



## Research Article

# A New Aggregation Operator Based on Uninorms in $L^*$ -Fuzzy Set

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## ABSTRACT

In practical applications, some existing multi-attribute decision-making methods based on the  $L^*$  fuzzy set theory suffer from a lot of shortcomings, namely, incorrect choice preference orders of alternatives are obtained in some cases. In this paper, we construct a new aggregation operator based on uninorms in  $L^*$ -fuzzy set theory. The aggregation result depends on arguments that are larger than neutral value, less than neutral value, and incomparable with neutral value, as the frontier between good scores and bad scores. The detailed decision-making procedure based on the proposed aggregation operator, which is shown to be reasonable and effective through examples, is introduced.

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## 1. INTRODUCTION

In order to deal with fuzzy information in practical applications, Zadeh established the fuzzy set theory [1]. Information aggregation is a common activity in real life, and many fuzzy aggregation operators have been proposed, among which, the weighted averaging (WA) operator [2], the ordered weighted averaging (OWA) operator [3], the weighted geometric averaging (WGA) operator [3], and the ordered weighted geometric averaging (OWGA) operator [3] are the most familiar ones for aggregating information. Recently, adhibition of bipolar scales has been widely used in decision-making [4,5]. There are psychological evidence that in many cases, scores given by humans lie on a bipolar scales, that is to say a scale with a neutral value serving as frontier between good scores and bad scores. In general, our behavior with the good scores are different from the bad scores, so it becomes important to define aggregation operators, which can reflect the diversity of aggregation behavior on bipolar scales. This situation is already considered for uninorm [6], with identity element  $e$  serving as frontier. Based on uninorm, a class of very flexible aggregation operators are constructed [5]. This type of information aggregation allows us to command the aggregation process based on the allocation of parameters. But, until now information of bipolar scales is based on fuzzy set introduced by Zadeh [1], which only provides the membership degree of each element in the universe, and the nonmembership degree is 1 minus the membership degree, it is not enough to depict the ambiguity and simulate the life scene.

Although fuzzy set has been widely used, it still cannot handle uncertain information well. As a generalization of fuzzy set,

intuitionistic fuzzy set was introduced [7], which is composed of membership degree and nonmembership degree. Intuitionistic fuzzy set can describe not only the fuzziness of objects, but also the uncertainty. It has been shown that intuitionistic fuzzy set is  $L^*$ -fuzzy set w.r.t. the lattice  $(L^*, \leq_{L^*})$  [8]. Then, some intuitionistic fuzzy aggregation operators were developed. The intuitionistic fuzzy weighted averaging (IFWA) operator, the intuitionistic fuzzy ordered weighted averaging (IFOWA) operator, and some intuitionistic fuzzy aggregation hybrid averaging (IFHA) operator were proposed [9]. Based on the algebraic sum, algebraic product, and operational laws on intuitionistic fuzzy sets, the intuitionistic fuzzy weight geometric (IFWG) operator, the intuitionistic fuzzy ordered weighted geometric (IFOWG) operator, and the intuitionistic fuzzy hybrid (IFGH) operator were presented by Xu and Yager [10]. Algebraic product and algebraic sum, as the basic operations of intuitionistic fuzzy sets, are not the only operations to establish the intersection and union model of intuitionistic fuzzy sets. Some algebraic operators such as the Einstein sum operator, the Einstein product operator and the Einstein scalar multiplication operator were introduced by Wang and Liu [11]. In line with these algebraic operators, intuitionistic fuzzy Einstein weighted geometric operator (IFWG $^\varepsilon$ ) and the intuitionistic fuzzy Einstein ordered weighted geometric (IFOWG $^\varepsilon$ ) operator were proposed. As those aggregation operators [10,11] have the drawback that if there is only one membership degree of intuitionistic fuzzy sets equal to zero, the membership degree of the aggregation result of  $n$  intuitionistic fuzzy sets is zero even if the membership degree of  $n - 1$  intuitionistic fuzzy sets are not zero. To conquer the defects of aggregation operations, the intuitionistic fuzzy weighted geometric interaction averaging (IFWGIA) operator, the intuitionistic

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fuzzy ordered weighted geometric interaction averaging (IFOWGIA) operator, and the intuitionistic fuzzy hybrid geometric interaction averaging (IFHGIA) operator were proposed by He *et al.* [12]. These aggregation operations may result in unconscionable decision results in some cases. Chen and Chang [13] proposed the intuitionistic fuzzy weighted geometric averaging (IFWGA) operator, the intuitionistic fuzzy ordered weighted geometric averaging (IFOWGA) operator, and the intuitionistic fuzzy hybrid geometric averaging (IFHGA) operator. In practical applications, some existing methods aggregate information directly rather than classifying the information into different classes, which results in the incorrect order of preference for alternatives in some cases. For example,  $A$  and  $B$  be two sets of intuitionistic fuzzy values, and  $\omega_1 = 0.12$ ,  $\omega_2 = 0.13$ ,  $\omega_3 = 0.18$ ,  $\omega_4 = 0.10$ ,  $\omega_5 = 0.17$ ,  $\omega_6 = 0.18$ ,  $\omega_7 = 0.12$ . Information are shown in the following table.

	<i>A</i>	<i>B</i>
	(0.3000, 0.6000)	(0.8000, 0.2000)
	(1.0000, 0.0000)	(0.7000, 0.2000)
	(0.2000, 0.6000)	(0.7000, 0.2000)
	(0.1000, 0.7000)	(0.8000, 0.1000)
	(0.3000, 0.7000)	(0.9000, 0.0000)
	(0.1000, 0.9000)	(0.1000, 0.8000)
	(0.0000, 0.8000)	(0.6000, 0.3000)
<i>IFWGA</i>	(1.0000, 0.0000)	(0.7009, 0.2991)
<i>IFOWGA</i>	(1.0000, 0.0000)	(0.7889, 0.2111)
<i>IFHGA</i>	(1.0000, 0.0000)	(0.7175, 0.2825)

Note: IFHGA, intuitionistic fuzzy hybrid geometric averaging; IFWGA, intuitionistic fuzzy weighted geometric averaging; IFOWGA, intuitionistic fuzzy ordered weighted geometric averaging.

It can be seen from the table that the intuitionistic fuzzy values in  $A$  are smaller than those in  $B$ , the preference order should be  $B > A$ , but in fact the preference order is  $A > B$  according to three aggregation operations, this is counterintuitive. Most of the time, our behavior with the good items are not the same with the bad items, categorizing information first, and then aggregating them. As uninorms are introduced to model bipolar behavior, in this paper, we will construct aggregation operator using uninorms in  $L^*$ -fuzzy set theory.

The rest of this paper consists of the following. Part 2 reviews some basic definitions. Some aggregation operators are constructed based on the uninorm in  $L^*$ -fuzzy set theory in part 3. In part 4, we develop the decision-making approach based on the proposed aggregation operator, and apply it to decision experiments. Conclusions are stated in part 5.

## 2. PRELIMINARIES

In this part, we retrospect some concepts of intuitionistic fuzzy set,  $L^*$ -fuzzy set, score function, uninorms, and aggregation operator.

**Definition 1.** [7] An object of the form  $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$  on a universe  $X$  is called intuitionistic fuzzy set, where  $\mu_A(x) \in [0, 1]$  is called the degree of membership of  $x$  in  $A$ ,  $\nu_A(x) \in [0, 1]$  is called the degree of nonmembership of  $x$  in  $A$ , and  $\mu_A(x), \nu_A(x) \in [0, 1]$  satisfy the following condition:  $0 \leq \mu_A(x) + \nu_A(x) \leq 1 (\forall x \in X)$ .

Consider the set  $L^* = \{(x_1, x_2) \mid (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 + x_2 \leq 1\}$ ,  $(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \geq y_2, \forall (x_1, x_2), (y_1, y_2) \in L^*$ .  $(L^*, \leq_{L^*})$  is a complete lattice, and  $0_{L^*} = (0, 1)$ ,  $1_{L^*} = (1, 0)$ . Note that, for  $x = (x_1, x_2), y = (y_1, y_2) \in L^*$ , if  $x_1 < y_1$  and  $x_2 < y_2$ , or  $x_1 > y_1$  and  $x_2 > y_2$ , then  $x$  and  $y$  are incomparable w.r.t.  $\leq_{L^*}$ , denote as  $x \parallel_{L^*} y$  [8].

Deschrijver and Kerre introduced uninorm in  $L^*$ -fuzzy set theory [14], which are generalizations of t-norm and t-conorm in  $L^*$ -fuzzy set theory.

**Definition 2.** [14]  $(L^*, \leq_{L^*})$  is called a complete lattice, where  $L^* = \{(x_1, x_2) \mid (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 + x_2 \leq 1\}$ . A uninorm  $\mathcal{U}$  on  $L^*$  is an increasing, associative and communicative  $(L^*)^2 \rightarrow L^*$  mapping that satisfies  $(\exists (e_1, e_2) \in L^*) (\forall (x, y) \in L^*) (\mathcal{U}((e_1, e_2), (x, y)) = (x, y))$ .

The element  $(e_1, e_2)$  corresponding to a uninorm  $\mathcal{U}$  is clearly unique and is called the identity element of  $\mathcal{U}$ . If  $(e_1, e_2) = 0_{L^*}$ , then we obtain a t-conorm  $\mathcal{U}$  on  $L^*$ , while in case  $(e_1, e_2) = 1_{L^*}$ , we obtain a t-norm  $\mathcal{T}$  on  $L^*$ .

The set of all positive integers is denoted by  $\mathbb{N}^*$ . In fuzzy set theory, aggregation operators are defined as follows.

**Definition 3.** [15] A mapping  $A : \bigcup_{n \in \mathbb{N}^*} [0, 1]^n \rightarrow [0, 1]$  is called an aggregation operator  $A$  on  $[0, 1]$  if the following properties are met:

1.  $A(x) = x$ , for all  $x \in [0, 1]$ ;
2. If  $x_i \leq y_i$  for all  $i \in \{1, 2, \dots, n\}$ , then  $A(x_1, x_2, \dots, x_n) \leq A(y_1, y_2, \dots, y_n)$ , for all  $n \in \mathbb{N}^*$  and for all  $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \in [0, 1]^n$ ;
3.  $A(\underbrace{0, 0, \dots, 0}_{n \text{ times}}) = 0$  for all  $n \in \mathbb{N}^*$ ;
4.  $A(\underbrace{1, 1, \dots, 1}_{n \text{ times}}) = 1$  for all  $n \in \mathbb{N}^*$ .

In [16], this definition was extended to  $L^*$ -fuzzy set theory. Let  $(L^*, \leq_{L^*})$  be a complete lattice, where  $L^* = \{(x_1, x_2) \mid (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 + x_2 \leq 1\}$ .

**Definition 4.** [16] A mapping  $\mathcal{A} : \bigcup_{n \in \mathbb{N}^*} (L^*)^n \rightarrow L^*$  is called an aggregation operator  $\mathcal{A}$  on  $L^*$  the following properties are met:

1.  $\mathcal{A}((x, y)) = (x, y)$ , for all  $(x, y) \in L^*$ ;
2. If  $(x_i, y_i) \leq_{L^*} (x'_i, y'_i)$  for all  $i \in \{1, 2, \dots, n\}$ , then  $\mathcal{A}((x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)) \leq_{L^*} \mathcal{A}((x'_1, y'_1), (x'_2, y'_2), \dots, (x'_n, y'_n))$ , for all  $n \in \mathbb{N}^*$  and for all  $(x_1, y_1), \dots, (x_n, y_n), (x'_1, y'_1), \dots, (x'_n, y'_n) \in L^*$ ;
3.  $\mathcal{A}(\underbrace{(0, 1), (0, 1), \dots, (0, 1)}_{n \text{ times}}) = (0, 1)$  for all  $n \in \mathbb{N}^*$ ;
4.  $\mathcal{A}(\underbrace{(1, 0), (1, 0), \dots, (1, 0)}_{n \text{ times}}) = (1, 0)$  for all  $n \in \mathbb{N}^*$ .

**Definition 5.** [17] Let  $\tilde{\alpha} = (a, b)$  be an intuitionistic fuzzy number, the score function is defined by:

$$S(\tilde{\alpha}) = \frac{a - b}{2} \quad (1)$$

Next, we will recall some intuitionistic fuzzy aggregation operators.

**Definition 6.** (IFWG operator) [10] Let  $\tilde{a}_j = (\mu_{\tilde{a}_j}, \nu_{\tilde{a}_j})$ ,  $\mu_{\tilde{a}_j} \in [0, 1]$ ,  $\nu_{\tilde{a}_j} \in [0, 1]$ ,  $0 \leq \mu_{\tilde{a}_j} + \nu_{\tilde{a}_j} \leq 1$  and  $1 \leq j \leq n$ . The IFWG operator is defined as follow:

$$\text{IFWG}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = \left( \prod_{j=1}^n \mu_{\tilde{a}_j}^{\omega_j}, \prod_{j=1}^n (1 - \nu_{\tilde{a}_j})^{\omega_j} \right) \quad (2)$$

where  $\omega_j$  is the weight of  $\tilde{a}_j$ ,  $\omega_j \in [0, 1]$ ,  $1 \leq j \leq n$  and  $\sum_{j=1}^n \omega_j = 1$ .

**Definition 7.** (IFOWG operator) [10] Let  $\tilde{a}_j = (\mu_{\tilde{a}_j}, \nu_{\tilde{a}_j})$ ,  $\mu_{\tilde{a}_j} \in [0, 1]$ ,  $\nu_{\tilde{a}_j} \in [0, 1]$ ,  $0 \leq \mu_{\tilde{a}_j} + \nu_{\tilde{a}_j} \leq 1$  and  $1 \leq j \leq n$ . Let  $\omega = \{\omega_1, \omega_2, \dots, \omega_n\}^T$  be the weighting vector of the Einstein ordered weighted geometric operator, where  $\omega_j \in [0, 1]$ ,  $1 \leq j \leq n$  and  $\sum_{j=1}^n \omega_j = 1$ . The IFOWG operator is defined as follow:

$$\text{IFOWG}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = \left( \prod_{j=1}^n \mu_{\tilde{a}_{\sigma(j)}}^{\omega_j}, \prod_{j=1}^n (1 - \nu_{\tilde{a}_{\sigma(j)}})^{\omega_j} \right) \quad (3)$$

where “ $\sigma(1), \sigma(2), \dots$ , and  $\sigma(n)$ ” is a permutation of “ $1, 2, \dots$ , and  $n$ ,” such that  $\tilde{a}_{\sigma(j)}$  is the  $j$ th largest intuitionistic fuzzy value among  $\tilde{a}_1, \tilde{a}_2, \dots$ , and  $\tilde{a}_n$ , where  $1 \leq j \leq n$ .

**Definition 8.** (IFHG operator) [10] Let  $\tilde{a}_j = (\mu_{\tilde{a}_j}, \nu_{\tilde{a}_j})$ ,  $\mu_{\tilde{a}_j} \in [0, 1]$ ,  $\nu_{\tilde{a}_j} \in [0, 1]$ ,  $0 \leq \mu_{\tilde{a}_j} + \nu_{\tilde{a}_j} \in [0, 1]$  and  $1 \leq j \leq n$ . Let  $\omega_1, \omega_2, \dots$ , and  $\omega_n$  be the weights of the intuitionistic fuzzy values  $\tilde{a}_1, \tilde{a}_2, \dots$ , and  $\tilde{a}_n$ , respectively, where  $\omega_j \in [0, 1]$ ,  $1 \leq j \leq n$  and  $\sum_{j=1}^n \omega_j = 1$ . Let  $\omega = \{\omega_1, \omega_2, \dots, \omega_n\}^T$  be the weighting vector of the hybrid geometric, where  $\omega_j \in [0, 1]$ ,  $1 \leq j \leq n$  and  $\sum_{j=1}^n \omega_j = 1$ . Let  $\tilde{a}_j = (\tilde{a})^{nw_j}$ , where  $n$  is the balancing coefficient, the intuitionistic fuzzy hybrid geometric (IFHG) operator is defined as follow:

$$\text{IFHG}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = \left( \prod_{j=1}^n \mu_{\tilde{a}_{\sigma(j)}}^{\omega_j}, \prod_{j=1}^n (1 - \nu_{\tilde{a}_{\sigma(j)}})^{\omega_j} \right) \quad (4)$$

where “ $\sigma(1), \sigma(2), \dots$ , and  $\sigma(n)$ ” is a permutation of “ $1, 2, \dots$ , and  $n$ ,” such that  $\tilde{a}_{\sigma(j)}$  is the  $j$ th largest intuitionistic fuzzy value among  $\tilde{a}_1, \tilde{a}_2, \dots$ , and  $\tilde{a}_n$ , where  $1 \leq j \leq n$ .

**Definition 9.** (IFWG  $^\varepsilon$  operator) [11] Let  $\tilde{a}_j = (\mu_{\tilde{a}_j}, \nu_{\tilde{a}_j})$ ,  $\mu_{\tilde{a}_j} \in [0, 1]$ ,  $\nu_{\tilde{a}_j} \in [0, 1]$ ,  $0 \leq \mu_{\tilde{a}_j} + \nu_{\tilde{a}_j} \leq 1$  and  $1 \leq j \leq n$ . The IFWG  $^\varepsilon$  operator is defined as follow:

$$\text{IFWG}^\varepsilon(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = \left( \frac{2 \prod_{j=1}^n \mu_{\tilde{a}_j}^{\omega_j}}{\prod_{j=1}^n (2 - \mu_{\tilde{a}_j})^{\omega_j} + \prod_{j=1}^n \mu_{\tilde{a}_j}^{\omega_j}}, \frac{\prod_{j=1}^n (1 + \nu_{\tilde{a}_j})^{\omega_j} - \prod_{j=1}^n (1 - \nu_{\tilde{a}_j})^{\omega_j}}{\prod_{j=1}^n (1 + \nu_{\tilde{a}_j})^{\omega_j} + \prod_{j=1}^n (1 - \nu_{\tilde{a}_j})^{\omega_j}} \right) \quad (5)$$

where  $\omega_j$  is the weight of  $\tilde{a}_j$ ,  $\omega_j \in [0, 1]$ ,  $1 \leq j \leq n$  and  $\sum_{j=1}^n \omega_j = 1$ .

**Definition 10.** (IFOWG  $^\varepsilon$  operator) [11] Let  $\tilde{a}_j = (\mu_{\tilde{a}_j}, \nu_{\tilde{a}_j})$ ,  $\mu_{\tilde{a}_j} \in [0, 1]$ ,  $\nu_{\tilde{a}_j} \in [0, 1]$ ,  $0 \leq \mu_{\tilde{a}_j} + \nu_{\tilde{a}_j} \leq 1$  and  $1 \leq j \leq n$ . Let  $\omega = \{\omega_1, \omega_2, \dots, \omega_n\}^T$  be the weighting vector of the Einstein ordered weighted geometric operator, where  $\omega_j \in [0, 1]$ ,  $1 \leq j \leq n$  and  $\sum_{j=1}^n \omega_j = 1$ . The IFOWG  $^\varepsilon$  operator is defined as follow:

$$\text{IFOWG}^\varepsilon(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) \quad (6)$$

$$= \left( \frac{2 \prod_{j=1}^n \mu_{\tilde{a}_{\sigma(j)}}^{\omega_j}}{\prod_{j=1}^n (2 - \mu_{\tilde{a}_{\sigma(j)}})^{\omega_j} + \prod_{j=1}^n \mu_{\tilde{a}_{\sigma(j)}}^{\omega_j}}, \frac{\prod_{j=1}^n (1 + \nu_{\tilde{a}_{\sigma(j)}})^{\omega_j} - \prod_{j=1}^n (1 - \nu_{\tilde{a}_{\sigma(j)}})^{\omega_j}}{\prod_{j=1}^n (1 + \nu_{\tilde{a}_{\sigma(j)}})^{\omega_j} + \prod_{j=1}^n (1 - \nu_{\tilde{a}_{\sigma(j)}})^{\omega_j}} \right)$$

where “ $\sigma(1), \sigma(2), \dots$ , and  $\sigma(n)$ ” is a permutation of “ $1, 2, \dots$ , and  $n$ ,” such that  $\tilde{a}_{\sigma(j)}$  is the  $j$ th largest intuitionistic fuzzy value among  $\tilde{a}_1, \tilde{a}_2, \dots$ , and  $\tilde{a}_n$ , where  $1 \leq j \leq n$ .

**Definition 11.** (IFWGIA operator) [12] Let  $\tilde{a}_j = (\mu_{\tilde{a}_j}, \nu_{\tilde{a}_j})$ ,  $\mu_{\tilde{a}_j} \in [0, 1]$ ,  $\nu_{\tilde{a}_j} \in [0, 1]$ ,  $0 \leq \mu_{\tilde{a}_j} + \nu_{\tilde{a}_j} \leq 1$  and  $1 \leq j \leq n$ . The IFWGIA operator is defined as follow:

$$\text{IFWGIA}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = \left( \prod_{j=1}^n (1 - \nu_{\tilde{a}_j})^{\omega_j} - \prod_{j=1}^n (1 - (\mu_{\tilde{a}_j} + \nu_{\tilde{a}_j}))^{\omega_j}, 1 - \prod_{j=1}^n (1 - \nu_{\tilde{a}_j})^{\omega_j} \right) \quad (7)$$

where  $\omega_j$  is the weight of  $\tilde{a}_j$ ,  $\omega_j \in [0, 1]$ ,  $1 \leq j \leq n$  and  $\sum_{j=1}^n \omega_j = 1$ .

**Definition 12.** (IFOWGIA operator) [12] Let  $\tilde{a}_j = (\mu_{\tilde{a}_j}, \nu_{\tilde{a}_j})$ ,  $\mu_{\tilde{a}_j} \in [0, 1]$ ,  $\nu_{\tilde{a}_j} \in [0, 1]$ ,  $0 \leq \mu_{\tilde{a}_j} + \nu_{\tilde{a}_j} \leq 1$  and  $1 \leq j \leq n$ . Let  $\omega = \{\omega_1, \omega_2, \dots, \omega_n\}^T$  be the weighting vector of the ordered weighted geometric interaction averaging operator, where  $\omega_j \in [0, 1]$ ,  $1 \leq j \leq n$  and  $\sum_{j=1}^n \omega_j = 1$ . The IFOWGIA operator is defined as follow:

$$\text{IFOWGIA}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = \left( \prod_{j=1}^n (1 - \nu_{\tilde{a}_{\sigma(j)}})^{\omega_j} - \prod_{j=1}^n (1 - (\mu_{\tilde{a}_{\sigma(j)}} + \nu_{\tilde{a}_{\sigma(j)}}))^{\omega_j}, 1 - \prod_{j=1}^n (1 - \nu_{\tilde{a}_{\sigma(j)}})^{\omega_j} \right) \quad (8)$$

where “ $\sigma(1), \sigma(2), \dots$ , and  $\sigma(n)$ ” is a permutation of “ $1, 2, \dots$ , and  $n$ ,” such that  $\tilde{a}_{\sigma(j)}$  is the  $j$ th largest intuitionistic fuzzy value among  $\tilde{a}_1, \tilde{a}_2, \dots$ , and  $\tilde{a}_n$ , where  $1 \leq j \leq n$ .

**Definition 13.** (IFHGIA operator) [12] Let  $\tilde{a}_j = (\mu_{\tilde{a}_j}, \nu_{\tilde{a}_j})$ ,  $\mu_{\tilde{a}_j} \in [0, 1]$ ,  $\nu_{\tilde{a}_j} \in [0, 1]$ ,  $0 \leq \mu_{\tilde{a}_j} + \nu_{\tilde{a}_j} \leq 1$  and  $1 \leq j \leq n$ . Let  $\omega_1, \omega_2, \dots$ , and  $\omega_n$  be the weights of the intuitionistic fuzzy values  $\tilde{a}_1, \tilde{a}_2, \dots$ , and  $\tilde{a}_n$ , respectively, where  $\omega_j \in [0, 1]$ ,  $1 \leq j \leq n$  and  $\sum_{j=1}^n \omega_j = 1$ . Let  $\omega = \{\omega_1, \omega_2, \dots, \omega_n\}^T$  be the weighting vector of the hybrid geometric interaction averaging operator, where  $\omega_j \in [0, 1]$ ,  $1 \leq j \leq n$  and  $\sum_{j=1}^n \omega_j = 1$ . The IFHGIA operator is defined as follow:

Let  $\omega = \{\omega_1, \omega_2, \dots, \omega_n\}^T$  be the weighting vector of the hybrid geometric interaction averaging operator, where  $\omega_j \in [0, 1]$ ,  $1 \leq j \leq n$  and  $\sum_{j=1}^n \omega_j = 1$ .

$j \leq n$  and  $\sum_{j=1}^n \omega_j = 1$ . Let  $\tilde{a}_j = (\tilde{a})^{nw_j}$ , where  $n$  is the balancing coefficient, the IFHGA operator is defined as follow:

$$\begin{aligned} IFHGA(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) &= \left( \prod_{j=1}^n (1 - \nu_{\tilde{a}_{\sigma(j)}})^{\omega_j} \right. \\ &\quad \left. - \prod_{j=1}^n (1 - (\mu_{\tilde{a}_{\sigma(j)}} + \nu_{\tilde{a}_{\sigma(j)}}))^{\omega_j}, 1 - \prod_{j=1}^n (1 - \nu_{\tilde{a}_{\sigma(j)}})^{\omega_j} \right) \end{aligned} \quad (9)$$

where “ $\sigma(1), \sigma(2), \dots, \text{and } \sigma(n)$ ” is a permutation of “1, 2,  $\dots$ , and  $n$ ”, such that  $\tilde{a}_{\sigma(j)}$  is the  $j$ th largest intuitionistic fuzzy value among  $\tilde{a}_1, \tilde{a}_2, \dots, \text{and } \tilde{a}_n$ , where  $1 \leq j \leq n$ .

**Definition 14.** (IFWGA operator) [13] Let  $A = \{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n\}$ ,  $\tilde{a}_j = (\mu_{\tilde{a}_j}, \nu_{\tilde{a}_j})$ ,  $\mu_{\tilde{a}_j} \in [0, 1]$ ,  $\nu_{\tilde{a}_j} \in [0, 1]$ ,  $0 \leq \mu_{\tilde{a}_j} + \nu_{\tilde{a}_j} \leq 1$  and  $1 \leq j \leq n$ . Let  $\omega_1, \omega_2, \dots, \text{and } \omega_n$  be the weights of the intuitionistic fuzzy values  $\tilde{a}_1, \tilde{a}_2, \dots, \text{and } \tilde{a}_n$ , respectively, where  $\omega_j \in [0, 1]$ ,  $1 \leq j \leq n$  and  $\sum_{j=1}^n \omega_j = 1$ . The IFWGA operator is as follow:

$$\begin{aligned} IFWGA(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) &= \left( 1 - \prod_{j=1}^n (1 - \mu_{\tilde{a}_j}) \right)^{\omega_j}, \\ &\quad \prod_{j=1}^n (1 - \mu_{\tilde{a}_j})^{\omega_j} - \prod_{j=1}^n (1 - \mu_{\tilde{a}_j} - \nu_{\tilde{a}_j})^{\omega_j} \end{aligned} \quad (10)$$

**Definition 15.** (IFOWGA operator) [13] Let  $A = \{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n\}$ ,  $\tilde{a}_j = (\mu_{\tilde{a}_j}, \nu_{\tilde{a}_j})$ ,  $\mu_{\tilde{a}_j} \in [0, 1]$ ,  $\nu_{\tilde{a}_j} \in [0, 1]$ ,  $0 \leq \mu_{\tilde{a}_j} + \nu_{\tilde{a}_j} \leq 1$  and  $1 \leq j \leq n$ . Let  $\omega = \{\omega_1, \omega_2, \dots, \omega_n\}^T$  be the weighting vector of the OWGA operator, where  $\omega_j \in [0, 1]$ ,  $1 \leq j \leq n$  and  $\sum_{j=1}^n \omega_j = 1$ . The IFOWGA operator is as follow:

$$\begin{aligned} IFOWGA(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) &= \left( 1 - \prod_{j=1}^n (1 - \mu_{\tilde{a}_{\sigma(j)}}) \right)^{\omega_j}, \\ &\quad \prod_{j=1}^n (1 - \mu_{\tilde{a}_{\sigma(j)}})^{\omega_j} - \prod_{j=1}^n (1 - \mu_{\tilde{a}_{\sigma(j)}} - \nu_{\tilde{a}_{\sigma(j)}})^{\omega_j} \end{aligned} \quad (11)$$

where “ $\sigma(1), \sigma(2), \dots, \text{and } \sigma(n)$ ” is a permutation of “1, 2,  $\dots$ , and  $n$ ”, such that  $\tilde{a}_{\sigma(j)}$  is the  $j$ th largest intuitionistic fuzzy value among  $\tilde{a}_1, \tilde{a}_2, \dots, \text{and } \tilde{a}_n$  and  $S(\tilde{a}_{\sigma(j-1)}) > S(\tilde{a}_{\sigma(j)})$ , where  $1 \leq j \leq n$ .

**Definition 16.** (IFHGA operator) [13] Let  $A = \{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n\}$ ,  $\tilde{a}_j = (\mu_{\tilde{a}_j}, \nu_{\tilde{a}_j})$ ,  $\mu_{\tilde{a}_j} \in [0, 1]$ ,  $\nu_{\tilde{a}_j} \in [0, 1]$ ,  $0 \leq \mu_{\tilde{a}_j} + \nu_{\tilde{a}_j} \leq 1$  and  $1 \leq j \leq n$ . Let  $\omega_1, \omega_2, \dots, \text{and } \omega_n$  be the weights  $\tilde{a}_1, \tilde{a}_2, \dots, \text{and } \tilde{a}_n$ , respectively, where  $\omega_j \in [0, 1]$ ,  $1 \leq j \leq n$  and  $\sum_{j=1}^n \omega_j = 1$ . Let  $\omega = \{\omega_1, \omega_2, \dots, \omega_n\}^T$  be the weighting vector of the hybrid geometric averaging operator, where  $\omega_j \in [0, 1]$ ,  $1 \leq j \leq n$  and  $\sum_{j=1}^n \omega_j = 1$ . Let  $\tilde{a}_j = (\tilde{a}_j)^{nw_j}$ , where  $n$  is the balancing coefficient. The IFHGA operator is as follow:

$$\begin{aligned} IFHGA(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) &= \left( 1 - \prod_{j=1}^n (1 - \mu_{\tilde{a}_{\sigma(j)}}) \right)^{\omega_j}, \\ &\quad \prod_{j=1}^n (1 - \mu_{\tilde{a}_{\sigma(j)}})^{\omega_j} - \prod_{j=1}^n (1 - \mu_{\tilde{a}_{\sigma(j)}} - \nu_{\tilde{a}_{\sigma(j)}})^{\omega_j} \end{aligned} \quad (12)$$

where “ $\sigma(1), \sigma(2), \dots, \text{and } \sigma(n)$ ” is a permutation of “1, 2,  $\dots$ , and  $n$ ”, such that  $\tilde{a}_{\sigma(j)}$  is the  $j$ th largest intuitionistic fuzzy value among  $\tilde{a}_1, \tilde{a}_2, \dots, \text{and } \tilde{a}_n$  and  $S(\tilde{a}_{\sigma(j-1)}) > S(\tilde{a}_{\sigma(j)})$ , where  $1 \leq j \leq n$ .

### 3. AGGREGATION OPERATORS IN $L^*$ -FUZZY SETS

Let  $(L^*, \leq_{L^*})$  be a complete lattice, where  $L^* = \{(x_1, x_2) | (x_1, x_2) \in [0, 1]^2, x_1 + x_2 \leq 1\}, (x_1, x_2) \leq_{L^*} (y_1, y_2)$  iff  $x_1 \leq y_1$  and  $x_2 \geq y_2$ , for  $(x_1, x_2), (y_1, y_2) \in L^*$ . We know that uninorms where all arguments are below or above identity behave like t-norms and t-conorms. The proposed aggregation operation has neutral value hold the post of frontier between these four types of scores. As information be classified into different classes before aggregating is more intuitive, a neutral value serving as frontier used to classify information is needed.

**Theorem 1.** Let  $(e, f) \in L^*$  be a neutral value serving as frontier used to classify information. Arguments  $(x_{11}, y_{11}), (x_{12}, y_{12}), \dots, (x_{1k}, y_{1k}), (x_{21}, y_{21}), (x_{22}, y_{21}), \dots, (x_{2l}, y_{21}), (x_{31}, y_{31}), (x_{32}, y_{32}), \dots, (x_{3m}, y_{3m}), (x_{41}, y_{41}), (x_{42}, y_{42}), \dots, (x_{4n}, y_{4n}) \in L^*$  are ordered:

$$\begin{aligned} &(x_{11}, y_{11}), (x_{12}, y_{12}), \dots, (x_{1k}, y_{1k}) \leq_{L^*} (e, f); \\ &(e, f) \leq_{L^*} (x_{21}, y_{21}), (x_{22}, y_{21}), \dots, (x_{2l}, y_{21}); \\ &e < x_{31}, x_{32}, \dots, x_{3m}, f \leq y_{31}, y_{32}, \dots, y_{3m}; \\ &x_{41}, x_{42}, \dots, x_{4n} < e, y_{41}, y_{42}, \dots, y_{4n} \leq f. \end{aligned}$$

$$\begin{aligned} &\mathcal{R}_G^*((x_{11}, y_{11}), \dots, (x_{1k}, y_{1k}), (x_{21}, y_{21}), \dots, \\ &\quad (x_{2l}, y_{2l}), (x_{31}, y_{31}), \dots, (x_{3m}, y_{3m}), \\ &\quad (x_{41}, y_{41}), \dots, (x_{4n}, y_{4n})) \\ &= \mathcal{A}(\mathcal{U}_1((x_{11}, y_{11}), \dots, (x_{1k}, y_{1k})), \\ &\quad \mathcal{U}_2((x_{21}, y_{21}), \dots, (x_{2l}, y_{2l})), \\ &\quad \mathcal{U}_3((x_{31}, y_{31}), \dots, (x_{3m}, y_{3m})), \\ &\quad \mathcal{U}_4((x_{41}, y_{41}), \dots, (x_{4n}, y_{4n}))) \\ &= \mathcal{A}((X_1, Y_1), (X_2, Y_2), (X_3, Y_3), (X_4, Y_4)) \\ &= (A(X_1, X_2, X_3, X_4), A(Y_1, Y_2, Y_3, Y_4)), \end{aligned}$$

where  $A(X_1, X_2, X_3, X_4) = \frac{kX_1 + lX_2 + mX_3 + nX_4}{k+l+m+n}$ ,  $A(Y_1, Y_2, Y_3, Y_4) = \frac{kY_1 + lY_2 + mY_3 + nY_4}{k+l+m+n}$ ,  $(X_1, Y_1) = \mathcal{U}_1((x_{11}, y_{11}), \dots, (x_{1k}, y_{1k}))$ ,  $\mathcal{U}_1$  is an uninorm in  $L^*$  with identity element  $(e_1, f_1)$ , and  $(e, f) \leq_{L^*} (e_1, f_1)$ .  $(X_2, Y_2) = \mathcal{U}_2((x_{21}, y_{21}), \dots, (x_{2l}, y_{2l}))$ ,  $\mathcal{U}_2$  is an uninorm in  $L^*$  with identity element  $(e_2, f_2)$ , and  $(e_2, f_2) \leq_{L^*} (e, f)$ .  $(X_3, Y_3) = \mathcal{U}_3((x_{31}, y_{31}), \dots, (x_{3m}, y_{3m}))$ ,  $\mathcal{U}_3$  is an uninorm in  $L^*$  with identity element  $(e_3, f_3)$ , and  $e_3 \leq e, f_3 \leq f$ .  $(X_4, Y_4) = \mathcal{U}_4((x_{41}, y_{41}), \dots, (x_{4n}, y_{4n}))$ ,  $\mathcal{U}_4$  is an uninorm in  $L^*$  with identity element  $(e_4, f_4)$ , and  $e \leq e_4, f \leq f_4$ . Then  $\mathcal{R}^*$  is an aggregation operator.

**Proof.** (1) It is obvious that for all  $(x, y) \in L^*$ ,  $\mathcal{R}^*((x, y)) = (x, y)$ .

(2) Monotonicity. Suppose that  $(x_{11}, y_{11}) \leq_{L^*} (x'_{11}, y'_{11}), \dots, (x_{1k}, y_{1k}) \leq_{L^*} (x'_{1k}, y'_{1k}), (x_{21}, y_{21}) \leq_{L^*} (x'_{21}, y'_{21}), \dots, (x_{2l}, y_{21}) \leq_{L^*} (x'_{2l}, y'_{2l}), (x_{31}, y_{31}) \leq_{L^*} (x'_{31}, y'_{31}), \dots, (x_{3m}, y_{3m}) \leq_{L^*} (x'_{3m}, y'_{3m}), (x_{41}, y_{41}) \leq_{L^*} (x'_{41}, y'_{41}), \dots, (x_{4n}, y_{4n}) \leq_{L^*} (x'_{4n}, y'_{4n})$ , according to the monotonicity of uninorms, we have

$$\begin{aligned} \mathcal{U}_1((x_{11}, y_{11}), \dots, (x_{1k}, y_{1k})) &\leq_{L^*} \mathcal{U}_1((x'_{11}, y'_{11}), \dots, (x'_{1k}, y'_{1k})), \\ \mathcal{U}_2((x_{21}, y_{21}), \dots, (x_{2l}, y_{2l})) &\leq_{L^*} \mathcal{U}_2((x'_{21}, y'_{21}), \dots, (x'_{2l}, y'_{2l})), \\ \mathcal{U}_3((x_{31}, y_{31}), \dots, (x_{3m}, y_{3m})) &\leq_{L^*} \mathcal{U}_3((x'_{31}, y'_{31}), \dots, (x'_{3m}, y'_{3m})), \\ \mathcal{U}_4((x_{41}, y_{41}), \dots, (x_{4n}, y_{4n})) &\leq_{L^*} \mathcal{U}_4((x'_{41}, y'_{41}), \dots, (x'_{4n}, y'_{4n})) \end{aligned}$$

then

$$\frac{kX_k + lX_l + mX_m + nX_n}{k + l + m + n} \leq \frac{kX'_k + lX'_l + mX'_m + nX'_n}{k + l + m + n}$$

$$\frac{kY_k + lY_l + mY_m + nY_n}{k + l + m + n} \geq \frac{kY'_k + lY'_l + mY'_m + nY'_n}{k + l + m + n}$$

it is obvious that

$$\begin{aligned} \mathcal{R}^*((x_{11}, y_{11}), \dots, (x_{1k}, y_{1k}), (x_{21}, y_{21}), \\ \dots, (x_{2l}, y_{2l}), (x_{31}, y_{31}), \dots, (x_{3m}, y_{3m}), \\ (x_{41}, y_{41}), \dots, (x_{4n}, y_{4n})) \\ \leq \mathcal{R}^*((x'_{11}, y'_{11}), \dots, (x'_{1k}, y'_{1k}), (x'_{21}, y'_{21}), \\ \dots, (x'_{2l}, y'_{2l}), (x'_{31}, y'_{31}), \dots, (x'_{3m}, y'_{3m}), \\ (x'_{41}, y'_{41}), \dots, (x'_{4n}, y'_{4n})) \end{aligned}$$

$$(3) \quad \mathcal{R}^*(\underbrace{(0, 1), (0, 1), \dots, (0, 1)}_{n \text{ times}}) = \mathcal{U}(\underbrace{(0, 1), (0, 1), \dots, (0, 1)}_{n \text{ times}}) = (0, 1).$$

$$(4) \quad \mathcal{R}^*(\underbrace{(1, 0), (1, 0), \dots, (1, 0)}_{n \text{ times}}) = \mathcal{U}(\underbrace{(1, 0), (1, 0), \dots, (1, 0)}_{n \text{ times}}) = (1, 0).$$

By Theorem 1, we have two special aggregation operators:

**Corollary 2.** Take  $\mathcal{U}_1 = \mathcal{U}_2 = \mathcal{U}_3 = \mathcal{U}_4 = \mathcal{U}$  with identity element  $(e, f)$ , which is neutral value serving as frontier used to classify information, we have

$$\begin{aligned} \mathcal{R}_G^*((x_{11}, y_{11}), \dots, (x_{1k}, y_{1k}), (x_{21}, y_{21}), \dots, \\ (x_{2l}, y_{2l}), (x_{31}, y_{31}), \dots, (x_{3m}, y_{3m}), \\ (x_{41}, y_{41}), \dots, (x_{4n}, y_{4n})) \\ = \mathcal{A}(\mathcal{U}((x_{11}, y_{11}), \dots, (x_{1k}, y_{1k})), \mathcal{U}((x_{21}, y_{21}), \\ \dots, (x_{2l}, y_{2l}))) \mathcal{U}((x_{31}, y_{31}), \dots, (x_{3m}, y_{3m})), \\ \mathcal{U}((x_{41}, y_{41}), \dots, (x_{4n}, y_{4n}))), \\ = \mathcal{A}((X_1, Y_1), (X_2, Y_2), (X_3, Y_3), (X_4, Y_4)) \\ = (A(X_1, X_2, X_3, X_4), A(Y_1, Y_2, Y_3, Y_4)) \end{aligned}$$

**Corollary 3.** Obviously, t-norms and t-conorms are uninorms with identity elements are  $(1, 0)$  and  $(0, 1)$ , respectively. Then take  $\mathcal{U}_1 = \mathcal{T}$ ,  $\mathcal{U}_2 = \mathcal{S}$ ,  $\mathcal{U}_3 = \mathcal{U}_4 = \mathcal{U}$  with identity element  $(e, f)$ , which is neutral value serving as frontier used to classify information,

we have

$$\begin{aligned} & \mathcal{R}_G^*((x_{11}, y_{11}), \dots, (x_{1k}, y_{1k}), (x_{21}, y_{21}), \dots, \\ & (x_{2l}, y_{2l}), (x_{31}, y_{31}), \dots, (x_{3m}, y_{3m}), (x_{41}, y_{41}), \\ & \dots, (x_{4n}, y_{4n})) \\ &= \mathcal{A}(\mathcal{T}((x_{11}, y_{11}), \dots, (x_{1k}, y_{1k})), \mathcal{S}((x_{21}, y_{21}), \\ & \dots, (x_{2l}, y_{2l}))), \mathcal{U}((x_{31}, y_{31}), \dots, (x_{3m}, y_{3m})), \\ & \mathcal{U}((x_{41}, y_{41}), \dots, (x_{4n}, y_{4n})) \\ &= \mathcal{A}((X_1, Y_1), (X_2, Y_2), (X_3, Y_3), (X_4, Y_4)) \\ &= (A(X_1, X_2, X_3, X_4), A(Y_1, Y_2, Y_3, Y_4)) \end{aligned}$$

## 4. APPLICATION FOR MULTI-ATTRIBUTES DECISION-MAKING

### 4.1. Multi-attributes Decision Method

Mathematically speaking, the multi-attributes decision-making problem of  $n$  alternatives with  $m$  attributes can be expressed as

$$A_i \left( \begin{array}{cccc} c_1 & c_2 & \cdots & c_m \\ (x_{11}, y_{11}) & (x_{12}, y_{12}) & \cdots & (x_{1m}, y_{1m}) \\ (x_{21}, y_{21}) & (x_{22}, y_{22}) & \cdots & (x_{2m}, y_{2m}) \\ \vdots & \vdots & \ddots & \vdots \\ (x_{n1}, y_{n1}) & (x_{n2}, y_{n2}) & \cdots & (x_{nm}, y_{nm}) \end{array} \right)$$

where  $U = \{A_1, A_2, \dots, A_n\}$  is the set of alternatives,  $C = \{c_1, c_2, \dots, c_m\}$  is the set of attributes,  $(x_{ij}, y_{ij})$  ( $i = 1, 2, \dots, n, j = 1, 2, \dots, m$ ) is the evaluation information of alternatives  $A_i$  under  $c_j$  provided by experts.

The decision procedure is summarized as follow:

**Step 1:** For every  $A_i$ ,  $1 \leq i \leq n$ , according to the neutral value serving as frontier used to classify information,  $(x_{i1}, y_{i1}), \dots, (x_{im}, y_{im})$  are classified into four classes.

**Step 2:** For the four classes, aggregating them using different uninorms, respectively, then we can obtain  $(X_1, Y_1), (X_2, Y_2), (X_3, Y_3), (X_4, Y_4)$ .

**Step 3:** By Theorem 1, aggregating  $(X_1, Y_1), (X_2, Y_2), (X_3, Y_3), (X_4, Y_4)$ , then we can obtain the aggregation result  $\mathcal{R}^*(A_i)$  ( $1 \leq i \leq n$ ), respectively.

**Step 4:** Compute the score functions  $S(\mathcal{R}^*(A_i))$  ( $1 \leq i \leq n$ ) by Eq. (1), respectively.

**Step 5:** Rank these score functions  $S(\mathcal{R}^*(A_i))$  ( $1 \leq i \leq n$ ) and choose the best one.

### 4.2. Application for Multi-attributes Decision-Making

In this part, we compare the proposed method with other existing methods by some examples. Assume the neutral value serving as frontier used to classify information is  $(0.5, 0.5)$ ,  $\mathcal{U}_1((x_1, y_1), (x_2, y_2)) = (x_1 \wedge x_2, y_1 \vee y_2)$ ,

$$\begin{aligned} \mathcal{U}_2((x_1, y_1), (x_2, y_2)) &= (x_1 \vee x_2, y_1 \wedge y_2), \quad \mathcal{U}_3((x_1, y_1), (x_2, y_2)) = \\ &\left\{ \begin{array}{ll} \frac{x_1 x_2}{1-x_1-x_2+2x_1 x_2}, & \text{if } \{x_1, x_2\} \neq \{0, 1\} \\ , 1 \wedge (y_1 + y_2), & \text{otherwise.} \end{array} \right. \\ \mathcal{U}_4((x_1, y_1), (x_2, y_2)) &= (x_1 x_2, 0 \vee (y_1 + y_2 - 1)), \\ \mathcal{U}_5((x_1, y_1), (x_2, y_2)) &= (x_1 x_2, y_1 \vee y_2), \quad \mathcal{U}_6((x_1, y_1), (x_2, y_2)) = \\ &(x_1 \vee x_2, y_1 y_2). \end{aligned}$$

**Example 1.** [12] Let's say an investment company wants to invest some money in the most promising firm. Through market analysis, three possible choices A, B, and C will be considered, as shown below: A is a car firm, B is a food firm, and C is a computer firm. The five attributes  $c_1, c_2, c_3, c_4$ , and  $c_5$  are used to evaluate the three alternatives, as shown below:  $c_1$  denotes the risk analysis,  $c_2$  denotes the growth analysis,  $c_3$  denotes the social-political impact analysis,  $c_4$  denotes the environment impact analysis, and  $c_5$  denotes the development of the money. The intuitionistic fuzzy decision matrix is shown as Table 1.

Assume that the weight  $w_1, w_2, w_3, w_4$ , and  $w_5$  of the attributes  $c_1, c_2, c_3, c_4$ , and  $c_5$  are 0.112, 0.236, 0.304, 0.236, and 0.112, respectively, that is,  $w_1 = 0.112, w_2 = 0.236, w_3 = 0.304, w_4 = 0.236$ , and  $w_5 = 0.112$ . Assume that IFOWGA operator has the weighting vector  $\omega = \{0.25, 0.20, 0.15, 0.18, 0.22\}^T$ , and IFHGA operator has the weighting vector  $\omega = \{0.25, 0.20, 0.15, 0.18, 0.22\}^T$ . Table 2 compares the preference order of the different methods.

From Table 2, we can see that the method of Xu and Yager [10], the method of Wang and Liu [11], the method of He et al. [12], the method of Chen and Chang [13], and the proposed method get the same preference order of the alternatives  $A > B > C$ , although different uninorms are chosen. Therefore, there is no direct relationship between the result of decision and the choice of uninorms.

In the following research, we discovered that methods in Xu and Yager [10], Wang and Liu [11], He et al. [12], Chen and Chang [13] are flawed in that they fail to differentiate the preference order of the alternatives in some situation.

**Example 2.** Let's say the investor wants to invest some money in the most promising firm. Through market analysis, three possible choices A, B, and C will be considered, as shown below: A is a car firm, B is a food firm, and C is a computer firm. The five attributes  $c_1, c_2, c_3, c_4$ , and  $c_5$  are used to evaluate the three alternatives, as shown below:  $c_1$  denotes the risk analysis,  $c_2$  denotes the growth analysis,  $c_3$  denotes the social-political impact analysis,  $c_4$  denotes the environment impact analysis, and  $c_5$  denotes the development of the money. The intuitionistic fuzzy decision matrix is shown as Table 3.

Assume that the weight  $w_1, w_2, w_3, w_4$ , and  $w_5$  of the attributes  $c_1, c_2, c_3, c_4$ , and  $c_5$  are 0.2, 0.2, 0.2, 0.2, and 0.2, respectively, that is,  $w_1 = w_2 = w_3 = w_4 = w_5 = 0.2$ . Assume that IFOWGA operator has the weighting vector  $\omega = \{0.2, 0.2, 0.2, 0.2, 0.2\}^T$ , and IFHGA operator has the weighting vector  $\omega = \{0.2, 0.2, 0.2, 0.2, 0.2\}^T$ . Table 4 compares the preference order of the different methods.

Table 4 shows a comparison of the preference order of the alternatives A, B, and C for different methods. We can see that the method of Xu and Yager [10], the method of Wang and Liu [11] and the method of He et al. [12] cannot distinguish the preference order among the alternatives A, B, and C, and the method of Chen and Chang [13] cannot distinguish the preference order among the alternatives A and B. On the contrary, although different uninorms are chosen, the proposed method can distinguish the preference order of the alternatives A > B > C. Therefore, the method based

**Table 1** | The decision matrix represented by intuitionistic fuzzy values.

	$c_1$	$c_2$	$c_3$	$c_4$	$c_5$
A	(0.2000, 0.5000)	(0.4000, 0.2000)	(0.5000, 0.4000)	(0.3000, 0.3000)	(0.7000, 0.1000)
B	(0.2000, 0.7000)	(0.6000, 0.3000)	(0.4000, 0.3000)	(0.4000, 0.4000)	(0.6000, 0.1000)
C	(0.2000, 0.7000)	(0.5000, 0.3000)	(0.4000, 0.5000)	(0.3000, 0.4000)	(0.6000, 0.2000)

**Table 2** | A comparison of the preference orders of the alternatives for the different methods.

Methods	Aggregation Operators	Preference order	Results
Method [10]	IFWG	$A > B > C$	A
	IFOWG	$A > B > C$	A
	IFHG	$A > B > C$	A
Method [11]	IFWG $^\varepsilon$	$A > B > C$	A
	IFOWG $^\varepsilon$	$A > B > C$	A
Method [12]	IFWGIA	$A > B > C$	A
	IFOWGIA	$A > B > C$	A
	IFHGA	$A > B > C$	A
Method [13]	IFWGA	$A > B > C$	A
	IFOWGA	$A > B > C$	A
	IFHGA	$A > B > C$	A
The proposed method ( $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \mathcal{U}_4$ )	$\mathcal{R}^*$	$A > B > C$	A
The proposed method ( $\mathcal{U}_5, \mathcal{U}_6, \mathcal{U}_3, \mathcal{U}_4$ )	$\mathcal{R}^*$	$A > B > C$	A

Note: IFHG, intuitionistic fuzzy hybrid geometric; IFHGA, intuitionistic fuzzy hybrid geometric averaging; IFHGIN, intuitionistic fuzzy hybrid geometric interaction averaging; IFOWG, intuitionistic fuzzy ordered weighted geometric; IFOWGE, intuitionistic fuzzy Einstein ordered weighted geometric; IFOWGA, intuitionistic fuzzy ordered weighted geometric averaging; IFWGIA, intuitionistic fuzzy ordered weighted geometric interaction averaging; IFWG, intuitionistic fuzzy weight geometric; IFWG $^\varepsilon$ , intuitionistic fuzzy Einstein weighted geometric operator; IFWGA, intuitionistic fuzzy weighted geometric averaging; IFWGIA, intuitionistic fuzzy weighted geometric interaction averaging.

**Table 3** | The decision matrix represented by intuitionistic fuzzy values.

	$c_1$	$c_2$	$c_3$	$c_4$	$c_5$
A	(0.3416, 0.2300)	(0.6900, 0.3100)	(0.0000, 0.8000)	(0.2000, 0.7500)	(0.0000, 0.8700)
B	(0.2000, 0.7500)	(0.7100, 0.2300)	(0.1200, 0.3100)	(0.2000, 0.8000)	(0.0000, 0.8700)
C	(0.1000, 0.8700)	(0.2500, 0.7500)	(0.0000, 0.8000)	(0.36000, 0.2300)	(0.0000, 0.3100)

**Table 4** | A comparison of the preference orders of the alternatives for the different methods.

Methods	Aggregation Operators	Preference Order	Results
Method [10]	IFWG	$A = B = C$	Don't know
	IFOWG	$A = B = C$	Don't know
	IFHG	$A = B = C$	Don't know
Method [11]	$IFWG^{\varepsilon}$	$A = B = C$	Don't know
	$IFOWG^{\varepsilon}$	$A = B = C$	Don't know
Method [12]	IFWGIA	$A = B = C$	Don't know
	IFOWGIA	$A = B = C$	Don't know
	IFHGIA	$A = B = C$	Don't know
Method [13]	IFWGA	$A = B > C$	Don't know
	IFOWGA	$A = B > C$	Don't know
	IFHGA	$A = B > C$	Don't know
The proposed method ( $U_1, U_2, U_3, U_4$ )	$\mathcal{R}^*$	$A > B > C$	A
The proposed method ( $U_5, U_6, U_3, U_4$ )	$\mathcal{R}^*$	$A > B > C$	A

Note: IFHG, intuitionistic fuzzy hybrid geometric; IFHGA, intuitionistic fuzzy hybrid geometric averaging; IFHGIA, intuitionistic fuzzy hybrid geometric interaction averaging; IFOWG, intuitionistic fuzzy ordered weighted geometric; IFOWG $\varepsilon$ , intuitionistic fuzzy Einstein ordered weighted geometric; IFOWGA, intuitionistic fuzzy ordered weighted geometric averaging; IFOWGIA, intuitionistic fuzzy ordered weighted geometric interaction averaging; IFWG, intuitionistic fuzzy weight geometric; IFWG $\varepsilon$ , intuitionistic fuzzy Einstein weighted geometric operator; IFWGA, intuitionistic fuzzy weighted geometric averaging; IFWGIA, intuitionistic fuzzy weighted geometric interaction averaging.

on the proposed aggregation operator can conquer the defects of Xu and Yager's [10], Wang and Liu's [11], He *et al.*'s [12] and Chen and Chang's [13] approaches. Moreover, The result shows that different uninorms do not affect the ranking of alternatives.

#### 4.3. Analysis of the Effectiveness of the Proposed Method

In practical application, some existing methods aggregate information directly rather than classifying the information into different classes, which resulted to the defects that they get wrong preference orders of alternatives or cannot be ranked in some situations. Most of the time, our behavior with the good items are not the same with the bad items, therefore these information be classified into different classes before aggregating is reasonable. The proposed aggregation operator based on uninorms in  $L^*$ -fuzzy set theory is effective and reasonable, because uninorm can model the data of bipolar behavior. Moreover, we show that there is no direct relationship between the result of decision and the choice of uninorms.

### 5. CONCLUSIONS

In this study, a new aggregation operator for decision-making based on uninorms in  $L^*$ -fuzzy set theory is proposed. The aggregation operator allows us to control the aggregation relying on allocation of arguments larger than neutral value, less than neutral value and incomparable with neutral value. Moreover, examples are given to expound the practicability of the decision-making approach. It have been shown multi-attributes decision-making approach based on the proposed aggregation operator is effective. As aggregation

operators play vital role in decision-making, in next work, we will investigate other aggregation operators in  $L^*$ -fuzzy set theory.

### CONFLICTS OF INTEREST

The authors declare no conflict of interest.

### AUTHORS' CONTRIBUTIONS

M.X. Luo initiated the research and provide the framework of this paper. Y. Zhang and B. Liu wrote and complete this paper with M.X. Luo's validity and helpful suggestions.

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