbb{F}_{\Gamma,U,E,E}^T$  (see Figure 6) for  $n = 5, b = 100$  are provided. In the case of the small initial sample, the power of the  $C_2$ -test is significantly lower for the classical approach if compared with the VA method (in the case of  $\mathbb{F}_{N,U,U}^T$ ) or both of the new resampling methods (for  $\mathbb{F}_{\Gamma,U,E,E}^T$ ).

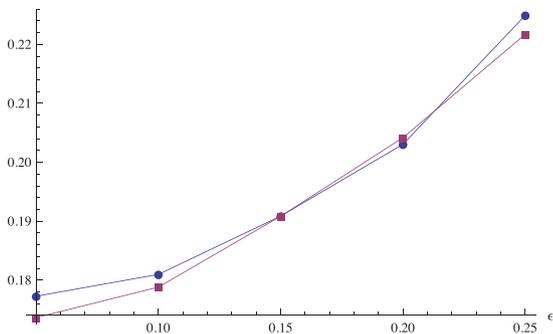
### 7.3. Standard Error of the Estimators

To evaluate the proposed method for generating bootstrap samples we also analyze the standard error of the estimators of the mean obtained from the secondary samples based on the classical approach, the VA and the VAF method. We consider the following two estimators of the variance (12), namely

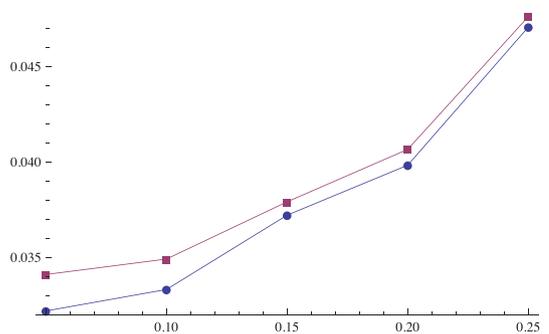
$$SE_1(m) = \sqrt{\frac{1}{m-1} \sum_{i=1}^m D_{\theta}^{\lambda}(X_i^*, \mathcal{E}(X))}, \quad (23)$$

**Table 14** | C-test power analysis for  $\mathbb{F}_{\Gamma,U,E,E}^T$

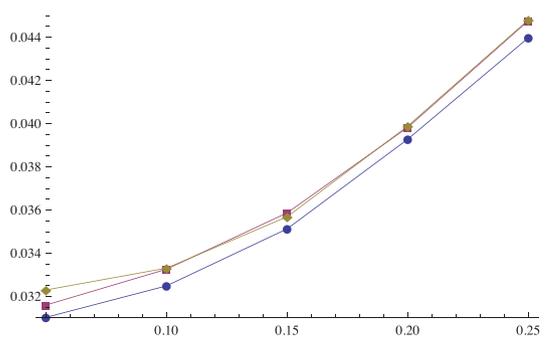
$n$	5	10	30	100
$\epsilon, b$	0.25, 100			
Boot.	0.04212	<b>0.0515</b>	0.05583	<b>0.12237</b>
VA	<b>0.04307</b>	0.04998	0.05405	0.11691
VAF	0.04296	0.05006	<b>0.05595</b>	0.11479
$\epsilon, b$	0.25, 200			
Boot.	0.03758	<b>0.04651</b>	<b>0.05078</b>	<b>0.11361</b>
VA	0.0373	0.04573	0.04919	0.10843
VAF	<b>0.03785</b>	0.04569	0.04761	0.1092
$\epsilon, b$	0.25, 1000			
Boot.	0.03521	<b>0.04368</b>	<b>0.04592</b>	<b>0.10715</b>
VA	0.03519	0.04228	0.04447	0.09967
VAF	<b>0.03526</b>	0.04226	0.04523	0.10037
$\epsilon, b$	0.5, 100			
Boot.	0.03465	<b>0.04794</b>	<b>0.10707</b>	<b>0.40581</b>
VA	<b>0.0361</b>	0.04705	0.10309	0.392
VAF	0.03574	0.04742	0.10425	0.38931
$\epsilon, b$	0.5, 200			
Boot.	0.03052	<b>0.04251</b>	<b>0.09776</b>	<b>0.39256</b>
VA	<b>0.03056</b>	0.04204	0.09466	0.38094
VAF	0.03006	0.04246	0.09413	0.38102
$\epsilon, b$	0.5, 1000			
Boot.	<b>0.0291</b>	0.03849	<b>0.09125</b>	<b>0.38409</b>
VA	0.02877	0.03804	0.08657	0.36825
VAF	0.02841	<b>0.03853</b>	0.08848	0.36717
$\epsilon, b$	0.75, 100			
Boot.	0.03143	<b>0.05988</b>	<b>0.22716</b>	<b>0.77303</b>
VA	<b>0.03216</b>	0.05876	0.22043	0.75998
VAF	0.03192	0.05932	0.21974	0.75842
$\epsilon, b$	0.75, 200			
Boot.	0.02642	<b>0.05292</b>	<b>0.21511</b>	<b>0.76798</b>
VA	<b>0.02719</b>	0.05186	0.20683	0.75334
VAF	0.02675	0.05283	0.20829	0.75342
$\epsilon, b$	0.75, 1000			
Boot.	0.02489	<b>0.04895</b>	<b>0.20417</b>	<b>0.76269</b>
VA	<b>0.02511</b>	0.04722	0.19661	0.74743
VAF	0.02478	0.04744	0.19837	0.74969



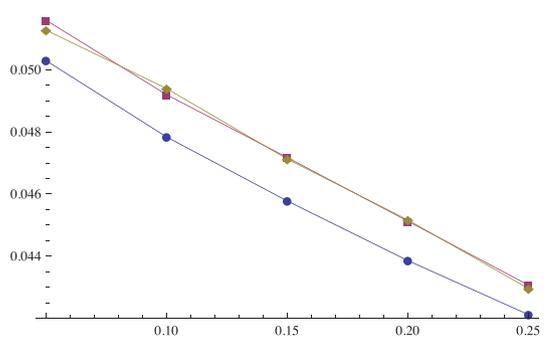
**Figure 1** | Power curves of the  $K$ -test for  $\mathbb{F}_{N,U,U}^\Delta$  for  $n = 5$ ,  $b = 100$  (the classical bootstrap—circles, the VA method—squares).



**Figure 2** | Power curves of the  $M$ -test for  $\mathbb{F}_{N,U,U}^\Delta$  for  $n = 5$ ,  $b = 100$  (the classical bootstrap—circles, the VA method—squares).



**Figure 3** | Power curves of the  $C$ -test for  $\mathbb{F}_{N,U,U,U}^T$  for  $n = 5$ ,  $b = 100$  (the classical bootstrap—circles, the VA method—squares, the VAF method—diamonds).



**Figure 4** | Power curves of the  $C$ -test for  $\mathbb{F}_{\Gamma,U,E,E}^T$  for  $n = 5$ ,  $b = 100$  (the classical bootstrap—circles, the VA method—squares, the VAF method—diamonds).

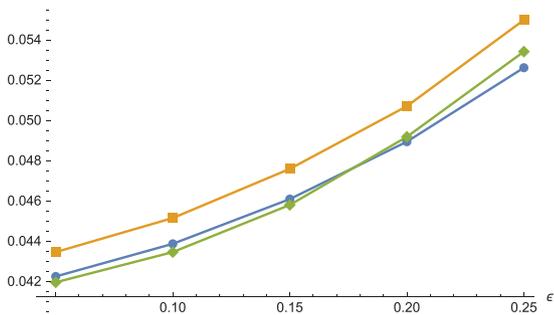
and

$$SE_2(m) = \sqrt{D_\theta^\lambda \left( \frac{1}{m} \sum_{i=1}^m X_i^*, \mathcal{E}(X) \right)}, \quad (24)$$

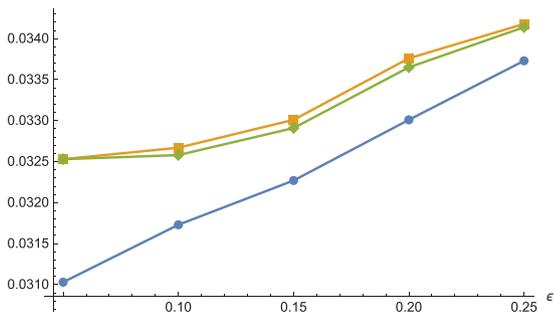
where  $m$  denotes the size of the secondary sample (see also Ref. [28]),  $\theta = 1$  and  $\lambda$  stands for the Lebesgue measure.

To shorten the paper, we provide the results only for the two cases of a moderate size ( $n = 50$ ) of the primary sample from  $\mathbb{F}_{N,U,U,U}^T$  and  $\mathbb{F}_{\Gamma,U,E,E}^T$  (see Figures 7–10).

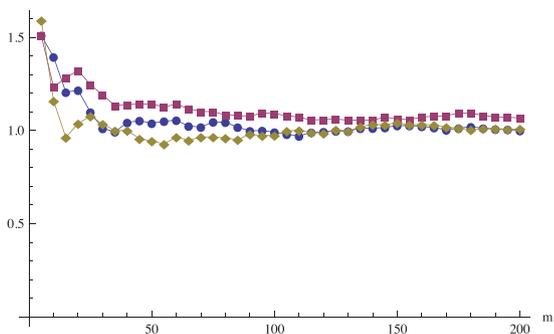
It seems that the obtained variabilities tend to be similar for all sampling methods, but in the case of  $\mathbb{F}_{\Gamma,U,E,E}^T$  they are smaller for both the VA and the VAF method if they are compared with the classical approach.



**Figure 5** | Power curves of the C2-test for  $\mathbb{F}_{N,U,U,U}^T$  for  $n = 5$ ,  $b = 100$  (the classical bootstrap—circles, the VA method—squares, the VAF method—diamonds).



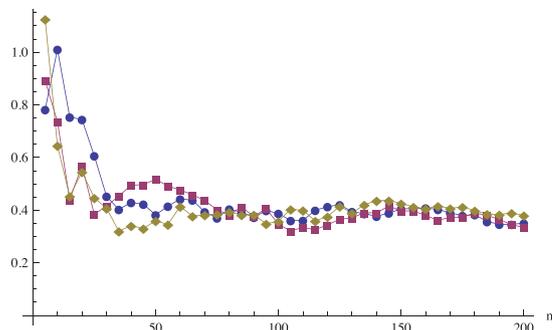
**Figure 6** | Power curves of the C2-test for  $\mathbb{F}_{\Gamma,U,E,E}^T$  for  $n = 5$ ,  $b = 100$  (the classical bootstrap—circles, the VA method—squares, the VAF method—diamonds).



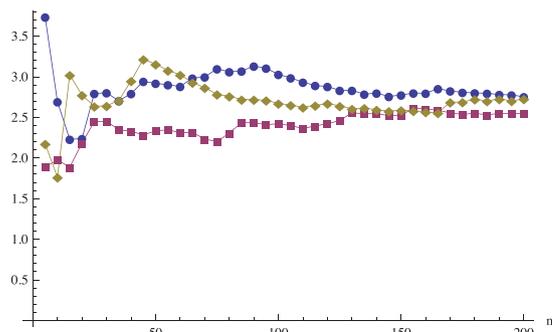
**Figure 7** |  $SE_1(m)$  for the moderate primary sample  $\mathbb{F}_{N,U,U,U}^T$  (the classical bootstrap—circles, the VA method—squares, the VAF method—diamonds).

## 8. CONCLUSIONS

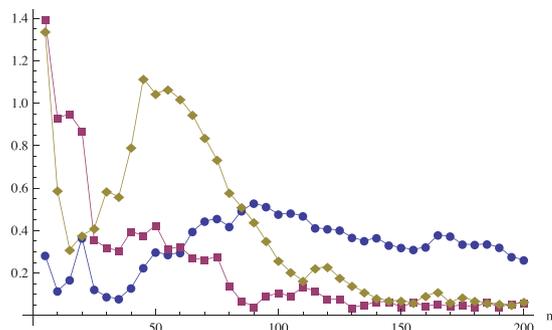
Various methods to avoid undesired repetitions in the bootstrap samples have been proposed in the literature. In this paper, we suggest another approach, called the flexible bootstrap, which can help to enrich bootstrap samples of fuzzy numbers. Our main idea is to substitute the process of drawing randomly observations from the primary samples and inserting them directly into the secondary sample by the more sophisticated approach, where the members of the secondary sample may somehow differ from those that appear



**Figure 8** |  $SE_2(m)$  for the moderate primary sample  $\mathbb{F}_{N,U,U,U}^T$  (the classical bootstrap—circles, the VA method—squares, the VAF method—diamonds).



**Figure 9** |  $SE_1(m)$  for the moderate primary sample  $\mathbb{F}_{\Gamma,U,E,E}^T$  (the classical bootstrap—circles, the VA method—squares, the VAF method—diamonds).



**Figure 10** |  $SE_2(m)$  for the moderate primary  $\mathbb{F}_{\Gamma,U,E,E}^T$  (the classical bootstrap—circles, the VA method—squares, the VAF method—diamonds).

in the primary one. We simply allow for some modifications in their membership functions, however only such which do not disturb the canonical representation of the initial fuzzy numbers. We consider both two-parameter (the value and the ambiguity) and three-parameter (the value, the ambiguity and the fuzziness) canonical representations. Then we present three algorithms for generating bootstrap samples of triangular or trapezoidal fuzzy numbers which preserve the canonical representations of observations from the primary sample.

The aforementioned algorithms for generating bootstrap samples are presented in the form ready for direct use by the practitioners. Moreover, the proposed methods were examined through the broad simulation study focused on two statistical tests and compared with the classical bootstrap. The promising results indicate that the suggested methodology can be useful in statistical inference for imprecise data.

Obviously, many questions and problems remain open. In particular, one may ask about the statistical efficiency of the proposed flexible bootstrap in applications other than testing, e.g., in constituting confidence intervals, classification, etc. It seems also that the general idea of generating bootstrap samples which preserve some characteristics of fuzzy numbers, other than these discussed in this paper, may lead to alternative flexible resampling procedures, which is worth a further study.

## CONFLICTS OF INTEREST

The authors declare no conflict of interest.

## AUTHORS' CONTRIBUTIONS

Przemysław Grzegorzewski and Maciej Romaniuk developed the main results and were in charge of writing the paper. All authors have agreed to the final version of the contribution.

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