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Assessing Optimal Retention With Quantile and Expectile Risk Measure

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Abstract— **The form of claim severity from the reinsurance contracts is skewed to the right and heavy tail. The aims of this research are to determine the optimal risk measure on a reinsurance contracts. This risk measure is the coverage limit between an insurance company and a reinsurance company called optimal retention. Optimal retention in this research is estimated with 2 risk measures methods, which are the quantile-based Value-at-Risk (QVaR) and the expectile-based Value-at-Risk (EVaR). The Comparison results of the both methods by analytically and numerically show that EVaR is coherent and has a more optimal value than QVaR. As an illustration, we will use data of claim severity on a reinsurance contracts to determine the value of QVaR and EVaR.**

Keywords— Pareto Distribution, Reinsurance Contract, Quantile, Expectile, Coherent Risk Measures

I. Introduction

A general approach in loss modeling data on reinsurance claim severity is the distribution that skewed to the right. Claims with large losses undermine the viability of reinsurance contracts. Thus, in modeling losses special efforts must be made in analyzing the behavior of extreme values. For claim severity, the extreme values occur in the upper (right hand) tail of the distribution. A distribution with hight probability of heavy loss is said to a heavy tail, which may be interpreted in the relative or absolute sense. One of the distributed candidates used to model these is the Pareto distribution. Pareto distribution is a good policy to model income distribution and insurance distribution with threshold. Therefore, the use of this distribution is suitable for reinsurance data.

Data of claim severity are modelled with Pareto distribution (α,γ). This literature follows the pioneering work of Cai et al. [5] and Gray et al. [8]. Suppose that a random loss is $X\sim$ Pareto (α,γ). The claim severity that can be insured are data with a value of $E[X]$ finite. Moreover, the data that follows the Pareto distribution must have $\alpha > 1$ such that $E[X]$ exists. The values on the tail Pareto distribution are the values to be reinsured. This is a loss in excess of risk measure (ℓ) or denoted by $X-\ell|X>\ell.X-\ell|X>\ell\sim$Pareto$(\alpha,\ \gamma+\ell)$. This shows that the claim severity distribution of reinsurance also has Pareto distribution.

Gray and Pitts [8] show that reinsurance contracts are the same as insurance contracts. In a reinsurance contract, the reinsurance company bears claims that the insurance company can- not pay. In general, reinsurance is divided into proportional and non-proportional reinsurance (see Lempeart [11]). We will focus on non-proportional reinsurance. In non-proportional reinsurance, the random loss of the claims reported by the policyholder (X) will be divided into 2 parts, namely the large random loss of claims paid by the insurance company (Y) and the large random loss of claims paid by the reinsurance company (Z). The value sharing limit between Y and Z is called optimal retention (M).

Optimal retention is a risk measure. Optimal retention of this research will be estimated with 2 risk measures methods, which are the quantile-based Value-at-Risk (QVaR) and the expectile- based Value-at-Risk (EVaR). From these two risk measures we will compare the best risk measure to optimal retention. At the end of the research, an optimal retention will be estimated using QVaR and EVaR with data of claim severity on a reinsurance contracts.

II. Optimal Retention

A. Quantile VaR (QVaR) vs Expectile-based VaR (EVaR)

*1) Quantile VaR (QVaR)*Let X be a continuous variable denote a claim severity with distribution function continue F_X. Given $\delta \in (0, 1)$ is the confidence level. Newey and Powell [12] is well known that δ can be obtained by minimizing asymmetrically weighted mean absolute deviations:

$$E\left[\left|\delta-1_{X\leq q}\right|\cdot|X-q|\right] \tag{1}$$

where 1_A is the indicator of event A.

$$1_A=\begin{cases}1, & X\leq q\\ 0, & x>q\end{cases}$$

The first order condition of minimizing (1) is QVaR with confidence level δ. $QVaR_\delta(X) = q_\delta$ is the quantile with confidence level δ.

$$QVaR_\delta(X)=q_\delta=F_X^{-1}(\delta) \tag{2}$$

where F^{-1} is the inverse of the distribution function F.

QVaR as a measure of quantile risk depend only on the *probability* of extreme losses but not their magnitude. QVaR is a statistical measure based on a measure of coherent risk. QVaR as a measure of risk does not following of axiom subadditivity ($\ell(X+Y)<\ell(X)+\ell(Y)$).

2) Expectile-based VaR (EVaR) Expectile-based VaR (EVaR) is an improvement of QVaR where EVaR does not only depend on the probability of extreme losses but also sensitive to their magnitude . Before discussing EVaR, we will first discuss expectile. The expectile function was first introduced by Newey and Powell in 1987 [12]. This function is defined as a single solution from minimizing quadratic loss function which is defined as

$$E[\rho_\theta(x-v)] = E[|\theta - 1_{X\leq v}| \cdot |x-v|^2] \quad (3)$$

The first order condition of minimizing (3) is

$$\theta \int_v^\infty |y-v|\, f_Y(y)dy + (\theta-1)\int_{-\infty}^v |y-v| f_Y(y)dy = 0 \quad (4)$$

from (4) it can also be verified that expectile from random variable X is

$$v_\theta = \beta E[X|X > v_\theta] + (1-\beta)E[X|X \leq v_\theta] \quad (5)$$

where

$$\beta = \frac{\theta[1-F_X(v_\theta)]}{\theta[1-F_X(v_\theta)]+(1-\theta)F_X(v_\theta)}$$

may be interpreted as a weighted probability of $X > v_\theta$. Thus, v_θ is an average that balances between $E[X|X \leq v_\theta]$ (conditional upside mean) and $E[X|X \leq v_\theta]$ (conditional downside mean). From (5) Kuan et al. [10] states EVaR as in the following equation

$$EVaR_\theta(X) = |v_\theta(X)|$$

where $\theta < \frac{1}{2}$.

B. Optimal Retention

Optimal retention is minimum value of risk measure proposed by Cai et al. [4], Cai et al. [5] and Assa [2].

$$\min_{M>0} \ \ell(M,\delta) \quad (6)$$

where ℓ is risk measure, M retention and $\delta \in (0,1)$ determines the confidence level of risk measure. As an illustration to see which risk measure is better to be the optimal retention between two risk measures described earlier, see Fig. 1.

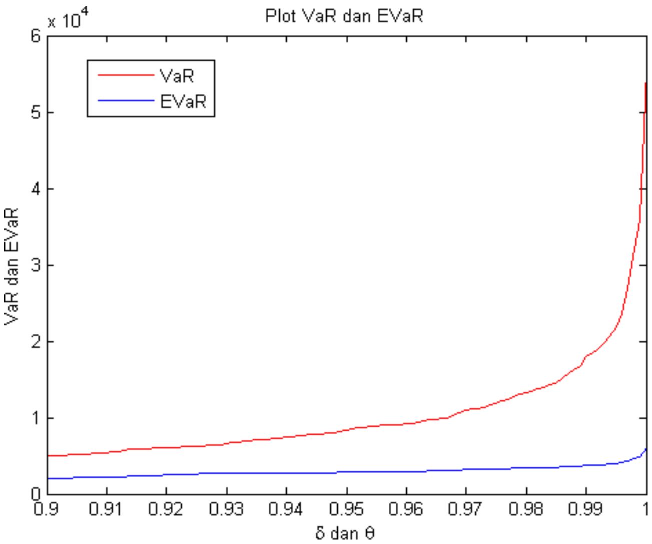

Fig. 1. Illustration of QVaR and EVaR values of The Pareto distribution

From the Fig. 1 it can be seen that the EVaR value is less than the QVaR for the same δ and θ. Based on this, the EVaR value is more optimal than QVaR, so EVaR is considered better to be an optimal risk measure.

The properties discussed above suggest that EVaR, which takes into account the magnitude of loss, may serve as a better measure for tail risk. We can be seen from the value of θ. Artzer et al. [1] suggest four axioms of measures of risk. They argue that these axioms "should hold for any risk measure that is to be used to effectively regulate or manage risks." A risk measure that satisfies these four axiom is said to be coherent. The risk measure of EVaR is satisfies axioms of coherent.

Theorem 2.1. *$EVaR_\theta(X)$ is a coherent risk measure.*

Proof.

i. Axiom Translational invariance and Positive homogeneity.

Suppose that $\tilde{X} = kX + a$ where $k \geq 0$ and $a \geq 0$, so that the expectile-based VaR

$$\begin{aligned} v_\theta(\tilde{X}) &= \beta E[\tilde{X}|\tilde{X} > v_\theta] + (1-\beta)E[\tilde{X}|\tilde{X} \leq v_\theta] \\ &= \beta E[kX+a|kX+a > v_\theta] + (1-\beta)E[kX+a|kX+a \leq v_\theta] \\ &= \beta(kE[X|X > v_\theta] + a) + (1-\beta)(kE[X|X \leq v_\theta] + a) \\ &= k(\beta E[X|X > v_\theta] + (1-\beta)E[X|X \leq v_\theta] + a \\ &= kv\_\theta\ (X) + a \end{aligned}$$

ii. Axiom monotonicity.

If we have X_1 and X_2 such that $X_1 \leq X_2$, any constant $a \geq 0$ then evidently we have that $X_2 = X_1 + a \Leftrightarrow X_2 - a = X_1$ so that

$$\begin{aligned} v_\theta(X_1) &= \beta E[X_1|X_1 > v_\theta] + (1-\beta)E[X_1|X_1 \leq v_\theta] \\ &= \beta E[X_2 - a|X_2 - a > v_\theta] + (1-\beta)E[X_2 - a|X_2 - a \leq v_\theta] \\ &= \beta(E[X_2|X_2 > v_\theta] - a) + (1-\beta)(E[X_2|X_2 \leq v_\theta] - a) \\ &= \beta(E[X_2|X_2 > v_\theta] + (1-\beta)(E[X_2|X_2 \leq v_\theta] - a \\ &= v_\theta(X_2) - a \end{aligned}$$

iii. Axiom Subadditivity.

Before proving that EVaR fulfills the axiom of Subadditivity, it will first prove a theorem that will be used to prove axiom of Subadditivity.

Theorem 2.2 *Suppose that X and x such that $1-F_X(x)>0$ for any event A such that $Pr(A) = 1 - F_X(x)$, then*

$$E[X|A] \leq E[X|X > x]$$

Proof

$$\begin{aligned} E[X|X > x] &= E[X|X > x, A]Pr(A|X > x) + E[X|X > x, A^c]Pr(A^c|X > x) - xPr(A|X > x) - xPr(A^c|X > x) + x \\ &= x + E[X|X > x, A]Pr(A|X > x) - xPr(A|X > x) + E[X|X > x, A^c]Pr(A^c|X > x) - xPr(A^c|X > x) \\ &= x + E[X - x|X > x, A]Pr(A|X > x) + E[X - x|X > x, A^c]Pr(A^c|X > x) \\ &\leq x + E[X - x|X > x, A]Pr(A|X > x) \end{aligned}$$

because $Pr(A) = Pr(X > x)$ so that

$$\begin{aligned} x + E[X - x|X > x, A]Pr(A|X > x) &= x + E[X - x|X > x, A]Pr(X > x|A) \\ &\geq x + E[X - x|X > x]Pr(X > x|A) + E[X - x|X \leq x]Pr(X \leq x|A) \\ &= x + E[X|X > x]Pr(X > x|A) + E[X|X \leq x]Pr(X \leq x|A) - xPr(X > x|A) - xPr(X \leq x|A) \\ &= E[X|A] \end{aligned}$$

the axiom subadditivity can also be verified that

$$
\begin{aligned}
v_\theta(X_1+X_2) &= \beta E[X_1|X_1+X_2>v_\theta] + (1-\beta)E[X_2|X_1+X_2\le v_\theta] \\
&= \beta E[X_1|X_1+X_2>v_\theta] + E[X_2|X_1+X_2>v_\theta] + \\
&\quad (1-\beta)E[X_1|X_1+X_2>v_\theta] + E[X_2|X_1+X_2>v_\theta] \\
&\le \beta E[X_1|X_1>v_\theta] + E[X_2|X_2>v_\theta] + (1- \\
&\quad \beta)E[X_1|X_1>v_\theta] + E[X_2|X_2>v_\theta] \\
&= \beta E[X_1|X_1>v_\theta] + (1-\beta)E[X_1|X_1>v_\theta] + \\
&\quad E[X_2|X_2>v_\theta] + +(1-\beta)E[X_2|X_2>v_\theta] \\
&= v_\theta(X_1) + v_\theta(X_2)
\end{aligned}
$$

III. Empirical Study

A. Data and Computation

As an illustration were used claim severity data on car reinsurance claims in 2004-2005 which were calculated in dollars ($). The fitted model from the data with the Pareto distribution can be seen in Figure 2 and Figure 3.

Parameter estimates using the maximum likelihood method are obtained $\alpha = 2.0063$ and $\gamma = 2.12502\text{x}10^3$. It can be seen from Figure 2 and 3. From Figure 2 the probability function of the Pareto distribution is validated to determine adequately conforms to extreme data. The Pareto distribution function is also very close to the empirical function of the data in Figure 3.

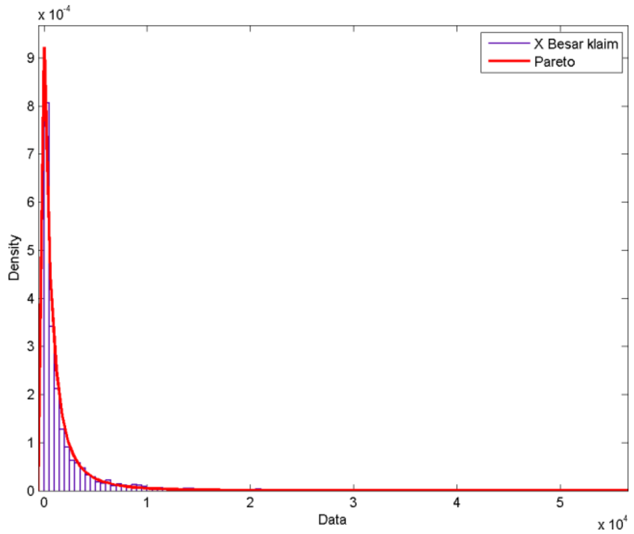

Fig. 2. Histogram of the data and probability density function of Pareto distribution

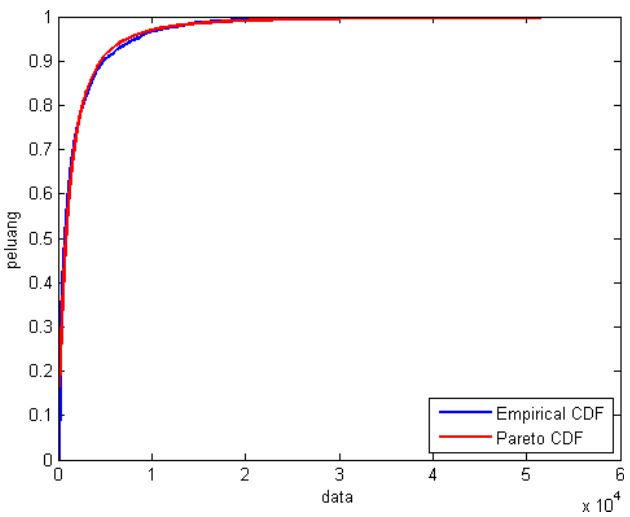

Fig. 3. Plot of distribution function of Pareto and plot of empirical function of data

B. Empirical Results

To estimate the optimal retention from claim severity data on car reinsurance claims in 2004- 2005, using (2) and (5). For the empirical result, we can see at Table 1.

The calculation data in Table 1 show that the difference in EVaR and QVaR values is far enough. for any $\alpha = \theta$, the EVaR value will be minimum than QVaR, especially on the tail distributed (confidence level at 0.999). Where the greater of the confidence level, different value of EVaR and QVaR will be more significant. In our empirical analysis, for any confidence level $\alpha = \theta$ the value of EVaR has a minimum than the value of QVaR. Based on (6), we know that the best optimal retention risk measure for this case is EVaR.

TABLE I. PREDICTION OF QVAR AND EVAR ($)

α = θ	**0.9**	**0.95**	**0.99**	**0.999**
QVaR	4.94191×10^3	8.26453×10^3	17.87407×10^3	35.93875×10^3
EVaR	2.06663×10^3	2.82103×10^3	3.75068×10^3	4.86854×10^3

IV. CONCLUDING REMARKS

In this paper researcher propose an expectile-based value at risk (EVaR), that is more sensitive to the magnitude of extreme losses than quantile-based value at risk (QVaR). To implement this measure, researcher assessing the optimal retention for data of reinsurance claim severity with EVaR and QVaR. The results from these research has been shown that EVaR is coherent risk measure and has a more optimal value than QVaR. In addition, it was concluded that EVaR is the best optimal risk measure for the claim severity data in the reinsurance contract.

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