

Research Article

Hidden Conflicts of Belief Functions

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ARTICLE INFO

Article History

Received 06 May 2020

Accepted 23 Sep 2020

Keywords

 Belief function
 Internal conflict
 Conflict between belief functions
 Hidden conflict
 Auto-conflict
 Non-conflictiness

ABSTRACT

In this paper, hidden conflicts of belief functions are observed in the case where the sum of all multiples of the conflicting belief masses equals zero. Degrees of hidden conflicts and a degree of non-conflictiness are defined and analyzed, including full non-conflictiness. The notion of a hidden conflict extends the traditional approach to conflicts of belief functions. A hidden conflict between two belief functions is distinguished from internal hidden conflict(s) of the individual belief function(s). Upper bounds for both degrees of hidden conflict of two belief functions and degrees of hidden internal conflict of a belief function are presented, together with other properties of both types of hidden conflicts. The presented results include the relationship of hidden conflicts and of Martin's auto-conflicts together with an extended overview of properties of auto-conflicts. Finally, computational issues of hidden conflicts and non-conflictiness are presented.

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1. INTRODUCTION

When combining belief functions (BFs) by the conjunctive rules of combination, some conflicts often appear (they are assigned either to \emptyset by non-normalized conjunctive rule \otimes or distributed among other belief masses by normalization in Dempster's rule of combination \oplus). A combination of conflicting BFs and interpretation of their conflicts are often questionable in real applications.

A new interpretation of conflicts was introduced in Ref. [1] where two quantitatively different parts of conflicts were distinguished: (i) an internal conflict of an individual BF (due to its internal inconsistency) and (ii) a conflict between two different BFs (due to a conflict/contradiction of evidence represented by these BFs).

The sum of all multiples of conflicting belief masses (denoted by $m_{\otimes}(\emptyset)$) was interpreted as a conflict between BFs in the classic Shafer's approach [2]. Nevertheless, examples of mutually non-conflicting BFs with high $m_{\otimes}(\emptyset)$ were observed as early as in 1990s. Attempts to cover the nature of conflicts of BFs and to classify various types of conflicts resulted in a series of papers, e.g., Refs. [3–13].

A slightly different topic, i.e., management of conflicting situations and combinations of conflicting BFs, was investigated, e.g., by Refs. [14–16].

Note that the zero sum of all multiples of conflicting belief masses is usually considered as non-conflictiness of the BFs within all of the mentioned approaches. Nevertheless, considering the

conflict between BFs based on their non-conflicting parts [5,17], a positive value of a conflict was observed even in a situation when the sum of all multiples of the conflicting belief masses equals zero. The observed conflicts are not high, but they go against the generally accepted important classification of BFs as either mutually conflicting or mutually non-conflicting!

This paradox brought up a series of new questions; we will focus on two of them in this text: *How to interpret the sum of conflicting masses? Is the conflict based on non-conflicting parts of BFs correct?* Partial answers provided in this text are positive in favor of the conflict based on non-conflicting parts, leading us to the definition of a hidden conflict of BFs.

Considering hidden conflicts, we will extend the traditional approach to the $m_{\otimes}(\emptyset)$ conflict also to conflicting cases where $m_{\otimes}(\emptyset) = 0$.

Going further, different levels/degrees of hidden conflicts are defined and the maximum degree of hidden conflict is investigated in Section 4. Analogous to the degrees of hidden conflicts, there also exist different degrees of non-conflictiness. Full non-conflictiness and conditions under which BFs are fully non-conflicting [18] are defined and presented in this paper.

In accordance with the approach from Ref. [1], not only BFs with hidden conflict between them are observed and presented, but also those with internal hidden conflicts inside an individual BF (Section 5). Moreover, we distinguish between different levels/degrees of internal conflictiness as introduced in (Section 5.5). Further, we show the relationship of a hidden conflict to Martin's auto-conflict [12,19,20] and an extended overview of Martin's

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auto-conflict properties is presented (Section 6). Finally, computational aspects of hidden conflict are briefly presented.

2. THE STATE OF THE ART

In this section, we will recall some basic notation needed in this paper and we will briefly summarize some previous works on conflicts of BFs.

2.1. Notation

Assume a finite frame of discernment Ω is given. In the case of $|\Omega| = n$, we will note this fact using a subscript as Ω_n and assume $\Omega_n = \{\omega_1, \omega_2, \dots, \omega_n\}$. $\mathcal{P}(\Omega) = \{X | X \subseteq \Omega\}$ is the *power-set* of Ω .

A *basic belief assignment (bba)* is a mapping $m : \mathcal{P}(\Omega) \rightarrow [0, 1]$ such that $\sum_{A \subseteq \Omega} m(A) = 1$; the values of the bba are called *basic belief masses (bbm)*; $m(\emptyset) = 0$ is usually assumed.

There are other equivalent representations of m : A *BF* is a mapping $Bel : \mathcal{P}(\Omega) \rightarrow [0, 1]$, $Bel(A) = \sum_{\emptyset \neq X \subseteq A} m(X)$. A *plausibility function* $Pl : \mathcal{P}(\Omega) \rightarrow [0, 1]$, $Pl(A) = \sum_{\emptyset \neq A \cap X, X \subseteq \Omega} m(X)$. Because there is a unique correspondence between m and the corresponding Bel and Pl , we often speak about m as a BF.

Let $X \subseteq \Omega$ and Bel be a BF defined by bba m on Ω . If $m(X) > 0$ for $X \subseteq \Omega$, we say X is a *focal element* of Bel . If all focal elements are *singletons* ($|X| = 1, X \subseteq \Omega$), then we speak about a *Bayesian belief function (BBF)*; in fact, it is a probability distribution on Ω . If there are only focal elements such that $|X| = 1$ or $|X| = n$, the BF is called *quasi-Bayesian (qBBF)*. In the case of $m(\Omega) = 1$, we speak about the *vacuous BF (VBF)* and about a *non-vacuous BF* otherwise. In the case of $m(X) = 1$ for $X \subset \Omega$, the BF is called *categorical*; if all focal elements have a non-empty intersection, the BF is called *consistent*; and if all of the focal elements are nested, the BF is called *consonant*.

Dempster's (normalized conjunctive) rule of combination \oplus :

$$(m_1 \oplus m_2)(A) = \sum_{X \cap Y = A; X, Y \subseteq \Omega} K m_1(X) m_2(Y)$$

for $A \neq \emptyset$, where $K = \frac{1}{1-\kappa}$,

$$\kappa = \sum_{X \cap Y = \emptyset; X, Y \subseteq \Omega} m_1(X) m_2(Y), \quad (1)$$

and $(m_1 \oplus m_2)(\emptyset) = 0$, see Ref. [2]. Putting $K = 1$ and $(m_1 \circledast m_2)(\emptyset) = \kappa$, we obtain the *non-normalized conjunctive rule of combination* \circledast , see, e. g., Ref. [21]:

$$(m_1 \circledast m_2)(A) = \sum_{X \cap Y = A; X, Y \subseteq \Omega} m_1(X) m_2(Y) \quad (2)$$

for any $A \subseteq \Omega$. To simplify formulas, we often use $\circledast_1^3 m = m \circledast m \circledast m$, and also $\circledast_1^k(m_1 \circledast m_2) = (m_1 \circledast m_2) \circledast \dots \circledast (m_1 \circledast m_2)$, where $(m_1 \circledast m_2)$ is repeated k -times.

Smets' *pignistic probability* is given by $BetP(\omega_i) = \sum_{\omega_j \in X \subseteq \Omega} \frac{1}{|X|} \frac{m(X)}{1-m(\emptyset)}$, see, e.g., Ref. [21]. *Normalized plausibility of singletons* of m is a probability distribution Pl_P such that $Pl_P(\omega_i) = \frac{Pl(\{\omega_i\})}{\sum_{\omega \in \Omega} Pl(\{\omega\})}$ [22,23] where Pl is the corresponding

plausibility function. Note that plausibility of singletons is called a *contour function* by Shafer in Ref. [2] : $f(\omega) = Pl(\{\omega\})$; thus $Pl_P(\omega)$ is just the respective normalized contour function.

2.2. Conflicts of BFs

A conflict between/among BFs is a situation when two or generally more BFs are somehow against each other. Definitely, when the BFs support different decisions.

The original Shafer's definition of the conflict measure between two BFs [2] is nothing else than the normalization constant κ from (1) used in the definition of Dempster's combination rule \oplus (more precisely, it is its transformation $\log \frac{1}{1-\kappa}$). Thus $\kappa = (m_1 \circledast m_2)(\emptyset)$ and we abbreviate it as $\kappa = m_{\circledast}(\emptyset)$. Whenever we refer to $m_{\circledast}(\emptyset)$, we think of two BFs m_1, m_2 and their conjunctive combination, respectively the sum of all multiples of basic belief masses of their disjoint focal elements:

$$m_{\circledast}(\emptyset) = (m_1 \circledast m_2)(\emptyset) = \sum_{X \cap Y = \emptyset; X, Y \subseteq \Omega} m_1(X) m_2(Y) \quad (3)$$

Unfortunately, it is not always possible to interpret $m_{\circledast}(\emptyset)$ as (a size of) a conflict between BFs. The problematic issue of an essence of a conflict between BFs was first mentioned by Almond [24] already in 1995, and discussed further by Liu [11] in 2006. Almond's counterexample has been overcome by W. Liu's progressive approach. Unfortunately, the substance of the issue has not been solved that time as a positive conflict value still may be detected for a pair of mutually non-conflicting BFs.

The very important W. Liu's approach [11] was followed by a series of other approaches and their modifications. W. Liu suggested a two-dimensional conflict measure composed of $m_{\circledast}(\emptyset)$ and $DifBetP_{m_j}^{m_i}$ —the maximum difference of pignistic probabilities $BetP_{m_i}(A), BetP_{m_j}(A)$ for BFs Bel_i, Bel_j and their bbas m_i, m_j over the focal elements $A \subseteq \Omega$ (thus a kind of a distance of BFs); as it was shown, neither $m_{\circledast}(\emptyset)$ nor any distance of BFs may solely be used as a convenient measure of conflict of BFs.

Among other approaches, we can mention the axiomatic foundation laid by Destercke and Burger [8], resulting in a conflict between BFs being measured as the inconsistency of their conjunctive combination, further elaborated with Burger's geometric approach [3]. The axiomatic approach given by Destercke and Burger served as a basis for the work of Pichon *et al.* [25]—they introduced a whole parameterized family of consistency measures covering several existing definitions of consistency of a BF. They further tried to interpret the consistency of a BF as a distance to a BF representing total inconsistency. Nevertheless, note that a conflict between BFs itself cannot be simply defined as any distance between these BFs because the conflict in general violates the basic property of any distance measure, the triangle inequality.

Another axiomatic approach to conflict of BFs was done by Martin in Ref. [12].

An internal conflict of a BF is naturally always unary (related to only one BF), whereas a conflict with another BF(s) may in general be n -ary—related to several BFs. Nevertheless, in accordance with

both the classic approach [2] and all of the previous authors' publications, including [1,5], we are only interested in up to binary conflicts between two BFs in this study.

In 2010, Daniel distinguished an internal conflict inside an individual BF from a conflict with another BF ([1]) and defined three new approaches to conflicts. The most promising prospective approach (see Ref. [1])—the so-called *plausibility conflict*—was further elaborated in Refs. [4,7]. We have to also mention the authors' consonant conflict measures [26], which use consonant approximations of BFs when computing the corresponding measures of conflict. Finally, Daniel's *conflict based on non-conflicting parts of BFs* was introduced in Ref. [5]. In addition to the works on the relationship of $m_{\odot}(\emptyset)$ to conflict of BFs, we should also mention studies of its relationship to the Open World Assumption [27].

The last-mentioned measure of conflict has motivated our current research of hidden conflicts and their special case—a hidden auto-conflict of BFs [28].

Definition 1. A conflict of BFs Bel' , Bel'' , based on their non-conflicting parts Bel'_0 , Bel''_0 , is defined by the expression

$$Conf(Bel', Bel'') = (m'_0 \odot m''_0)(\emptyset),$$

where m_0 is the bba corresponding to Bel_0 , a non-conflicting part of BF Bel , which is nothing else than a unique consonant BF (focal elements are nested) with the same normalized plausibility of singletons as Bel , i.e., $Pl_{P_0} = Pl_P$.

Algorithm 1 for how to compute m_0 (originally suggested in Ref. [5]) follows.

3. HIDDEN CONFLICTS

When analyzing properties of the conflict based on the non-conflicting parts, the following questions arise. Does $(m' \odot m'')(\emptyset) = 0$ really represent non-conflictiveness of respective BFs as it is usually assumed? Is the definition of the conflict based on non-conflicting parts correct? Surprisingly, as one can see in the following example, there are BFs with $(m' \odot m'')(\emptyset) = 0$ while $Conf(m', m'') > 0$.

3.1. An Introductory Example

Let us assume there are two simple consistent BFs Bel' , Bel'' on $\Omega_3 = \{\omega_1, \omega_2, \omega_3\}$ given by the bbas $m'(\{\omega_1, \omega_2\}) = 0.6$, $m'(\{\omega_1, \omega_3\}) = 0.4$, and $m''(\{\omega_2, \omega_3\}) = 1.0$ (see Figure 1). Then $(m' \odot m'')(\emptyset) = 0$, which seems—and is usually considered—to be a proof of non-conflictiveness of Bel' and Bel'' . Surprisingly, especially with respect to the previous work [5], there is a positive conflict based on the non-conflicting parts $Conf(Bel', Bel'') = 0.4$. This issue was further elaborated in a recent paper [29] where the size of the conflict based on the non-conflicting parts of BFs $Conf$ was compared with the sum of all multiples of bbms of disjoint focal elements of the BFs in question $m_{\odot}(\emptyset)$ with the goal to reach a simple upper bound for $Conf$.

We can easily verify this statement: there is an only focal element of m'' that has a non-empty intersection with both focal elements of m' , thus $\sum_{(X \cap Y) = \emptyset} m'(X)m''(Y)$ is an empty sum, hence $(m' \odot m'')(\emptyset) = 0$; m'' is consonant, thus $m''_0 = m''$. $Pl'(\{\omega_1\}) =$

Algorithm 1

(Computing the non-conflicting part of a BF). Take all element(s) with the maximum contour (plausibility of singletons) value; they create the smallest (by cardinality) focal element of m_0 with a mass that equals the difference between the max and “next to the max” (different) contour values.

A cycle. Among the remaining elements (if there are any), take again all the element(s) with the maximum contour value and add them to the previous focal element, you thus obtain a new focal element of m_0 , and its mass corresponds to the difference between the different contour values again.

Repeat the cycle **until** Ω_n is obtained as a focal element with a mass that equals the minimum contour value. For a positive minimum contour value,^a include Ω_n among focal elements of m_0 . In the case of a non-consistent BF ($Pl(\{\omega_i\}) < 1$ for any $\omega_i \in \Omega_n$), a final normalization of m_0 is needed.

More formally:

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FE := ∅; SFE := ∅; Ω := Ωn; m0 := array();
Max_Pl := Pl({ω}) such that Pl({ω}) ≥ Pl({ω'})
Min_Pl := Pl({ω}) such that Pl({ω}) ≤ Pl({ω'})
Max1 := {ω ∈ Ω | Pl({ω}) = Max_Pl}
Ω := Ω \ Max1
Max2 := {ω ∈ Ω | Pl({ω}) ≥ Pl({ω'})}
while Max2 ≠ ∅ do
  FE := FE ∪ Max1; SFE := SFE ∪ {FE}
  m0(FE) := Pl({ω1}) - Pl({ω2}),
    where ω1 ∈ Max1, ω2 ∈ Max2
  Max1 := Max2; Ω := Ω \ Max1
  Max2 := {ω ∈ Ω | Pl({ω}) ≥ Pl({ω'})}
end while
if Min_Pl > 0 then
  SFE := SFE ∪ {Ωn}
  /* because FE ∪ Max1 = Ωn * /
  m0(Ωn) := Min_Pl
end if
if ∑X ⊆ Ω m0(X) < 1 then
  normalize m0
  /* because ∑X ⊆ Ωn m0(X) = Max_Pl * /
end if
return m0

```

1, $Pl'(\{\omega_2\}) = 0.6$, $Pl'(\{\omega_3\}) = 0.4$, and $m'_0(\{\omega_1\}) = (1 - 0.6)/1 = 0.4$, $m'_0(\{\omega_1, \omega_2\}) = 0.2$, $m'_0(\{\omega_1, \omega_2, \omega_3\}) = 0.4$, hence $Conf(m', m'') = (m'_0 \odot m''_0)(\emptyset) = m'_0(\{\omega_1\})m''_0(\{\omega_2, \omega_3\}) = 0.4 \cdot 1 = 0.4$.

3.2. Interpretation of the Example—Observation of a Hidden Conflict

Are m' and m'' conflicting or non-conflicting? What does $(m' \odot m'')(\emptyset) = 0$ mean? How to interpret it? These and similar questions come in one's mind.

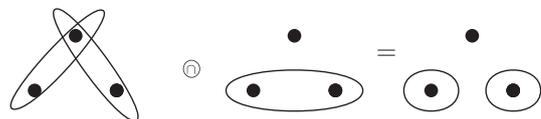


Figure 1 | Introductory Example: focal elements of m' , m'' , and of $m' \odot m''$.

^a Ω_n is not included among the focal elements of m_0 in the case of zero minimum contour value.

Let us continue in our reasoning. In the case of the Introductory Example, two non-conflicting BFs are combined. Thus, no conflict should arise there and also the combination $m' \oplus m''$ should not be conflicting with either m' or m'' .

This process can subsequently be repeated and no conflict should arise. Does this hold for BFs m' , m'' from our Example above? It holds true when we combine $m' \oplus m''$ with m'' one more time (assuming two instances of m'' coming from two independent belief sources). It follows from the idempotency of categorical m'' : $m' \oplus m'' \oplus m'' = m' \oplus m''$ and therefore $(m' \oplus m'' \oplus m'')(\emptyset) = 0$ again. On the other hand, we obtain $((m' \oplus m'') \oplus m'')(\emptyset) = (m' \oplus m'' \oplus m'')(\emptyset) = (m' \oplus m'' \oplus m'')(\emptyset) = 0.48$ (analogously assuming m' coming from two independent belief sources). See Table 1 and Figure 2. Similarly, when m'' and m' are combined once, we observe $m_{\oplus}(\emptyset) = 0$. When combining m'' with m' twice, then $((m' \oplus m'') \oplus (m' \oplus m''))(\emptyset) = 0.48$ holds. We observe a certain kind of a *hidden conflict* despite the fact that both individual BFs are consistent (there are no internal conflicts). Thus the conflict is a *hidden conflict between respective BFs*. It can serve as an argument for correctness of $Conf(m', m'') > 0$.

What is a decision-based interpretation of our BFs? The contours, i.e., the plausibilities of singletons are $Pl' = (1.0, 0.6, 0.4)$ and $Pl'' = (0.0, 1.0, 1.0)$; we obtain $Pl_{P'} = (0.5, 0.3, 0.2)$ and $Pl_{P''} = (0.0, 0.5, 0.5)$ by normalization; thus ω_1 is significantly preferred by m' , whereas it is one of ω_2, ω_3 in the case of m'' . This is also an argument for a positive value of a mutual conflict of the BFs. Considering Smets' pignistic probability, we obtain $BetP' = (0.5, 0.3, 0.2)$ and $BetP'' = (0.0, 0.5, 0.5)$, just the same values as in the case when the normalized plausibility of singletons (normalized contour) is used for decision. Both (in general different) probabilistic approximations $BetP$ and Pl_P give the highest value to different singletons for m' and m'' . Thus the argument for mutual conflictness of the BFs is supported and we obtain the same pair of incompatible decisions based on the BFs in both frequent decision-making approaches: using either the normalized contour (which is compatible with the conjunctive combination of BFs) or the pignistic probability (designed for betting). Note that there are other probability transformations of BFs (as summarized in Refs. [23,30]) but we do not consider them now.

Hence $(m' \oplus m'')(\emptyset)$ does not generally represent non-conflictness. It rather states just a simple or partial compatibility of the respective BFs, (focal elements).

3.3. Objections Against Our Interpretation

Thinking about this novel approach, one may have the following objections:

- “In the case of a combination of two identical BFs, an idempotent rule of combination should be used.” Yes, this would be right for BFs coming from two dependent belief sources. But this is not true for two or more numerically identical BFs coming from two or more independent belief sources.
- “The result is not surprising, because the conflict is increasing when combining several BFs.”
 - To be correct it should be stated non-“decreasing” instead of “increasing.”
 - More precisely, a conflict is non-decreasing when several conflicting BFs are combined. When truly non-conflicting BFs are combined, no positive conflict can ever arise there; e.g., $(\bigoplus_1^k m_1 \oplus \bigoplus_1^k m_2)(\emptyset) = 0$ for any $k > 0$ and m_i on Ω_3 given by $m_1(\{\omega_1\}) = 0.3, m_1(\{\omega_1, \omega_2\}) = 0.2, m_1(\{\omega_1, \omega_3\}) = 0.1, m_1(\{\omega_1, \omega_2, \omega_3\}) = 0.4, m_2(\{\omega_1, \omega_3\}) = 0.7, m_2(\{\omega_1, \omega_2, \omega_3\}) = 0.3$.
- “The result is rather unsurprising because one can clearly see that the hidden conflict occurs when the first combination results in disjoint focal elements.” Yes, in the very simple Introductory Example, this may be unsurprising for someone; but there are no disjoint sets after the first combination in the Little Angel Example below. Moreover, this should be surprising for all who accept the following assumption/axiom: BFs m' and m'' are non-conflicting whenever $(m' \oplus m'')(\emptyset) = 0$, e.g., Refs. [8,11,12] and the previous Daniel’s publications, e.g., Refs. [1,4,5,7].
- “It is obvious that a combination results in a conflict if a Bayesian BF ($m' \oplus m''$ in the Introductory Example) is combined with any other BF.” Yes, this is true in the very simple Introductory Example, but not in a general example, again see the Little Angel Example below.

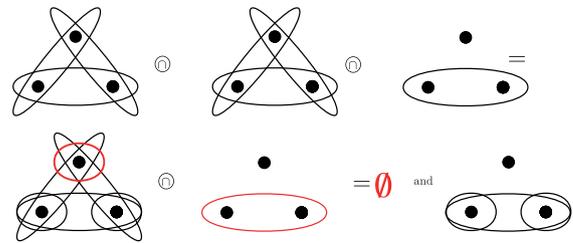


Figure 2 | Arising of a hidden conflict between belief functions (BFs) in the Introductory Example: focal elements of $m', m'', m'; m' \oplus m', m''; and of (m' \oplus m') \oplus m''$.

Table 1 | Hidden conflict in the Introductory Example

X	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2, \omega_3\}$	Ω_3	\emptyset
$m'(X)$	0.0	0.0	0.0	0.60	0.40	0.00	0.00	—
$m''(X)$	0.0	0.0	0.0	0.00	0.00	1.00	0.00	—
$(m' \oplus m'')(X)$	0.00	0.60	0.40	0.00	0.00	0.00	0.00	0.00
$(m' \oplus m'' \oplus m'')(X)$	0.00	0.60	0.40	0.00	0.00	0.00	0.00	0.00
$(m' \oplus m'' \oplus m')(X)$	0.00	0.36	0.16	0.00	0.00	0.00	0.00	0.48
$(m' \oplus m'' \oplus m' \oplus m'')(X)$	0.00	0.36	0.16	0.00	0.00	0.00	0.00	0.48

Analyzing these objections, one can see why it is not easy to observe the hidden conflicts. In simple cases, the observations seem to be obvious, thus not interesting. In more general examples, the conflict is hidden.

3.4. Definition of Hidden Conflict

Definition 2. Let us suppose two BFs Bel' , Bel'' are given by m' and m'' such that $(m' \odot m'')(\emptyset) = 0$. If there further holds

$$(m' \odot m'' \odot m')(\emptyset) > 0 \text{ or } (m' \odot m'' \odot m'')(\emptyset) > 0 \quad (4)$$

we say that there is a *hidden conflict* of the BFs.

Observation 1. Condition $(m' \odot m'' \odot m')(\emptyset) > 0$ or $(m' \odot m'' \odot m'')(\emptyset) > 0$ for the hidden conflict from Definition 2 simply implies the following weaker condition¹

$$(m' \odot m'' \odot m' \odot m'')(\emptyset) > 0. \quad (5)$$

Thus the new condition covers more cases of hidden conflicts than the original condition (4) from Definition 2.

We have to note that a hidden conflict is quite a new phenomenon, qualitatively different from ideas of all of Daniel's previous works on conflicts of BFs and also different from the other approaches we have referred to. To this date, it has been supposed that $m_{\odot}(\emptyset)$ includes entire conflict of respective BFs (all parts of the entire conflict): both the conflict between BFs and the internal conflicts of individual BFs. Thus the conflict between BFs was supposed to be less than or equal to $m_{\odot}(\emptyset)$. Here, we deal with a situation of a positive conflict between BFs while $m_{\odot}(\emptyset) = 0$.

We have already observed that $m_{\odot}(\emptyset) = 0$ does not mean full non-conflictiveness of respective BFs and that the condition (4), and even its weaker version (5) together with $(m' \odot m'')(\emptyset) = 0$, defines a hidden conflict. But it is still unknown whether condition (5) defines all hidden conflicts, i.e., whether its zero version

$$(m' \odot m'' \odot m' \odot m'')(\emptyset) = 0 \quad (6)$$

is sufficient for full non-conflictiveness of BFs m' and m'' . May some conflict still be hidden there?

Condition (6) seems to imply non-conflictiveness on Ω_3 , the frame of discernment used in the Introductory Example. To solve the question in general, we have to consider a larger frame of discernment.

¹(i) This is an update with respect to the text in Eusflat proceedings [36]: an equivalence of the conditions was originally stated there. The equivalence holds true for any BFs on a 3-element Ω_3 , but its left implication does not hold in general, for greater frames of discernment. There exists a counter-example already on Ω_4 : Let m_{c1}, m_{c2} be defined by $m_{c1}(\{\omega_1, \omega_3, \omega_4\}) = 0.7, m_{c1}(\{\omega_2, \omega_3, \omega_4\}) = 0.3; m_{c2}(\{\omega_1, \omega_2, \omega_3\}) = 0.6, m_{c2}(\{\omega_1, \omega_2, \omega_4\}) = 0.4$. Hence $(m_{c1} \odot m_{c2} \odot m_{c1})(\emptyset) = (m_{c1} \odot m_{c1} \odot m_{c2})(\emptyset) = 0$ and $(m_{c1} \odot m_{c2} \odot m_{c2})(\emptyset) = 0$ but $(m_{c1} \odot m_{c1} \odot m_{c2} \odot m_{c2})(\emptyset) = 0.0504 > 0$.

(ii) A question under which condition the equivalence holds on greater frames is still an open issue for future research.

3.5. Little Angel Example

For Ω_5 , one can find the following Little Angel Example—as defined in Table 2. Similarly to the Introductory Example, we have two consistent BFs Bel^i, Bel^{ii} defined by m^i and m^{ii} with disjoint sets of max-plausibility elements while the zero condition $(m^i \odot m^{ii})(\emptyset) = 0$ holds true.

In addition to the Introductory Example, $(m^i \odot m^{ii} \odot m^i \odot m^{ii})(\emptyset) = 0$ (see Table 2) while $Conf(m^i, m^{ii}) = 0.1$ is positive again. Positiveness of the $Conf$ value can be easily seen from the fact that the sets of max-plausibility elements are disjoint for the respective Pl^i and Pl^{ii} . Numerically, we have again $Bel_0^i = Bel^{ii}$, and $Pl_{-P^i} = (\frac{10}{39}, \frac{4}{39}, \frac{9}{39}, \frac{9}{39}, \frac{7}{39})$. We obtain $m_0^i(\{\omega_1\}) = 0.1, m_0^i(\{\omega_1, \omega_3, \omega_4\}) = 0.2, m_0^i(\{\omega_1, \omega_3, \omega_4, \omega_5\}) = 0.3, m_0^i(\{\Omega_5\}) = 0.4$, and $Conf(Bel^i, Bel^{ii}) = m_0^i(\{\omega_1\})m^{ii}(Z) = 0.1$. Analogous arguments hold true for the positive $Conf$ and the hidden conflict again (the conflict is more hidden this time; we will define it later as a hidden conflict of the 2nd degree). $BetP^i = (0.2583, 0.1083, 0.2250, 0.2250, 0.1833)$ which is numerically not the same as Pl_{-P^i} , but both prefer ω_1 , whereas $BetP^{ii} = Pl_{-P^{ii}} = (0.00, 0.25, 0.25, 0.25, 0.25)$.

For an existence of a hidden conflict, it is the structure of the focal elements that is important—not their belief masses. Belief masses are important for the size of a conflict. In general, we can take $m^i(A) = a, m^i(B) = b, m^i(C) = c$, for A, B, C defined in Table 2, and for any $a, b, c > 0$, such that $a + b + c = 1$, we obtain $m(\emptyset) = 6abc$ as a hidden conflict of the 2nd degree (a conflict hidden in the second degree); in our numeric case there is $6abc = 6 \cdot 0.1 \cdot 0.3 \cdot 0.6 = 0.108$. For graphical representation of the Little Angel Example, see Figure 3.

Degrees of a hidden conflict, its maximal value, and the issue of full non-conflictiveness will be analyzed in Section 4.

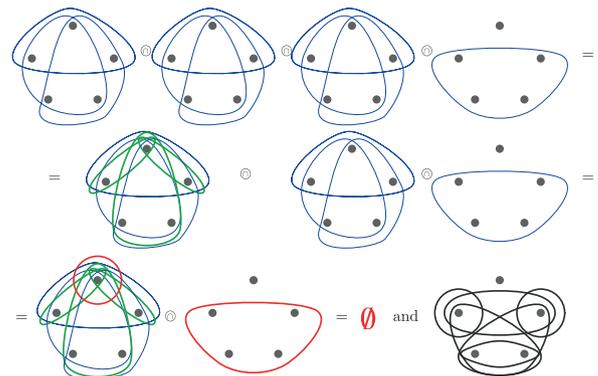


Figure 3 Arising of a hidden conflict between belief functions (BFs) in the Little Angel Example. Focal elements of $m^i, m^{ii}, m^i \odot m^i, m^i \odot m^i \odot m^i$, and of $(m^i \odot m^i \odot m^i) \odot m^{ii}$. Focal elements responsible for creation of the empty set in the last step are highlighted.

Table 2 | Hidden Conflict in the Little Angel Example.

X	$A =$ $\{\omega_1, \omega_2, \omega_5\}$	$B =$ $\{\omega_1, \omega_2, \omega_3, \omega_4\}$	$C =$ $\{\omega_1, \omega_3, \omega_4, \omega_5\}$	$Z =$ $\{\omega_2, \omega_3, \omega_4, \omega_5\}$	\emptyset
$m^i(X)$	0.10	0.30	0.60	0.00	—
$m^{ii}(X)$	0.00	0.00	0.00	1.00	—

X	$A \cap Z$	$B \cap Z$	$C \cap Z$	$A \cap B \cap Z$	$A \cap C \cap Z$	$B \cap C \cap Z$	\emptyset
$(m^i \circledast m^{ii})(X)$	0.10	0.30	0.60	0.00	0.00	0.00	0.00
$(m^i \circledast m^{ii} \circledast m^{ii})(X)$	0.10	0.30	0.60	0.00	0.00	0.00	0.00
$(m^i \circledast m^i \circledast m^{ii})(X)$	0.01	0.09	0.36	0.06	0.12	0.36	0.00
$(m^i \circledast m^i \circledast m^{ii} \circledast m^{ii})(X)$	0.01	0.09	0.36	0.06	0.12	0.36	0.00
$(m^i \circledast m^{ii} \circledast m^{ii} \circledast m^{ii})(X)$	0.010	0.090	0.360	0.060	0.120	0.360	0.000
$(m^i \circledast m^i \circledast m^i \circledast m^{ii})(X)$	0.001	0.027	0.216	0.036	0.126	0.486	0.108
$(m^i \circledast m^i \circledast m^i \circledast m^{ii} \circledast m^{ii})(X)$	0.001	0.027	0.216	0.036	0.126	0.486	0.108

4. DEGREES OF HIDDEN CONFLICT AND FULL NON-CONFLICTNESS

When analyzing examples from Section 3, we can observe two different degrees of hidden conflict. Let us formalize this observation:

Definition 3. Assume two BFs Bel^i, Bel^{ii} are defined by m^i and m^{ii} such that for some $k > 0$ $(\bigcirc_1^k(m^i \circledast m^{ii}))(\emptyset) = 0$. If there further holds $(\bigcirc_1^{k+1}(m^i \circledast m^{ii}))(\emptyset) > 0$, we say that there is a *conflict of BFs m^i and m^{ii} hidden in the k -th degree²* (hidden conflict of k -th degree).

In an analogy to particular degrees of a hidden conflict, there are degrees of non-conflictiness. The particular degrees of non-conflictiness are less important now. However, there is an important question whether there is a hidden conflict or not, i.e., whether or not the BFs in question are fully non-conflicting.

Definition 4. We say that BFs Bel^i, Bel^{ii} defined by m^i and m^{ii} are *fully non-conflicting* if $(m^i \circledast m^{ii})(\emptyset) = 0$ and, further, if there is no hidden conflict of any degree. I.e., if $(\bigcirc_1^k(m^i \circledast m^{ii}))(\emptyset) = 0$ for any $k > 0$.

There is thus a question of how many times we have to combine $(m^i \circledast m^{ii})$, i.e., for which k value of $(\bigcirc_1^k(m^i \circledast m^{ii}))(\emptyset)$ shows whether there is a hidden conflict of the BFs m^i and m^{ii} or not. For an answer to this question, see corollaries of the following two theorems.

Theorem 2 (maximum degree of hidden conflict). For any non-vacuous BFs Bel^i, Bel^{ii} defined by m^i and m^{ii} on Ω_n it holds that

$$(\bigcirc_1^{n-1}(m^i \circledast m^{ii}))(\emptyset) = 0 \quad \text{iff} \quad (\bigcirc_1^k(m^i \circledast m^{ii}))(\emptyset) = 0$$

for any $k \geq n - 1$.

Corollary 3. A hidden conflict of any non-vacuous BFs on any Ω_n always has a degree less than or equal to $n - 2$; i.e., the condition

$$(\bigcirc_1^{n-1}(m^i \circledast m^{ii}))(\emptyset) = 0 \quad (7)$$

²If there is already $(\bigcirc_1^{0+1}(m^i \circledast m^{ii}))(\emptyset) = (m^i \circledast m^{ii})(\emptyset) > 0$ for $k = 0$ then there is a conflict of the BFs which is not hidden or we can say that it is a conflict hidden in degree zero

always means the full non-conflictiness of any BFs m^i and m^{ii} on any Ω_n . Moreover, there is no hidden conflict³ on any two-element frame Ω_2 .

To prove the Theorem and its Corollary, let us first introduce several auxiliary assertions listed in the following Lemma.

- Lemma 4.**
- i. $m^i \circledast m^{ii}$: the focal elements (f.e.) of m^i, m^{ii} are kept and/or new smaller f.e. possibly appear(s) if the BFs are combined.
 - ii. $m \circledast m$: the focal elements (f.e.) of any m are kept + possibly new smaller f.e. appear(s) if the BF is combined with itself.
 - iii. $(m^i \circledast m^{ii}) \circledast m^{ii}$: the focal elements of $(m^i \circledast m^{ii})$ are kept + possibly new smaller f.e. appear(s) if $(m^i \circledast m^{ii})$ is combined with m^i or m^{ii} again.
 - iv. $\bigcirc_1^k(m^i \circledast m^{ii})$: the focal elements of $(m^i \circledast m^{ii})$ are kept + possibly new smaller f.e. appear(s) if $(m^i \circledast m^{ii})$ is combined with m^i, m^{ii} or with itself several times, thus with $\bigcirc_1^k m^i, \bigcirc_1^k m^{ii}$ or $\bigcirc_1^k(m^i \circledast m^{ii})$ for any $k \geq 1$.
 - v. If any focal element F of $(m^i \circledast m^{ii})$ is kept when $(m^i \circledast m^{ii})$ is combined with m^{ii} , then F is also kept when $(m^i \circledast m^{ii})$ is combined with $(m^{ii} \circledast m^{ii})$ and also with $\bigcirc_1^k m^{ii}$ for any $k \geq 1$.
 - vi. If any focal element F of $(m^i \circledast m^{ii})$ is kept when $(m^i \circledast m^{ii})$ is combined with $(m^i \circledast m^{ii})$, then F is also kept when $(m^i \circledast m^{ii})$ is combined with $\bigcirc_1^2(m^i \circledast m^{ii})$ and also with $\bigcirc_1^k(m^i \circledast m^{ii})$ for any $k \geq 1$.
 - vii. If any focal element F of $\bigcirc_1^k(m^i \circledast m^{ii})$ is kept when $\bigcirc_1^k(m^i \circledast m^{ii})$ is combined with $(m^i \circledast m^{ii})$, then F is also kept when combined with $\bigcirc_1^m(m^i \circledast m^i)$ for any $m > 1$.
 - viii. For any BFs m^i and m^{ii} it holds: $\bigcirc_1^k(m^i \circledast m^{ii})(\emptyset) > 0$ implies $\bigcirc_1^{k+1}(m^i \circledast m^{ii})(\emptyset) > 0$.

Proof. [of Theorem on maximum degree] Focal element (f.e.) Ω does not cause any conflict in any case. Proper focal elements (of cardinality not larger than $n - 1$) possibly produce a singleton after a maximum of $n - 2$ combinations \circledast which can possibly be

³ There may only be conflicts of degree $2 - 2 = 0$; meaning there are only conflicts $(m^i \circledast m^{ii})(\emptyset) > 0$ that are not hidden.

conflicting with a f.e. of another BF (using the Lemma above) and $\bigcirc_1^{n-1}(m^i \odot m^{ii})(\emptyset) > 0$.

The maximum possible degree is $n - 2$: it immediately follows from Example 1 that, in such a case, the statements hold true. If there is no unhidden conflict then, after at most $n - 2$ \odot combinations of $m^i \odot m^{ii}$ with themselves, a stable set of f.e. is reached. If there are any conflicting/disjoint f.e., they must thus appear no later than when the stable set of f.e. is reached.

Let us present an example of such a highly hidden conflict now. \boxtimes

Example 1. Example of a hidden conflict of the $(n - 2)$ -th degree: Let us suppose an n -element frame of discernment $\Omega_n = \{\omega_1, \omega_2, \dots, \omega_n\}$ is given, and m^i has $n - 1$ focal elements—all subsets of Ω_n of cardinality $n - 1$ containing ω_1 . m^{ii} has the only focal element $\Omega_n \setminus \{\omega_1\}$. Specifically, $m^i(\Omega_n \setminus \{\omega_n\}) = \frac{1}{n-1}, m^i(\Omega_n \setminus \{\omega_{n-1}\}) = \frac{1}{n-1}, m^i(\Omega_n \setminus \{\omega_{n-2}\}) = \frac{1}{n-1}, \dots, m^i(\Omega_n \setminus \{\omega_2\}) = \frac{1}{n-1}$. m^{ii} is categorical and $m^{ii}(\Omega_n \setminus \{\omega_1\}) = 1$.

At $m^i \odot m^i$ focal elements of cardinality $n - 2$ appear, at $m^i \odot m^i \odot m^i$ focal elements of cardinality $(n - 3)$ appear, at $\bigcirc_1^k m^i$ focal elements of cardinality $(n - k)$ appear, and at $\bigcirc_1^{n-2} m^i$ focal elements of cardinality 2 appear. All these focal elements have non-empty intersections with the only focal element of m^{ii} . Finally, at $\bigcirc_1^{n-1} m^i$, a singleton focal element $\{\omega_1\}$ appears and it has an empty intersection with the only focal element of m^{ii} — $Z = \{\omega_2, \omega_3, \dots, \omega_n\}$.

What does m^i express? It gives a big support to all elements of the frame, to the entire frame Ω_n , and even a greater support to ω_1 , which is included in all focal elements; ω_1 is preferred and, moreover, it has plausibility 1.

We can modify m^i and express this more easily: $\bar{m}^i(\Omega_n) = \frac{n-1}{n}$, $\bar{m}^i(\{\omega_1\}) = \frac{1}{n}$, or more generally, $\tilde{m}^i(\Omega_n) = 1 - a$, $\tilde{m}^i(\{\omega_1\}) = a$ for some $0 < a < 1$. We can easily see an evident conflict corresponding to positive $\bar{m}(\emptyset) = (\bar{m}^i \odot m^{ii})(\emptyset) = \frac{1}{n}$, $\tilde{m}(\emptyset) = (\tilde{m}^i \odot m^{ii})(\emptyset) = a$ for these modifications of m^i . Hence either a hidden conflict of the $(n - 2)$ -th degree of m^i and m^{ii} or positive $Conf(m^i, m^{ii}) = Conf(\bar{m}^i, m^{ii}) = \frac{1}{n}$ should not be very surprising.

Let us note that the Introductory Example is just a special instance of Example 1 in the case of $n = 3$.

Besides the previous Example, we can formulate a description of the structure of couples of BFs with their conflicts hidden in the $(n-2)^{th}$ degree as it follows:

Lemma 5. The only non-vacuous BFs on Ω_n with hidden conflicts of degree $(n - 2)$ are BFs with focal elements of cardinality $\geq n - 1$, such that one has at least $(n - 1)$ focal elements of cardinality $(n - 1)$ and the other one has just one focal element of cardinality $(n - 1)$. Moreover, every $(n - 1)$ -element subset of Ω_n must be a focal element of either one or both BFs.

Proof. (Let us denote f.e. as a focal element, $|F| = n$ as a f.e. F of cardinality n , and h.c. as a hidden conflict.) F.e. $|F| = n$ does not decrease (by intersection) neither cardinality of any other f.e. nor the degree of the h.c. Note that BFs from Example 1 and all their extensions with f.e.(s) $|F| = n$ have a h.c. of the $(n - 2)$ -th degree. Removing of any f.e. $|F| = n - 1$ from the BFs completely remove the h.c.

Without a loss of generality, we can assume m^i with at least $(n - 1)$ f.e. $|F| = n - 1$ and m^{ii} with just one f.e. $|F| = n - 1$. Addition of another f.e. $|F| = n - 1$ to m^{ii} excludes another element of Ω_n by m^{ii} (more precisely by $F \cap (\Omega_n \setminus \{\omega_1\})$ in $\bigcirc_1^2 m^{ii}$); thus the degree of h.c. decreases by 1. Note that an addition of the missing f.e. $|F| = n - 1$ to m^i from Example 1 keeps the degree $n - 2$ of the h.c. There is—of course—a decrease of the degree of the h.c. by adding a f.e. $|F| < n - 1$ to either m^i or m^{ii} . The degree of the h.c. is analogously decreased by decreasing the cardinality of any f.e. $|F| = n - 1$. Moving a f.e. $|F| = n - 1$ from m^i to m^{ii} : F decreases its cardinality in $\bigcirc_1^2 m^{ii}$; there are no necessary $n - 1$ combinations with m^i to obtain the empty set, hence there is a decrease in the degree of the h.c. by 1 again.

We have described a nice and useful general upper bound for a degree of a hidden conflict. Nevertheless, for a special set of BFs or for pairs of BFs we can go even further to show a tighter upper bound for the degree of hiddenness of the conflict. \boxtimes

Theorem 6. The following assertions hold true.

- i. Any non-vacuous BFs Bel^i, Bel^{ii} given by m^i, m^{ii} have a conflict hidden in at most $(c - 1)$ -th degree where $c = \min(c^i, c^{ii}) + \text{sgn}(|c^i - c^{ii}|)^4$, where c^i, c^{ii} are maximal cardinalities of the proper focal elements of m^i, m^{ii} (i.e., f.e. different from Ω). In other words,

$$(\bigcirc_1^c(m^i \odot m^{ii}))(\emptyset) = 0 \text{ iff } (\bigcirc_1^k(m^i \odot m^{ii}))(\emptyset) = 0$$

for any $k \geq c = \min(c^i, c^{ii}) + \text{sgn}(|c^i - c^{ii}|)$.

- ii. There are no hidden conflicts of any non-vacuous BFs on any two-element frame Ω_2 .
- iii. There are no hidden conflicts of any non-vacuous quasi-Bayesian BFs on any frame Ω_n .
- iv. For a BF m^i and a quasi-Bayesian BF m^{ii} there is a hidden conflict of (at most) the first degree; if it appears then it is, in fact, an internal conflict of m^{ii} .

Proof. (idea of) All these statements follow from Lemma 4, analogous to the proof of Theorem on maximum degree. Regarding (iv): it always holds $Conf(m^i, m^{ii}) = 0$ for a BF m^i and a qBBF m^{ii} such that $(m^i \odot m^{ii})(\emptyset) = 0$. \boxtimes

Corollary 7. i. Assume two non-vacuous BFs Bel^i, Bel^{ii} are given by m^i, m^{ii} on Ω_n . The zero value of the expression $(\bigcirc_1^c(m^i \odot m^{ii}))(\emptyset)$, i.e., the condition

$$(\bigcirc_1^c(m^i \odot m^{ii}))(\emptyset) = 0 \tag{8}$$

means the full non-conflictiness of the BFs for $c = \min(c^i, c^{ii}) + \text{sgn}(|c^i - c^{ii}|)$, where c^i, c^{ii} are maximum cardinalities of the proper focal elements of m^i, m^{ii} .

⁴Note that $\text{sgn}(|c^i - c^{ii}|)$ is equal to 0 for BFs with the same sizes of the maximum focal elements, and it is equal to 1 otherwise.

- ii. For any two non-vacuous quasi-Bayesian BFs m^i, m^{ii} on any frame of discernment Ω_n , the condition $(m^i \odot m^{ii})(\emptyset) = 0$ always means the full non-conflictiveness of the BFs.
- iii. For any BF m^i and any quasi-Bayesian BF m^{ii} , the condition $(\odot_1^2(m^i \odot m^{ii}))(\emptyset) = 0$ always means the full non-conflictiveness of the BFs.

5. INTERNAL HIDDEN CONFLICT

5.1. Internal Conflict and Conflict Between BFs

As a new interpretation of a general conflict of BFs was introduced in Ref. [1] where two types of conflict were distinguished (an internal conflict of an individual BF and a conflict between two BFs) it is also interesting to look at hidden conflicts from that point of view.

An entire or global⁵ conflict of two BFs usually contains a mixture⁶ of internal conflicts of individual BFs (if there are any) and a mutual conflict between them (if there is any). For mutually non-conflicting BFs, the global conflict is the composition of individual internal conflicts. On the other hand, for internally non-conflicting BFs it is just a conflict between them. Using this fact, we can show that it is reasonable to speak about a hidden internal conflict of a BF.

5.2. Observation of Internal Hidden Conflict

Modified Little Angel Example

Consider BFs given by m^i, m^{ii} from Table 2 again. Let us take m^{iii} instead of m^i such that $m^{iii}(A) = m^i(A)$, $m^{iii}(C) = m^i(C)$, and $m^{iii}(D) = m^{iii}(\{\omega_2, \omega_3, \omega_4\}) = 0.30$ instead of $m^i(B)$ where focal elements A, B, C are defined in Table 2. There is $(m^{ii} \odot m^{iii} \odot m^{iii})(\emptyset) = 0$, but

$$(m^{ii} \odot m^{iii} \odot m^{iii} \odot m^{iii})(\emptyset) = 0.108 > 0, \text{ and}$$

$$(m^{iii} \odot m^{iii} \odot m^{iii})(\emptyset) = 0.108 > 0, \text{ i.e. } (\odot_1^3 m^{iii})(\emptyset) > 0. \text{ For more details, see Ref. [31] and Figure 4.}$$

We observe a conflict of the BFs hidden in the 2nd degree again. Nevertheless, the situation of focal elements is different now: the only focal element Z (Table 2) of $m^{ii} = \odot_1^3 m^{iii}$ has a non-empty intersection with any focal element of $\odot_1^3 m^{iii}$, but $(\odot_1^3 m^{iii})(\emptyset) = 0.108 > 0$ now.

The only focal element (thus all focal elements) of both m^{ii} and $\odot_1^k m^{iii} = m^{ii}$ has a non-empty intersection with all focal elements of both m^{iii} and $\odot_1^k m^{iii}$ and vice versa. This therefore cannot be a

⁵Note that an entire or global conflict was called the total conflict in Ref. [1]. Unfortunately, a total conflict is used for cases of fully conflicting BFs, for conflicts which are equal to one (usually maximum) value of any measure of conflict by some other authors; that is why we do not use the notion of total conflict here.

⁶Let us note that this mixture is not a simple addition of two or three components; it is still not known how these part are composed together, and it is still an open issue whether it is possible to separate or somehow compute particular parts of an entire/global general conflict if all of its parts are positive.

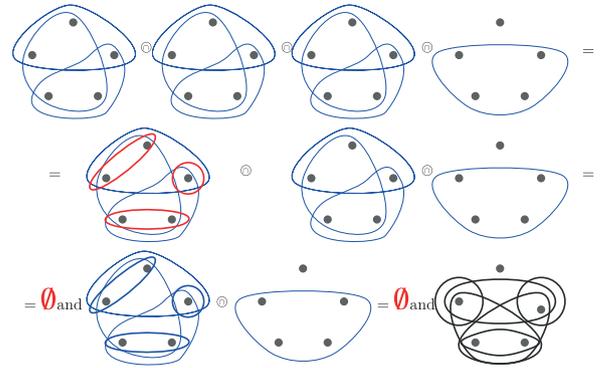


Figure 4 | Arising of an internal hidden conflict of belief function (BF) Bel^{iii} in the Modified Little Angel Example. Focal elements of $m^{iii}, m^{ii}, m^{iii} \odot m^{iii}, m^{iii} \odot m^{iii} \odot m^{iii}$, and of $(m^{iii} \odot m^{iii} \odot m^{iii}) \odot m^{ii}$. Focal elements responsible for creation of the empty set in the last step are highlighted.

hidden conflict between m^{ii} and m^{iii} , but a sort of *an internal hidden conflict* of m^{iii} .

Further, we can show that there is no conflict based on non-conflicting parts between m^{ii} and m^{iii} . Indeed, following Algorithm 1, $Conf(m^{ii}, m^{iii}) = 0$ due to non-empty intersection of sets with max-plausibility singletons: $Pl^{ii} = (0, 1, 1, 1, 1)$, $Pl^{iii} = (0.7, 0.4, 0.9, 0.9, 0.7)$, thus $Pl_P^{ii} = (0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, $Pl_P^{iii} = (\frac{7}{36}, \frac{4}{36}, \frac{9}{36}, \frac{9}{36}, \frac{7}{36})$, and $\{\omega_2, \dots, \omega_5\} \cap \{\omega_3, \omega_4\} \neq \emptyset$, and Algorithm 1 produces consonant BFs m_0^{ii} and m_0^{iii} that are non-conflicting. Hence there is another argument for mutual non-conflictiveness of m^{ii} with m^{iii} and an internal conflict of the individual BFs. m^{ii} is categorical, definitely internally non-conflicting; hence it must just be an internal conflict of m^{iii} , which is not consistent.

In general, we can observe an internal hidden conflict if at least one of BFs in consideration is not consistent.

5.3. Internal Hidden Conflict and Its Degrees

Considering the close relationship of the internal conflict of a BF to a hidden conflict of two BFs presented in the Modified Little Angel Example, we can define an internal hidden conflict with different degrees of hiddenness analogous to the definition of a hidden conflict.

Definition 5. Let us assume a BF is given by m such that $(\odot_1^2 m)(\emptyset) = 0$ and $(\odot_1^s m)(\emptyset) > 0$ for an $s > 2$. Then we say that there is *an internal hidden conflict* in m . More specifically, if $\exists k \geq 0$ such that $(\odot_1^{k+1} m)(\emptyset) = 0$ and $(\odot_1^{k+2} m)(\emptyset) > 0$, then we say that there is *an internal conflict of BF m hidden in k -th degree*⁷ (hidden internal conflict of k -th degree).

Let us show that we can find a simple example even on a 3-element frame of discernment Ω_3 . On the other hand, there are no internal conflicts of consistent BFs, see Lemma 8.

⁷Note that for $k = 0$ there is just $m(\emptyset) = 0$ and $(m \odot m)(\emptyset) > 0$; hence m is consistent and the conflict is not hidden or we can say *hidden in degree zero*; for $k = 1$ there is just $(m \odot m)(\emptyset) = 0$ and $(m \odot m \odot m)(\emptyset) > 0$ hence the conflict is hidden in the 1st degree.

Example 2. Let us suppose a BF on Ω_3 given by bba m''' from Table 3; we obtain $(\odot^2 m''')(\emptyset) = 0$ and $(\odot^3 m''')(\emptyset) = 0.18$ there. Hence the positive internal conflict found in $(\odot^3 m''')(\emptyset) = 0.18$ is again hidden by $(\odot^2 m''')(\emptyset) = 0$. For the focal elements and appearing of the hidden internal conflict of $\odot^3 m''' = m''' \odot m''' \odot m'''$, see Figure 5.

Lemma 8. There is no (hidden) internal conflict in any consistent BF.

Proof. All focal elements have non-empty intersections, thus there holds $(m \odot m)(\emptyset) = (\odot_1^k m)(\emptyset) = 0$ for any k and any BF defined by consistent bba m on any n -element frame Ω_n . \square

Corollary 9. Any (hidden) conflict of any consistent BFs is a (hidden) conflict between them.

5.4. Relationships between Hidden and Internal Hidden Conflicts

Let us turn our attention back to the original Little Angel Example. Similarly to the Introductory Example there is no internal hidden conflict of input BF, as $(\odot_1^k m)(\emptyset) = 0$ for all consistent BFs, thus for all BFs from the examples. Nevertheless, we can also find an internal hidden conflict in the original Little Angel Example: Let us compute the conjunctive combination of the input BFs m^i and m^{ii} : $m = m^i \odot m^{ii}$. We obtain $m(\{\omega_2, \omega_5\}) = 0.1$, $m(\{\omega_2, \omega_3, \omega_4\}) = 0.3$, $m(\{\omega_3, \omega_4, \omega_5\}) = 0.6$, $m(\emptyset) = 0$ (See line $(m^i \odot m^{ii})(X)$ in Table 2.)

Following Table 2, note that $(\odot_1^2 m)(\emptyset) = 0$ and $(\odot_1^3 m)(\emptyset) = 0.108$. There is an internal hidden conflict of degree 1.

For the respective focal elements and appearing of the hidden internal conflict of $m = m^i \odot m^{ii}$, see Figure 6.

Example 3. (A general example) We can take a conjunctive sum of any two BFs with a hidden conflict of 2nd degree, thus $m = m^i \odot m^{ii}$: there is $(\odot_1^2 m)(\emptyset) = (m^i \odot m^i \odot m^{ii} \odot m^{ii})(\emptyset) = 0$ and $(\odot_1^3 m)(\emptyset) = (m^i \odot m^i \odot m^i \odot m^{ii} \odot m^{ii} \odot m^{ii})(\emptyset) > 0$. There is thus a hidden internal conflict of 1st degree in $m = m^i \odot m^{ii}$.

Analogously, we can construct an internal conflict hidden in higher degrees from a conflict of two BFs hidden in degree $k > 2$. On the other hand, one can represent an internal hidden conflict as a conflict of two BFs:

Theorem 10. *i. Let us assume two BFs m^i, m^{ii} with hidden conflict of k -th degree for $k \geq 2$ and their conjunctive combination $m = m^i \odot m^{ii}$. Then there is an internal conflict of m hidden in $k - 1$ -th degree.*

ii. Any hidden conflict of any BF m of any degree $k > 1$ can be expressed as a hidden conflict of two BF of degree $k + 1$.

Table 3 | Example of a hidden internal conflict on Ω_3 .

X	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2, \omega_3\}$	Ω_3	\emptyset
$m'''(X)$	0	0	0	0.50	0.30	0.20	0	—
$\odot^2 m'''(X)$	0.30	0.20	0.12	0.25	0.09	0.04	0	0
$\odot^3 m'''(X)$	0.36	0.21	0.09	0.125	0.027	0.008	0	0.18

Proof.

- i. $(\odot_1^k m^i \odot m^{ii})(\emptyset) = 0 = (\odot_1^k m)(\emptyset)$ and $(\odot_1^{k+1} m)(\emptyset) = (\odot_1^{k+1} (m^i \odot m^{ii}))(\emptyset) > 0$. Thus the positive $(\odot_1^{k+1} m)(\emptyset)$ is hidden by $(\odot_1^s m)(\emptyset) = 0$ for $1 \leq s \leq k$, thus in $(k - 1)$ -th degree.
- ii. We can always use $m = m \odot VBF$; hence from $0 = (\odot_1^{k+1} m)(\emptyset) = (\odot_1^{k+1} (m \odot VBF))(\emptyset)$ and analogously $0 < (\odot_1^{k+2} m)(\emptyset) = (\odot_1^{k+2} (m \odot VBF))(\emptyset)$. \square

Note that, for $k = 1$ and $m = m^i \odot m^{ii}$, we obtain $0 < (\odot_1^{k+1} (m^i \odot m^{ii}))(\emptyset) = (\odot_1^2 m)(\emptyset) = (m \odot m)(\emptyset)$. Hence there is a conflict of m with itself, which is not hidden.

From this observation, we can understand why $k \geq 2$ must be in Theorem 10 and why the degree of hiddenness of internal conflict is by one smaller than the degree of hiddenness of the corresponding conflict of two BFs.

5.5. Maximum Degree of Internal Hidden Conflict

Due to a close relationship of hidden internal conflicts to hidden conflicts of BFs as mentioned in Theorem 10, it is obvious that the maximum degree of a hidden internal conflict is also bounded.

Theorem 11 provides an answer to the question of whether a BF m on Ω_n has any internal conflicts and whether its degree is either bounded or completely non-conflicting.

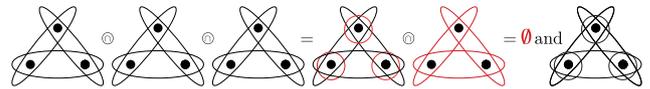


Figure 5 | Arising of a hidden auto-conflict of belief function (BF) m''' , focal elements of m''' , $\odot^2 m'''$, and $\odot^3 m'''$.

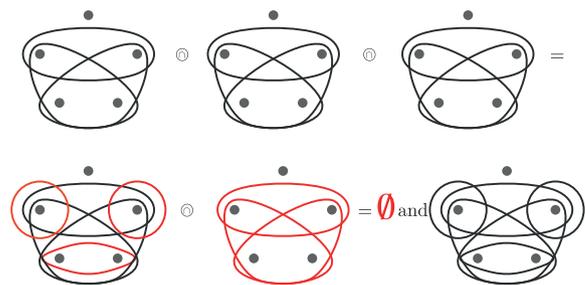


Figure 6 | Arising of a hidden auto-conflict of belief function (BF) $m^i \odot m^{ii}$, focal elements of $m^i \odot m^{ii}$, $\odot_1^2 (m^i \odot m^{ii})$, and $\odot_1^3 (m^i \odot m^{ii})$.

Theorem 11. For any BF given by bba m on Ω_n the following assertion holds:

$$(\odot_1^n m)(\emptyset) = 0 \text{ iff } (\odot_1^{n+k} m)(\emptyset) = 0$$

for any $k \geq 1$. The maximum degree of a hidden internal conflict is thus equal to $n - 2$.

Corollary 12. The zero value of the expression $(\odot_1^n m)(\emptyset)$, i.e., the condition

$$(\odot_1^n m)(\emptyset) = 0 \tag{9}$$

always means the full internal non-conflictiness of any BF given by any bba m on any Ω_n .

Proof. (idea of proof of Theorem 11) The size of new focal elements (f.e.) which appear when m is combined with itself is decreased until a fixed set of f.e. is reached. There is thus at most $n - 2$ decreases (creations of a new, smaller f.e.); at most $(n - 1)$ -th combination produces the empty intersection; n -times m , $(n - 1)$ -times \odot .

It means that the last possible combination such that $\odot_1^{k+1} m(\emptyset) = 0$ and $\odot_1^{k+2} m(\emptyset) > 0$ holds for $k + 1 = n - 1$, which proves the theorem by Definition 5.

For a given BF, we have an even tighter upper bound for the degree of hiddenness derived from sizes of its focal elements. \boxtimes

Theorem 13. For a given BF m on Ω_n , the following equality holds:

$$(\odot_1^{s+1} m)(\emptyset) = 0 \text{ iff } (\odot_1^{s+1+k} m)(\emptyset) = 0$$

for any $k \geq 1$, where s is the maximum cardinality of a proper focal element of m (a focal element different from Ω_n). Thus m may have a hidden internal conflict of a degree up to $s - 1$.

Corollary 14. The zero value the expression $(\odot_1^{s+1} m)(\emptyset)$, i.e.,

$$(\odot_1^{s+1} m)(\emptyset) = 0, \tag{10}$$

where s is the maximum cardinality of a proper focal element of bba m , always means the full internal non-conflictiness of any BF given by any bba m on any Ω_n .

Proof. (idea of proof of Theorem 13) Analogous to the proof of Theorem 11, there is at most s decreases for $s < n$. \boxtimes

The class of qBBFs has the maximum cardinality of a proper focal element 1, thus, according to this Theorem, we can see that the maximum degree of a hidden internal conflict is 0 for any qBBF. Similarly, according to Theorem 11, we can see that there is no hidden internal conflict of any BF on Ω_2 .

For a general BF on Ω_n , there are proper focal elements of cardinality up to $s = n - 1$. Thus $s + 1 = n$ and we can understand Theorem 11 as a special case of Theorem 13.

We have an upper bound for a degree of an internal hidden conflict. And we can ask the following questions: May this upper bound be reached? What do the BFs with the maximum degree of an internal hidden conflict look like?

Example 4. (A general example) Similar to hidden conflicts in general, the structure of focal elements is important for the existence and the particular degrees of hidden internal conflicts (not

values of belief masses; values are important for the resulting values of hidden internal conflicts). We can thus present a BF $m^{(n)}$ on Ω_n , having all subsets of Ω_n of cardinality $n - 1$ as its focal elements. For the sake of simplicity, we may assume $m^{(n)}(X) = 1/n$ for all $|X| = n - 1$. We obtain $(\odot_1^{n-1} m^{(n)})(\emptyset) = 0$ and $(\odot_1^n m^{(n)})(\emptyset) > 0$. There is a hidden internal conflict of $(n - 2)$ -th degree.

Example 5. (Specific examples) i. A simple example of the maximum degree of a hidden internal conflict on Ω_3 (thus the maximum degree is 1 there) has already been presented in Example 2. There is $(\odot_1^2 m''')(\emptyset) = 0$ and $(\odot_1^3 m''')(\emptyset) = 0.18$.

ii. Having $m^{(3)}$ from Example 4 with $m^{(3)}(X) = 1/3$, we obtain $(\odot_1^2 m^{(3)})(\emptyset) = 0$ and $(\odot_1^3 m^{(3)})(\emptyset) = 2/9 = 0.22\bar{2}$, $((\odot_1^3 m^{(3)})(\{\omega_i\})) = 2/9$ and $(\odot_1^3 m^{(3)})(X) = 1/27$ for $|X| = 2$.

iii. Let us suppose $m^{(16)}$ on Ω_{16} now: $m^{(16)}(X) = 1/16 = 0.0625$ for all $|X| = 15$ and $m^{(16)}(X) = 0$ otherwise. We obtain $(\odot_1^{15} m^{(16)})(\emptyset) = 0$ and $(\odot_1^{16} m^{(16)})(\emptyset) = 1.134227 \cdot 10^{-6}$. Thus there is very small hidden internal conflict of 14th degree. $(\odot_1^{24} m^{(16)})(\emptyset) = 5.460487 \cdot 10^{-3}$, $(\odot_1^{36} m^{(16)})(\emptyset) = 0.1480689$.

iv. Supposing m^{xvi} with bbms $m^{xvi}(X)$: 0.005, 0.010, 0.015, 0.020, 0.030, 0.040, 0.050, 0.060 0.065, 0.075, 0.085, 0.095, 0.105, 0.110, 0.115, 0.120 for $|X| = 15$ and $m^{xvi}(X) = 0$ otherwise we obtain: $(\odot_1^{15} m^{xvi})(\emptyset) = 0$ and $(\odot_1^{16} m^{xvi})(\emptyset) = 7.089108 \cdot 10^{-9}$.

Theorem 15. BFs on Ω_n with the maximum degree $n - 2$ of a hidden internal conflict are just BF from one of the following categories:

- i. $m^{(n)}$ with the structure as in Example 4,
- ii. $m^{(n)}$ with the structure as in Example 4 extended with focal element Ω_n .

There are no other BFs on Ω_n with a hidden internal conflict of degree $n - 2$.

Proof. Idea of proof. Both classes are obvious. A decreasing number of focal elements removes the hidden internal conflict; decreasing cardinality of any of the focal elements decreases the degree of the hidden conflict/internal conflict. Analogously, addition of any other focal element F (note that $|F|$ is less than $n - 1$) also decreases the degree.

Let us present a numerical example of a highly hidden internal conflict of the $(n - 2)$ -th degree on a general frame of discernment now. \boxtimes

Example 6. Let us consider the following modification of Example 1 on Ω_n . Instead of m^i , we take m^{iii} , having all focal elements of cardinality $n - 1$ such that $m^{iii}(\Omega_n \setminus \{\omega\}) = \frac{1}{n}$ for any $\omega \in \Omega_n$; m^{ii} same as in Example 1. m^{iii} is not consistent; $P^{iii}(\{\omega\}) = \frac{n-1}{n}$ for any $\omega \in \Omega_n$. We observe a hidden conflict of the $(n - 2)$ -th degree again. Because of the same plausibilities of all singletons, $m_0^{iii}(\Omega_n) = 1$ and $Conf(m^{iii}, m^{ii}) = 0$.

There is a positive hidden conflict of BFs m^{iii} and m^{ii} , but a zero conflict between them. This situation corresponds to the presence of an internal hidden conflict corresponding to the individual

Table 4 | Hidden conflict between belief functions (BFs) given by m^i and m^{ii} from Example 1 and hidden conflict of m^{ii} and m^{iii} from Example 6 which is an internal hidden conflict of m^{iii} , both on Ω_{16} .

Degree	Ω_{16}	$m' = m^i, m'' = m^{ii}$			$m' = m^{iii}, m'' = m^{ii}$		
	m_{\odot}	No of f. e.	Card. of f. e.	$m_{\odot}(\emptyset)$	No of f. e.	Card. of f. e.	$m_{\odot}(\emptyset)$
–	m'	15	15	–	16	15	–
–	m''	1	15	–	1	15	–
0	$m'_{\odot}m''$	15	14	0	16	14–15	0
1	$m'_{\odot}m''_{\odot}m'_{\odot}m''$	120	13–14	0	121	13–15	0
2	$\odot_1^3(m'_{\odot}m'')$	575	12–14	0	576	12–15	0
...	0	0
k	$\odot_1^{k+1}(m'_{\odot}m'')$...	(14-k)–14	0	...	(14-k)–15	0
...	0	0
13	$\odot_1^{14}(m'_{\odot}m'')$	32766	1–14	0	32767	1–15	0
14	$\odot_1^{15}(m'_{\odot}m'')$	32766	1–14	$2.98 \cdot 10^{-6}$	32767	1–15	$1.13 \cdot 10^{-6}$

non-consistency of respective BFs; m^{ii} is consistent; there is, consequently, an internal hidden conflict of BF m^{iii} in this case.

A numeric example was computed on Ω_{16} , see Table 4 for a comparison of focal elements and $(\odot_1^k m)(\emptyset)$ values of Examples 1 and 6.

6. HIDDEN CONFLICTS VS. AUTO-CONFLICTS

6.1. Auto-Conflicts

Let us note that an internal hidden conflict is closely related the so-called auto-conflict [12,19,20] $a(m)$ introduced by Osswald and Martin in Ref. [20]. Nevertheless, $1 - a(m)$ was referred to as the plausibility of a belief structure with itself by Yager [32] as a measure of consistency already in 1992. Let us recall its definition and some basic properties.

Definition 6. The *auto-conflict* of a BF m is given by $a(m) = (m_{\odot}m)(\emptyset)$; the *auto-conflict of order k* of a BF m is given by $a_k(m) = (\odot_1^k m)(\emptyset)$, where \odot is the non-normalized conjunctive combination.

Let us turn our attention to basic properties of an auto-conflict. The following implication holds true by Ref. [20] :

$$a_k(m) \leq a_{k+1}(m) \ \& \ a_2(m) > 0 \Rightarrow \lim_{k \rightarrow \infty} a_k(m) = 1.$$

It is stated in Ref. [19] that $\lim_{k \rightarrow \infty} a_k(m) = 1$ holds true (without any assumptions). This is generally not correct (any consistent BF is a counter-example). Nevertheless, it holds under a non-explicitly stated assumption of BFs considered in Ref. [19]—i.e., a special subclass of quasi-Bayesian BFs, BFs with all singletons plus Ω_n as their focal elements, i.e., qBBFs with exactly $n + 1$ focal elements. Such BFs have always $a_2(m) > 0$. On the other hand, for all consistent BFs it holds that $a(m) = a_2(m) = 0 = a_k(m)$, for any $k > 0$. Even $\lim_{k \rightarrow \infty} a_k(m) = 0$.

For any other BFs it holds that $a(m) \geq 0$ and $a_k(m) \leq a_{k+1}(m)$. Moreover it holds that $a_{k-1}(m) < a_k(m)$ implies $a_k(m) < a_{k+1}(m)$. This follows the numbers and cardinalities of focal elements of $a_k(m)$ (more precisely of $\odot_1^k m$; $a_k(m)$) has all focal elements of $a_{k-1}(m)$, plus possibly some additional focal elements (defined by intersections of focal elements of $a_{k-1}(m)$). Furthermore, if $a_{k-1}(m)$

and $a_k(m)$ have the same focal elements, this holds true also for any $a_s(m)$ where $s > k$. In particular,, whenever \emptyset is a “focal element” of $a_k(m)$ it is also a “focal element” of any $a_s(m)$ where $s > k$ and $a_s(m) < a_{s+1}(m)$.

6.2. Hidden Auto-Conflict

We can simply observe the relationship of our definition with Martin’s auto-conflict. It is simply possible to interpret/rewrite conditions from Definition 5 using auto-conflict as $(\odot_1^{k+1} m)(\emptyset) = a_{k+1}(m) = 0$ and $(\odot_1^{k+2} m)(\emptyset) = a_{k+2}(m) > 0$, thus we can define:

Definition 7. We say that BF Bel given by bba m has a *hidden auto-conflict*, if its auto-conflict is $a(m) = a_2(m) = 0$ and if $a_s(m) > 0$ holds for a certain $s > 0$.

We say that BF m have a *hidden auto-conflict of degree s* if $a_{s+1}(m) = 0$ and $a_{s+2}(m) > 0$.

Lemma 16. The hidden internal conflict of k -th degree is just the hidden auto-conflict of k -th degree.

Whenever we speak about a hidden internal conflict, we speak about a hidden auto-conflict, and vice versa. We must distinguish between two different terms here, order and degree, even though they are close to each other in their meanings and only differ in their values.⁸

Using this Lemma we can express a hidden conflict of BFs using auto-conflicts:

⁸Note that there is, moreover, only one degree of a hidden conflict for a given pair of BFs, the only one degree of a hidden internal conflict or a hidden auto-conflict of a given BF, nevertheless there is infinite number of orders of Martin’s auto-conflicts of the given BF. Let us assume a pair of BFs Bel^i, Bel^{ii} , and their conjunctive combination $Bel = Bel^i_{\odot}Bel^{ii}$; the auto-conflict of Bel of order equal to the degree of the hidden conflict of BFs Bel^i and Bel^{ii} (if there is a hidden conflict) is zero and similarly all the auto-conflicts of Bel of smaller orders, thus *order* \sim *degree* + 1, or rather *order* \geq *degree* + 1, more precisely, *order* of positive auto-conflict of $Bel^i_{\odot}Bel^{ii} \geq$ *degree* + 1. Analogously for the auto-conflict of Bel of order equal to the degree of the internal hidden conflict of Bel + 1 is zero and similarly all the auto-conflicts of Bel of smaller orders, hence *order* \sim *degree* + 2, or rather *order* (of positive auto-conflict) \geq *degree* + 2) in this case.

Corollary 17. Let us assume BFs Bel^i, Bel^{ii} are given by bbas m^i, m^{ii} .

- i. A hidden conflict of BFs Bel^i and Bel^{ii} of the first degree is just a positive auto-conflict (of second order) of their conjunctive combination $a(m^i \odot m^{ii})$ under the assumption that $(m^i \odot m^{ii})(\emptyset) = 0$ (i.e., that $a_1(m^i \odot m^{ii}) = 0$).
- ii. A hidden conflict of BFs Bel^i and Bel^{ii} of the k -th degree is a hidden auto-conflict of $k - 1$ degree; it is just a positive auto-conflict of $(k + 1)$ -th order of their conjunctive combination $a_{k+1}(m^i \odot m^{ii})$ under the assumption that $(\odot_1^k(m^i \odot m^{ii}))(\emptyset) = 0$ (i.e., that $a_k(m^i \odot m^{ii}) = 0$).

6.3. Extended Overview of Properties of Auto-Conflicts and Hidden Auto-Conflicts

Using our results on hidden internal conflicts and their relationships to Martin’s auto-conflicts, we can characterize properties of auto-conflicts and extend the presentation of their properties as follows.

Simply, any consistent BF has no auto-conflict of any order (its auto-conflicts of any orders are equal to 0) and, therefore, no hidden auto-conflict as well. (Any intersection of any number of focal elements of a consistent BF is non-empty, thus $(\odot_1^k m)(\emptyset) = 0$ and $a_k(m) = 0$ for any $k > 0$).

For a non-consistent BF m on Ω_n there are possible auto-conflicts of some orders. From the examples shown earlier in Section 5, we have seen that auto-conflict $a(m)$ may be equal to zero, hence there may be a hidden auto-conflict. From Theorem 11 we know that a degree of a hidden auto-conflict is at most $n - 2$, i.e., that any non-consistent BF has a (positive) auto-conflict of order n . And consequently it holds also for auto-conflicts of any orders higher than n .

The notion of an auto-conflict is utilized by Martin *et al.* as an alternative measure of internal conflict. Nonetheless, the values of auto-conflicts of higher orders have no reasonable interpretation and no way of comparing them with the values of an auto-conflict (of 2nd order). A hidden auto-conflict may be considered as an extension of this measure of an internal conflict. Unfortunately, analogous to the values of higher-order auto-conflicts, we have no procedure to compare the values of hidden auto-conflicts of different degrees.

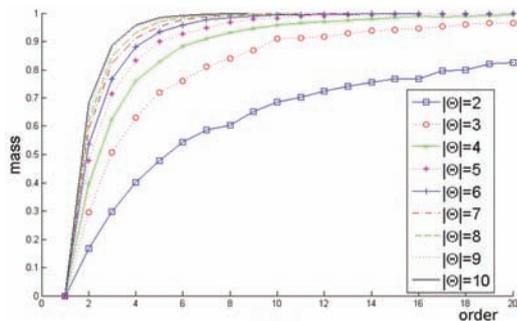


Figure 7 | Average auto-conflicts of quasi-Bayesian belief functions (qBBFs) from Ref. [19], $|\Theta| = |\Omega| = 2 - 10$. ($|X| = 1$ for all focal elements).

Theorem 18. For auto-conflicts of a BF m on Ω_n , the following assertions hold:

- i. An auto-conflict of a consistent BF of any order is zero (there is no auto-conflict in fact).
And also $\lim_{s \rightarrow \infty} a_s(m) = 0$ for any consistent BF m .
- ii. An auto-conflict of order 1 is equal to $m(\emptyset)$; hence it is zero for any normalized BF m .
- iii. Auto-conflicts of orders from 2 to $k + 1$ are zero for any non-consistent BF, where $k \leq n - 2$ is a degree of a hidden auto-conflict; in particular,
for $k = 0$: $a_2(m) > 0$ and also $a_s(m) > 0$ for all $s > 2$,
for $k = 1$: $a_2(m) = 0$ and $a_s(m) > 0$ for all $s > 2$,
for $k = 2$: $a_2(m) = a_3(m) = 0$ and $a_s(m) > 0$ for all $s > 3$, etc.
- iv. A positive auto-conflict is increasing: if $a_s(m) > 0$ then $a_s(m) < a_{s+1}(m)$, thus
 $a_1(m) = a_2(m) = \dots = a_{k+1}(m) = 0 < a_{k+2}(m) < a_{k+3}(m) < \dots$, for some k such that $0 \leq k \leq n - 2$; k is the degree of hidden auto-conflict.
And $\lim_{s \rightarrow \infty} a_s(m) = 1$ for any non-consistent BF.

Hidden auto-conflicts of qBBFs from Ref. [19] just follow the curves from $[1, 0]$ to $[\infty, 1]$ see Figure 7; $|X| = 1$ holds there for proper focal elements. Whereas hidden auto-conflicts of general non-consistent BFs are in the entire gray area on Figure 8, BFs with proper focal elements X up to $|X| = k$ in darker gray area above the curve from $[k, 0]$ to $[\infty, 1]$, and zero auto-conflicts of consistent BFs are on the straight line $[0, 1]$ to $[\infty, 0]$.

An auto-conflict is an intrinsic property of BFs as it is stated in Ref. [19], but $a(m) \geq 0$ in general. Hence it may be equal to zero while there may be no or some hidden auto-conflict.

We have seen that $a(m) > 0$ is an intrinsic property of a class of quasi-Bayesian BFs with just $n + 1$ focal elements,⁹ which was studied in Ref. [19]. More generally, it is also an intrinsic property of a class of non-consistent BFs with two or more disjunctive focal elements e.g., $X \cap Y = \emptyset$; $a(m) > 0$ always holds.

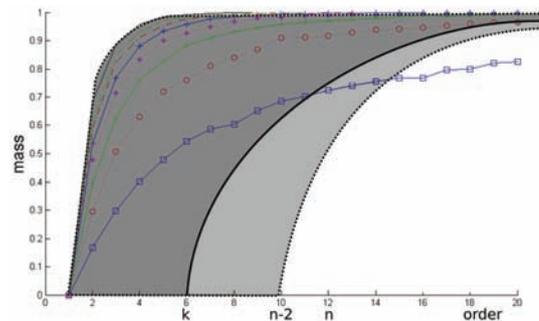


Figure 8 | Auto-conflicts of general non-consistent belief functions (BFs) and of non-consistent BFs with focal elements X up to $|X| = k$ (darker) on $|\Omega_{12}| = 12$

⁹More generally, it is an intrinsic property of a class of quasi-Bayesian BFs with more than one singleton.

7. COMPUTATIONAL COMPLEXITY AND COMPUTATIONS OF EXAMPLES

Based on Definition 3 and Theorem 2, the complexity of the computation of the degree of a hidden conflict between two BFs m^i and m^{ii} $O(n)$ of \odot operations on a general Ω_n . In the case of checking the existence of a hidden conflict of the BFs, we obtain the complexity $O(\log_2(n))$ of \odot operations utilizing a simplification of the computation based on $\odot_{j=1}^{2k}(m^i \odot m^{ii}) = \odot_{j=1}^k(m^i \odot m^{ii}) \odot \odot_{j=1}^k(m^i \odot m^{ii})$. Note that the complexity of the \odot operation depends on the number and structure of the focal elements.

In our analysis of hidden conflicts, a series of computations has been performed on frames of discernment with cardinality from 5 to 16. The numbers of the focal elements rapidly grow up to $|\mathcal{P}(\Omega)| = 2^{|\Omega|} - 1$ when the conjunctive combination \odot is repeated—see, e.g., 32766; and 32767 focal elements in the presented Examples 1 and 6 in Table 4. Because the degree of the hidden conflict and its existence depend on the number and the structure of the focal elements, not on their bbms, we have used the same bbms for all focal elements of a BF in our computations on frames of cardinality larger than 10.

All of our experiments have been performed in Language R [33] using R Studio [34]. We are currently developing an R package for dealing with the BFs on various frames of discernment. It is based on a relational database approach—nicely implemented in R, in the package called `data.table` ([35]).

8. SEVERAL IMPORTANT REMARKS

We have to underline that a hidden conflict of BFs is not a new measure of conflict. It just improves/extends the classic approach which defines a conflict by $m_{\odot}(\emptyset)$ in situations where $m_{\odot}(\emptyset) = 0$; it distinguishes fully non-conflicting BFs from those with a positive hidden conflict. This notion provides a deeper understanding of conflictness/non-conflictness. It enables us to point out the conflict also in situations where conflicts have not been expected, in situations where $m_{\odot}(\emptyset) = 0$; hence to point out and to help understand the conflicts which are hidden due to $m_{\odot}(\emptyset) = 0$.

Particular numeric values of a hidden conflict have no reasonable interpretation so far. For now, we are curious whether the value is zero (i.e., there is no conflict) or not. On the other hand, there are degrees of hiddenness, indicating how much the conflicts are hidden.

Repeated applications of the conjunctive combination \odot of a BF with itself is used here to simulate situations where different independent believers have numerically the same bbms. This approach has nothing to do with an idempotent belief combination (where, of course, no conflicts between two correctly created BFs are possible).

This study presents a brand new idea of hidden conflicts presented at conferences [18,28,29]. It extends the original Eusflat contribution [36]. The brand new interpretation of $m_{\odot}(\emptyset)$ distinguishes fully non-conflicting BFs from those with hidden conflicts. The assumption of non-conflictness when $m_{\odot}(\emptyset) = 0$ is relaxed, due to

the observation of a qualitatively new phenomena—observation of a hidden conflict in the cases where $m_{\odot}(\emptyset) = 0$. We want to point out the existence of hidden (auto-)conflicts in situations where no conflict has been expected until now. The definitions of hidden conflicts and hidden auto-conflicts [28] thus do not go against the previous Daniel's research and results on conflicts of BFs; e.g., Refs. [1,4,5]. Of course, some parts of the previous approaches should be updated to be fully consistent with the newly presented results on hidden conflicts and internal conflicts (auto-conflicts).

Our study has been motivated by the investigation of the conflict *Conf* of BFs based on their non-conflicting parts [5], we are thus interested in independent BFs for which a hidden conflict was observed. But we have to note that the conflictness/non-conflictness of BFs has nothing to do with dependence/independence of the BFs. There is or there is not a conflict between BFs regardless they are independent or not.

Repeated computations of several (up to n) numerically identical BFs, when looking for a hidden conflict is just a technical tool for computation of $m(\emptyset) = (\odot_1^k m_j)(\emptyset)$ or more precisely for computation of $\kappa = \sum_{X \cap Y = \emptyset} m_j(X) m_j(Y)$.

We are not interested in entire results of repeated applications of \odot ; we are only interested in $m(\emptyset)$ or more precisely in $\kappa = \sum_{X_1 \cap X_2 \cap \dots \cap X_k = \emptyset} m_j(X_1) m_j(X_2) \dots m_j(X_k)$. Our computation has nothing to do with any idempotent combination of BFs. And we can compute hidden conflicts using \odot_1^k (or κ) in the same way for both dependent and independent BFs. Neither is it necessary to include any independence assumption to our Definitions 2 and 3.

We have to realize that two BFs are conflicting (in a conflict or a hidden conflict) or non-conflicting (without any conflicts) regardless of their origins/sources and their relationship. Similarly, BFs are conflicting or non-conflicting regardless of whether one wants to combine them or not.

9. SUMMARY AND CONCLUSIONS

Hidden conflicts of BFs in situations where mutual intersections of any focal element of one BF with all focal element of the other BF are non-empty has been presented and analyzed. There may be a positive conflict in situations where sums of conflicting belief masses are empty, i.e., in situations which have been usually considered to be non-conflicting (till now).

Several levels—degrees of hidden conflict—have been observed, the maximum degree of hidden conflicts, which depends on the size of the corresponding frame of discernment, has been found. A variety of hidden conflicts of degrees 1 -- $(n - 2)$ has been described for an n -element frame of discernment. A necessary and sufficient condition for full non-conflictness of BFs in dependence on the maximum cardinality of their focal elements has been specified and their computational aspects analyzed. Analogous to the evident conflicts, internal hidden conflicts are distinguished from the hidden conflicts between BFs.

This qualitatively new phenomenon of conflicts of BFs moves us to a better understanding of the nature of conflicts of BFs in general and brings a challenge to elaborate and update the existing approaches to conflicts of BFs.

This may consequently serve as a basis for better combinations of conflicting BFs and better interpretations of the results of belief combinations whenever conflicting BFs appear in real applications.

CONFLICTS OF INTEREST

The authors declared that they have no conflicts of interest to this work.

ACKNOWLEDGMENTS

This work was supported by the institutional support RVO: 67985807 (first author) and by grant GA ČR 19-04579S (second author).

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