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Poly-Weighted Exponentiated Gamma Distribution with Application

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ABSTRACT

This paper proposes a weighting of the exponentiated gamma distribution with a polynomial function called the poly-weighted exponentiated gamma distribution (PWEGD). It shows that the modified distribution harnesses the multi-dimensional effects of the distribution. We provided an extensive mathematical treatment of this proposed distribution: obtained its parameters, estimated its statistical properties with applicable, tests and compared the estimates with existing distribution. The study estimated the cumulative distribution function, hazard function, survival function, skewness, kurtosis, mode, median and quartiles of the distribution and evaluated the distribution with Monte Carlo simulated data and the data of the wind direction (degrees) in Lagos, Nigeria. Empirical analysis showed that with increased polynomial function, the estimates and the statistical properties like the expectation, variance, standard error, median, mode, hazard and survival functions, cumulative distribution function (CDF), moments, skewness and kurtosis were significantly better than the existing root distributions. The MSE of the parameters decreased with increased power and the parameter is significant (p <0.05). It is concluded that the proposed distribution does not only provide better fitting but also establishes an efficient structure for lifetime data modelling.

1. INTRODUCTION

The field of modelling is still unexploited and is progressively attaining prominence in practically all fields of human endeavor like biological, financial, economic, medical, etc., which hitherto was overwhelming researchers and industry players. Contending with economic decline, climate change, high mortality and morbidity, low productivity, organism resistance, viral infections for which most traditional distributions developed have failed is global and efforts to identify and develop models that will curb these global problems is in the front burner. However, the problem of skewness or lack of symmetry inherent in real life events has given rise to the contemporary exploration of mixed, extended and modified distributions which birthed the current innovations and developments in the field of probability distributions.

The hype on a family of hybrid, mixed and modified distributions which are assumed to perform better than their traditional roots through hybridization, weightings and addition of parameters to these traditional distributions extends their usefulness and overcome their inflexibility, weaknesses skewedness [1–5].

This study proposes a polynomial weighting of the exponentiated gamma distribution (EGD) which is a mixture of the exponential and the gamma distributions with a weight with polynomial superscript. The modification makes the distribution more flexible than the root distributions and the length or area biased distributions in their estimates and variability of their parameters. The paper is organized as follows: Section 2 presents the proposed poly-weighted exponentiated gamma distribution (PWEGD), its estimates and statistical properties. Section 3 evaluates the performance using Monte Carlo simulation, analyzed real data and compared the results with existing baseline distribution. Finally, concluding remarks are made in Section 4.

New probability distributions developed from existing distributions are swiftly coming to limelight among theoretical statisticians because of the complexity of life problems faced daily which many prior established distributions could not handle. Researchers and statisticians have generally agreed that modified and enhanced distributions are more elastic as additional parameters are introduced to the base distributions [6]. Shahzad *et al.* [5] opined that the introduction of new parameters into existing distributions makes them flexible modelling complex life data accurately. Most real-life problems deviate from normality as earlier believed by statisticians but are skewed in reality. Hence, the newly developed class of distributions are flexible and deal with these observed complexities. Oguntunde *et al.* [1], noted that introducing additional shape or parameters to well-known standard distributions produce compound distributions which explores their tail properties

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and improve the goodness of fit of the root generator. Therefore, this study proposed an enhancement on the EGD by polynomially biasing or weighting it with an additional parameter to estimate.

The exponential distribution is very suitable when the rate of failure in some phenomena is constant whereas the gamma distribution is a more generalization of the exponential and chi-square distributions and reproduces the time until the occurrence of the next event and used as the "conjugate prior" to other distributions [7]. The weighted exponential distribution was proposed by Gupta and Kundu [8] and has been explored by other developers who adapted it in various ways like introducing parameters and using new methods to estimate its parameters I order to make it more flexible in application to real-life events [6,9–11]. It was Al-Kadim and Hussein [12] who noted that weighted distribution is probability distributions function that satisfy the following:

$$f_w(x) = \frac{w(x) f(x)}{\int_{-\infty}^{\infty} w(x) f(x)\, dx}, -\infty \le x \le \infty \tag{1}$$

They noted that distributions are length-biased when the weight function is contingent on the length of units of interest [12]. Ikegwu *et al.* [13] proposed the EGD as a fusion of the exponential and gamma distributions introducing a threshold parameter and explored and compared its length and area—biases and also obtained their properties and estimates. Wang *et al.* [14] explored an exponential polynomial distribution for modelling the frequency distribution of wind speed in the unimodal and multimodal form as it has lower variability for zero and low wind speed. The exponential polynomial was unique because it not only described the unimodal distribution but also accounts for the multimodal distribution exhibited by the phenomena under consideration. Chesneau *et al.* [15], fitted a polynomial—exponential model which allows for the event of zero—values in a probability distribution. This study explores a polynomial weighting of the EGD which mixes the exponential and the gamma distributions with a threshold parameter. This modified distribution will perform better than the individual, distributions, their mixture and length and area biases with respect to the estimates of their parameters and variability of these estimates.

Weighting distributions of interest with a $w(x) = x$ function in the first power yields the length biased distribution of the function. This infers that if X is lifetime distributed having a distribution function $f(x)$ and a finite expectation; Oyamakin and Durojaiye [3] expressed the length biased distribution of X as

$$f_{LB}(x;\theta) = \frac{xf(x;\theta)}{E\left[f(x;\theta)\right]}, x > \theta \tag{2}$$

Weighting the distribution of interests to with a quadratic function of the variable (i.e., $w(x) = x^2$) function results into the area biased distribution of the function. This implies that the random variable X as a lifetime distribution with pdf, $f(x)$ and a finite expectation according Oyamakin and Durojaiye [3] and Ikegwu [16] expressed the area biased distribution of X as

$$f_{AB}(x;\theta) = \frac{x^2 f(x;\theta)}{\int_{-\infty}^{\infty} x^2 f(x;\theta)\, dx}, x > \theta \tag{3}$$

Most researchers show the maximum likelihood estimates (MLEs) as the most applicable method of estimating the parameters of mixed or hybrid distributions, though others have approached it through other methods like the least square or method of moments [10,17]. Mahmoud *et al.* [11], observed that the maximum likelihood estimation is still the most feasible method of performing statistical inference as it required only the joint distribution of the observed values for estimation.

2. METHODOLOGY

The proposed distribution was developed with its cumulative density function, survival functions, hazard function and its statistical properties like the moments, mean, standard deviation, median, quartiles, skewness, kurtosis, etc., estimated. The parameter of the distribution was equally estimated by methods of maximum likelihood. Live and simulated data were used to explore the applicability of the proposed distribution.

The weighted distribution is probability distributions function that satisfy the following:

$$f_w(x) = \frac{w(x) f(x)}{\int_{-\infty}^{\infty} w(x) f(x)\, dx}, -\infty \le x \le \infty \tag{4}$$

2.1. The EGD

The EGD was introduced by Gupta *et al.* [18], to model failure rates and was applied by Shawky and Bakoban [17] using different methods of estimation. However, the EGD obtained from a mix of the exponential and gamma distributions by Ikegwu *et al.* [13] is adopted in this study and is given as

$$f(x, a, b, \theta) = \frac{2^a}{\Gamma a b^a} (x-\theta)^{a-1} e^{\frac{-2(x-\theta)}{b}}, x > \theta;\ a, b > 0 \tag{5}$$

with a cumulative density function (cdf) given as

$$F(X \leq t) = \frac{\Gamma\left(\frac{2(t-\theta)}{b}, a\right)}{\Gamma(a)}, t > \theta;\ a, b > 0 \tag{6}$$

where a is the shape parameter, b is the scale parameter, θ is the threshold parameter, Γ is the gamma function and is used to model positively scaled data. The threshold parameter, θ, estimates the initial failure occurrence and situates the distribution on a time scale bearing same unit random variable. Minitab [19], posit that a distribution with a threshold parameter is useful in estimating the earliest time-to-failure in a system while the distribution begins at the origin when $\theta = 0$, begins after the origin when $\theta > 0$ and before it when $\theta < 0$ which means the failure occurred before the test commenced. Amin and Hussian [20] noted that the threshold (θ) is the guaranteed time while b (the inverse of lambda (λ) is the mean time to failure.

This study extended the EGD by weighting the distribution in Equation (5) with a polynomial function (i.e., $w(x) = (x-\theta)^c$ using Equation (4) above. Then, the parameters of the proposed distribution are estimated, properties generated and its application harnessed with simulated as well as real-life data.

2.2. The Proposed PWEGD

The proposed PWEGD is given as

$$f(x; a, b, c, \theta) = \frac{(x-\theta)^c \frac{2^a}{\Gamma a b^a} (x-\theta)^{a-1} e^{\frac{-2(x-\theta)}{b}}}{\int_0^\infty (x-\theta)^c \frac{2^a}{\Gamma a b^a} (x-\theta)^{a-1} e^{\frac{-2(x-\theta)}{b}} dx}, 0 \leq x \leq \infty, x > \theta;\ a, b, c > 0 \tag{7}$$

where a is the shape parameter, b is the scale parameter, c is the power of the weighting function, θ is the threshold parameter, Γ is the gamma function and is used to model positively scaled data. The denominator in Equation (7) is

$$\int_0^\infty (x-\theta)^c \frac{2^a}{\Gamma a b^a} (x-\theta)^{a-1} e^{\frac{-2(x-\theta)}{b}} dx = \frac{2^a}{\Gamma a b^a} \int_0^\infty (x-\theta)^{a+c-1} e^{\frac{-2(x-\theta)}{b}} dx \tag{8}$$

Substituting,

$$= \frac{2^a}{\Gamma a b^a} \int_0^\infty \left(\frac{pb}{2}\right)^{a+c-1} e^{-p} . \frac{b}{2} dp = \frac{b^c}{\Gamma a 2^c} \int_0^\infty (p)^{a+c-1} e^{-p} dp \tag{9}$$

where

$$\int_0^\infty (p)^{a+c-1} e^{-p} dp = \Gamma(a+c) \tag{10}$$

$$= \frac{b^c}{\Gamma a 2^c} . \Gamma(a+c) \tag{11}$$

Putting Equation (11) into Equation (7) above

$$= \frac{2^{a+c}}{\Gamma(a+c)\, b^{a+c}} (x-\theta)^{a+c-1} e^{\frac{-2(x-\theta)}{b}};\ x > \theta; a, b, c > 0 \tag{12}$$

where a is the shape parameter, b is the scale parameter, c is the polynomial of the weighting function, θ is the threshold parameter, Γ is the gamma function and is used to model positively scaled data. Equation (12) is the proposed **PWEGD**.

2.3. The Properties of the PWEGD

This section explores the statistical properties of the PWEGD.

2.3.1. The cumulative distribution function of the PWEGD

The cumulative distribution function (CDF) of the PWEGD is given as

$$F(X) = p(X \leq x) = \int_0^x \frac{2^{a+c}}{\Gamma(a+c)\, b^{a+c}} (t-\theta)^{a+c-1}\, e^{\frac{-2(t-\theta)}{b}}\, dt \tag{13}$$

$$\therefore F(X) = \frac{\Gamma\left(\frac{2(t-\theta)}{b}, a+c\right)}{\Gamma(a+c)}; t > \theta \tag{14}$$

2.3.2. Survival function of the PWEGD

The survival function, (Sx), of the PWEGD is given as

$$s(x) = 1 - F(X) \tag{15}$$

$$= \frac{\Gamma(a+c) - \Gamma\left(\frac{2(t-\theta)}{b}, a+c\right)}{\Gamma(a+c)}; t > \theta \tag{16}$$

2.3.3. Hazard function of the PWEGD

The hazard function, $h(x)$, of the PWEGD is given as

$$h(x) = \frac{f(x)}{1-F(x)} = \frac{\frac{2^{a+c}}{\Gamma(a+c)b^{a+c}} (x-\theta)^{a+c-1}\, e^{\frac{-2(x-\theta)}{b}}}{\frac{\Gamma(a+c) - \Gamma\left(\frac{2(t-\theta)}{b}, a+c\right)}{\Gamma(a+c)}} \tag{17}$$

$$= \frac{2^{a+c}.\,(x-\theta)^{a+c-1}\, e^{\frac{-2(x-\theta)}{b}}}{b^{a+c}\left[\Gamma(a+c) - \Gamma\left(\frac{2(t-\theta)}{b}, a+c\right)\right]}; t > \theta \tag{18}$$

2.3.4. Moments of the PWEGD

The moment, μ_r' of the PWEGD is given as

$$\mu_r' = E\left[(x-\theta)^r\right] = \int_0^\infty (x-\theta)^r f(x, a, b, c, \theta)\, dx \tag{19}$$

$$= \frac{b^r.\Gamma(a+c+r)}{\Gamma(a+c)\, 2^r} \tag{20}$$

First moment $\left(\mu_1'\right)$, i.e., when r = 1

$$\mu_1' = Mean = E(x-\theta) = \frac{b.\Gamma(a+c+1)}{\Gamma(a+c).2} = \frac{b(a+c)}{2} \tag{21}$$

Second moment $\left(\mu_2'\right)$, i.e., when r = 2

$$\mu_2' = E\left[(x-\theta)^2\right] = \frac{b^2.\Gamma(a+c+2)}{\Gamma(a+c).2^2} = \frac{b^2(a+c+1)(a+c)}{2^2} \tag{22}$$

Variance of the PWEGD

$$Var(x-\theta)=\mu_2'-\mu_1'^2$$

$$=\frac{b^2(a+c+1)(a+c)}{2^2}-\left[\frac{b(a+c)}{2}\right]^2=\frac{b^2(a+c)}{2^2} \tag{23}$$

Third moment (μ_3'), i.e., when r = 3

$$\mu_3'=E[(x-\theta)^3]=\frac{b^3.\Gamma(a+c+3)}{\Gamma(a+c).2^3}=\left(\frac{b}{2}\right)^3(a+c+2)(a+c+1)(a+c) \tag{24}$$

Fourth moment (μ_4'), i.e., when r = 4

$$\mu_4'=E\left[(x-\theta)^4\right]=\frac{b^4.\Gamma(a+c+4)}{\Gamma(a+c).2^4}=\frac{b^4(a+c+3)(a+c+2)(a+c+1)(a+c)}{2^4} \tag{25}$$

2.4. Estimating the Parameters of the PWEGD

2.4.1. MLE of θ

The estimate of the parameter using the MLE is

$$L\left(\theta,a,b,c|x_i\right)=\prod_{i=1}^{n}f\left(x_i|a,b,c,\theta\right) \tag{26}$$

$$=\prod_{i=1}^{n}\frac{2^{a+c}}{\Gamma(a+c)\,b^{a+c}}(x-\theta)^{a+c-1}e^{\frac{-2(x-\theta)}{b}} \tag{27}$$

$$=\frac{2^{n(a+c)}}{(\Gamma(a+c))^n\,b^{n(a+c)}}\sum\nolimits_i^n(x_i-\theta)^{a+c-1}e^{\frac{-2}{b}\sum_{i=1}^{n}(x-\theta)} \tag{28}$$

Taking the log likelihood function, we have

$$\log\left(L\left(\theta,a,b,c|X_i\right)\right)=n(a+c)\log 2-n(a+c)\log b-n\,\log(\Gamma(a+c))$$
$$+(a+c-1)\log\sum\nolimits_i^n(x_i-\theta)-\frac{2}{b}\sum\nolimits_{i=1}^n(x-\theta) \tag{29}$$

differentiating with respect to θ and equating to zero

$$\hat{\theta}=\bar{x}-\frac{1}{n}\left[\tfrac{b}{2}(a+c-1)\right]^{\frac{1}{2}} \tag{30}$$

$$\hat{b}=\frac{2(\bar{x}-\theta)}{(a+c-1)} \tag{31}$$

2.4.2. Fisher's information of the PWEGD

The Fishers information (I_n) of θ, the parameter of the PWEGD is

$$I_n=-E\left[\frac{\partial^2\log\left[L\left(\theta,a,b,c|x_i\right)\right]}{\partial\theta^2}\right]=-E\left[\frac{\partial^2}{\partial\theta^2}\left[\frac{(a+c-1)}{n(\bar{x}-\theta)}-\frac{2n(\bar{x}-\theta)}{b}\right]\right] \tag{32}$$

$$\therefore I_n=0.5nb(a+c+1)(a+c)-\left[\frac{2}{b}\right]^2\frac{(a+c-1)}{n(a+c+1)(a+c)} \tag{33}$$

$$SE\left(\hat{\theta}\right)=\left[0.5nb(a+c+1)(a+c)-\left[\frac{2}{b}\right]^2\frac{(a+c-1)}{n(a+c+1)(a+c)}\right]^{-1/2} \tag{34}$$

2.5. Inferences about θ

2.5.1. The confidence interval of θ

The $(1-\alpha)$ 100% confidence interval of θ is

$$\hat{\theta} \pm Z_{\alpha/2}.SE\left(\hat{\theta}\right) \tag{35}$$

$$= \left[\bar{x} - \frac{1}{n}\left[\frac{b}{2}(a+c-1)\right]^{\frac{1}{2}}\right] \pm Z_{\alpha/2}\left[0.5nb(a+c+1)(a+c) - \left[\frac{2}{b}\right]^2 \frac{(a+c-1)}{n(a+c+1)(a+c)}\right]^{-1/2} \tag{36}$$

2.5.2. Hypothesis testing about θ

The student's t –test for θ is

$$t = \frac{\hat{\theta} - \theta_0}{SE\left(\hat{\theta}\right)} \tag{37}$$

2.6. Statistical Properties of the PWEGD

2.6.1. Skewness of the PWEGD

The skewness of the PWEGD is

$$\text{Skewness} = \frac{2(a+c+1)\left[(a+c)^2 + 1 - (a+c)\right]}{(a+c)^{\frac{1}{2}}} \tag{38}$$

2.6.2. Kurtosis of the PWEGD

The kurtosis of the PWEGD is

$$\text{Kurtosis} = \frac{3(a+c+2)}{(a+c)} \tag{39}$$

2.6.3. Mode of the PWEGD

The mode of the PWEGD is obtained when

$$f'(x) = \frac{d[f(x)]}{dx} = U\frac{dV}{dx} + V\frac{dU}{dx} = 0 \tag{40}$$

The mode (x_{mode}) of the distribution is at the point where

$$\therefore x_{mode} = \theta + \frac{b(a+c-1)}{2} \tag{41}$$

2.6.4. Quartiles of the PWEGD

First quartile (Q_1) of the PWEGD

The first quartile (q_1) is obtained at the point where $F\left(q_1\right) = \frac{1}{4}$ and that point is obtained as

$$x_{q_1} = \theta + \frac{b}{8} \tag{42}$$

Median of the PWEGD

The median is obtained at the point where $F(m) = \frac{1}{2}$ and is obtained as

$$x_{median} = \theta + \frac{b}{4} \tag{43}$$

Third quartile (Q_3) of the PWEGD

The third quartile (q_3) is obtained at the point where $F\left(q_3\right) = \frac{3}{4}$, and is obtained as

$$x_{q_3} = \theta + \frac{3b}{8} \tag{44}$$

3. EMPIRICAL RESULTS AND DISCUSSION

3.1. Data

The proposed PWEGD is applied to MC simulated data and the wind direction (degrees) in Lagos, Nigeria, obtained from the Center for Atmospheric Research (CAR) of the Center of the Nigerian National Space Research and Development Agency, NASRDA. The data collected for this study is analyzed using R software. The codes were written by the researchers and tested with the collected data.

Figures 1(a)–1(d) to 3 and 4(a)–4(d) and 5 shows the individual plots of the distribution functions, the CDF, the survival function and the hazard rate function, panel plots at different theta values of the MC simulated data and the Wind direction (degrees) at different values of c respectively.

The plot of the pdf of the PWEGD of the MC simulated data in Figure 1(a)–1(d) shows that it is a pdf and the plot tends to symmetry as the value of c increase. This is also corroborated by the CDF plot. The plot Figure 2 shows that it is a pdf and the plot tends to symmetry as the value of c increase. This is also corroborated by the CDF plot. Also revealed is the fact that with higher threshold value (θ), the symmetry is better.

The plot of the pdf of the PWEGD of the MC simulated data in Figure 3 shows that it is a pdf and the plot tends to symmetry as the value of c increase.

The plot of the pdf of the PWEGD of the wind direction (degrees) in Lagos, Nigeria, in Figure 4(a)–4(d) shows that it is a pdf and the plot tends to symmetry as the value of c increase. This is also corroborated by the CDF plot. The plot Figure 5 shows that it is a pdf and the plot tends to symmetry as the value of c increase. This is also corroborated by the CDF plot. Also revealed is the fact that with higher threshold value (θ), the symmetry is better.

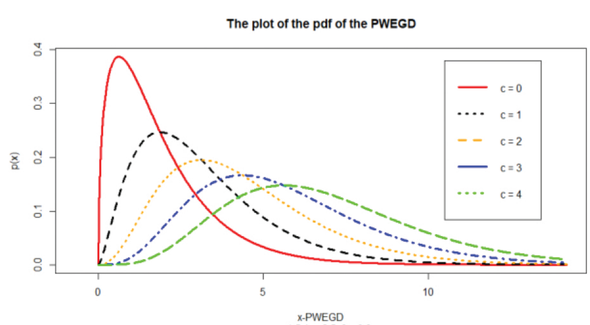

Figure 1a Plot of the pdf of the poly-weighted exponentiated gamma distribution (Probability Density Function) (n = 1491).

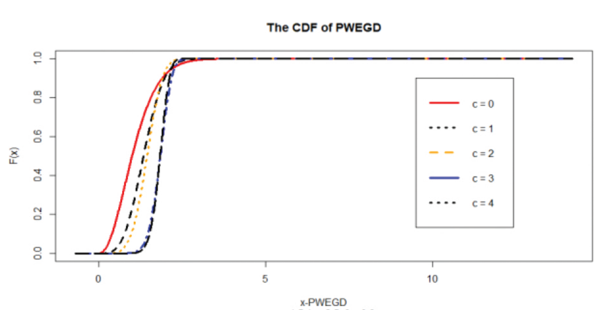

Figure 1b Plot of the cumulative distribution function (CDF) of the poly-weighted exponentiated gamma distribution (PWEGD) (n = 1491).

The plot of the pdf of the PWEGD of the wind direction in Lagos Nigeria in Figure 6 shows that it is a pdf and the plot tends to symmetry as the value of c increase.

3.2. Statistical Properties from the Data

This section presents the results of the statistical properties of the PWEGD obtained using MC simulated data and Wind direction (degrees) in Lagos, Nigeria.

Table 1 shows the summary of the probabilities obtained for the PWEGD using the Monte Carlo simulated data and the wind direction (degrees) in Lagos, Nigeria, at various values of polynomial parameter (c) compared with the base distribution (EGD at $c = 0$). The expected

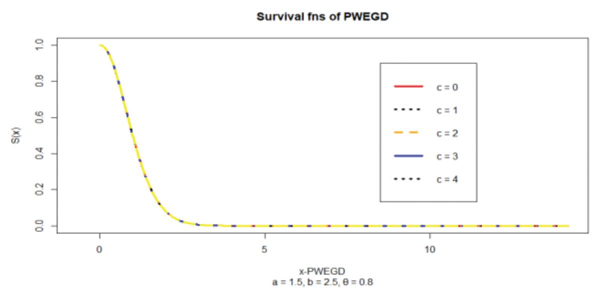

Figure 1c Plot of the survival fns - function of the poly-weighted exponentiated gamma distribution (PWEGD) (n = 1491).

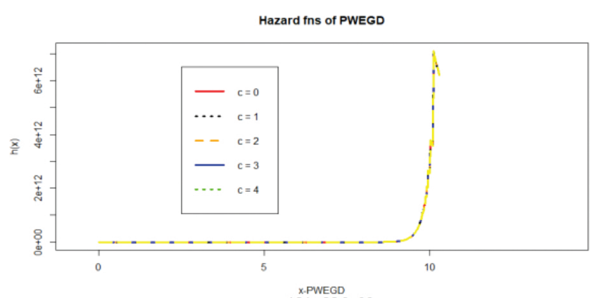

Figure 1d Plot of the hazard fns of the poly-weighted exponentiated gamma distribution (PWEGD) (n = 1491).

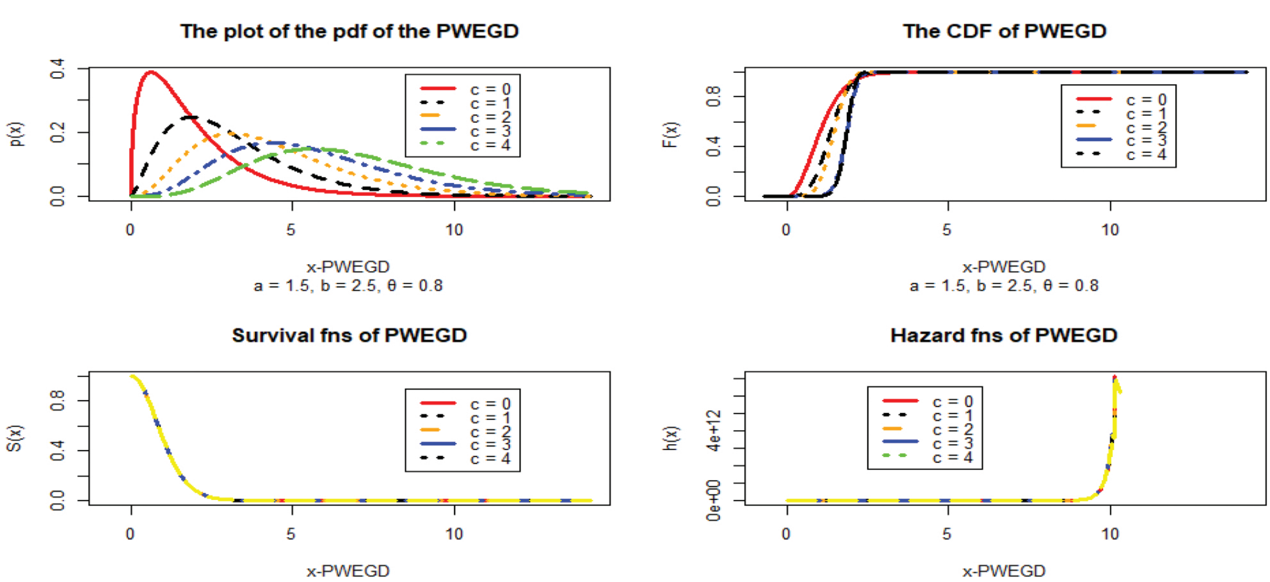

Figure 2 Panel plot of the poly-weighted exponentiated gamma distribution (n = 1491, a = 1.5, b = 2.5, θ = 0.8).

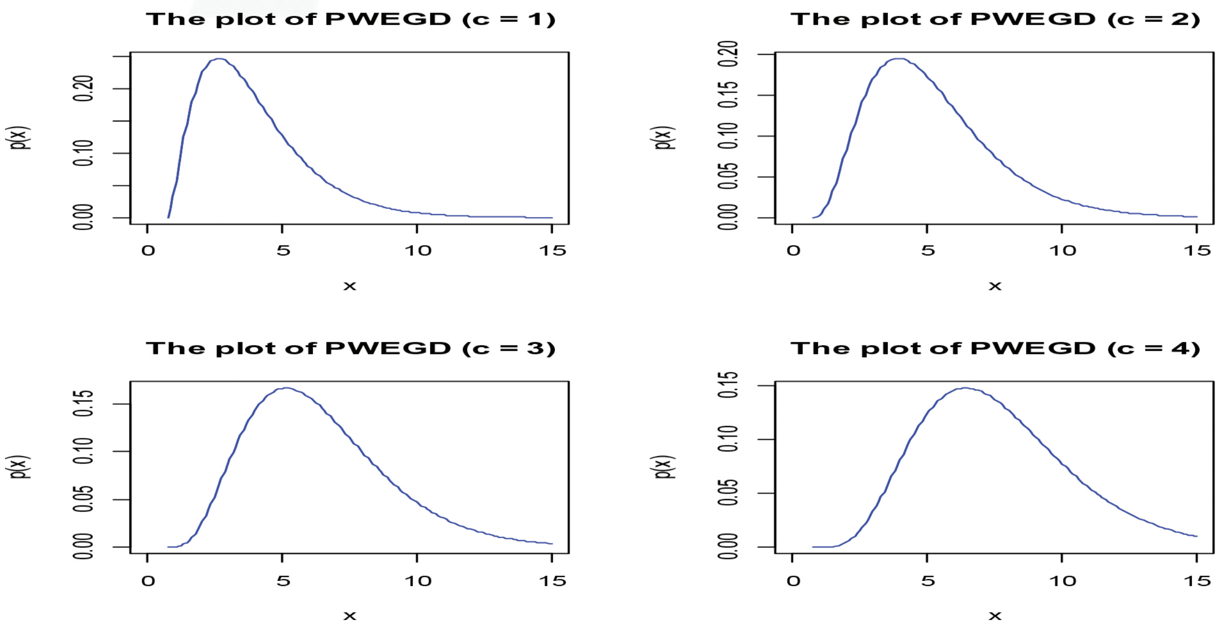

Figure 3 | Panel plot of the pdf of the poly-weighted exponentiated gamma distribution ($n = 1491$, $a = 1.5$, $b = 2.5$, $\theta = 0.8$).

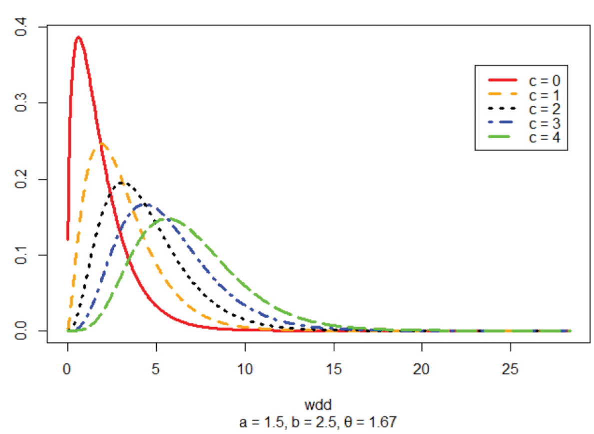

Figure 4a | Plot of the pdf of the wind direction (degrees).

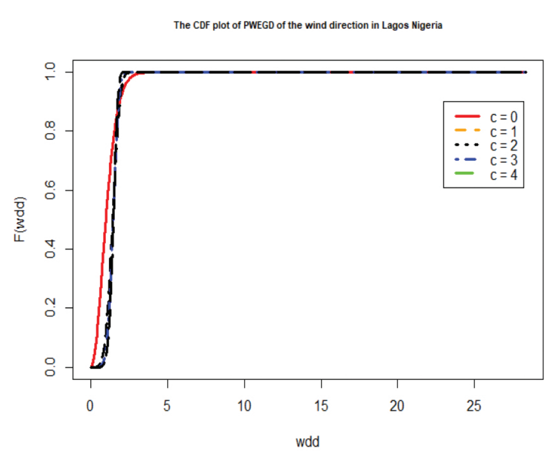

Figure 4b | Plot of the cumulative distribution function (CDF) of the wind direction (degrees).

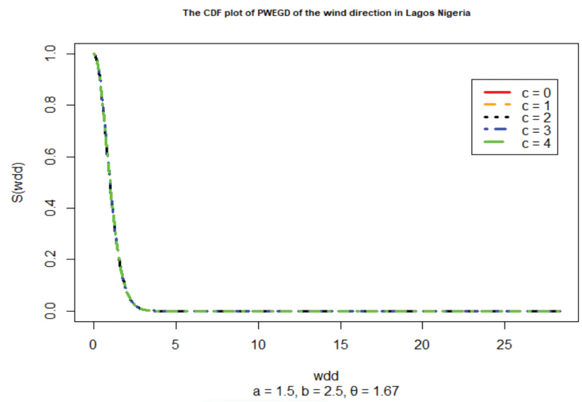

Figure 4c Plot of the survival fns of the wind direction (degrees).

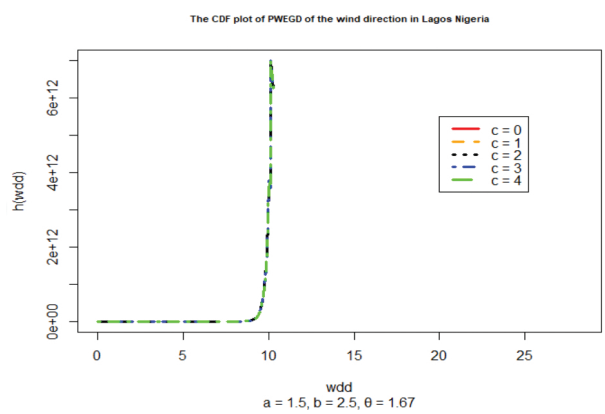

Figure 4d Plot of the hazard fns of the wind direction (degrees).

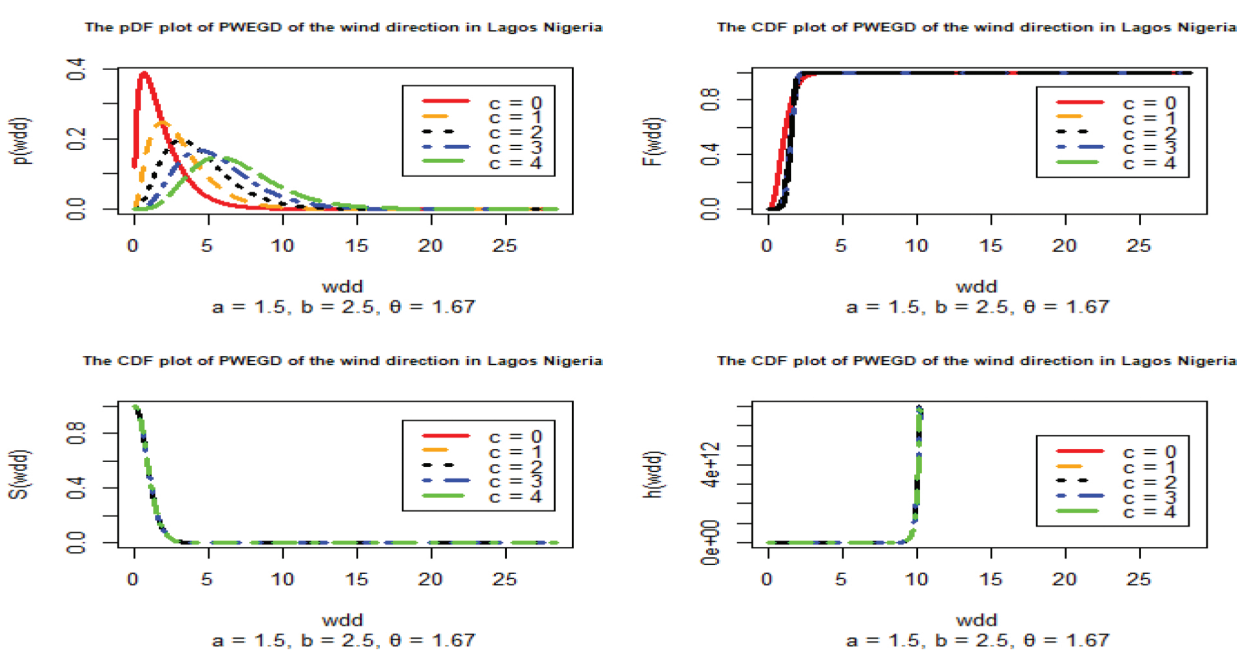

Figure 5 Panel plot of the poly-weighted exponentiated gamma distribution ($n = 2831$, $a = 1.5$, $b = 2.5$, $\theta = 1.67$).

value of the distribution increased slightly as c increase, while the coefficients of skewness and kurtosis tends to zero (0) as c increase also. Also revealed is the fact that as c increased, the mean and median approached the same value required for symmetry in a distribution.

Table 2 shows the summary of the statistical properties obtained for the PWEGD using the Monte Carlo simulated data and the wind direction (degrees) in Lagos, Nigeria, at various values of the polynomial parameter (c) compared with the base distribution—the EGD (column 1 where $c = 0$). The mean value of the distribution increased as c increase, while the coefficients of skewness and kurtosis tend to zero (0) as c increase also. However, the result of the AIC, BIC and the AICc were compared and while they increased from $c = 0$ to higher values of c, it actually reveals that at higher values of c had lower AIC, BIC and AICc for lower shape and scale parameters for the wind direction (degrees) in Lagos, Nigeria.

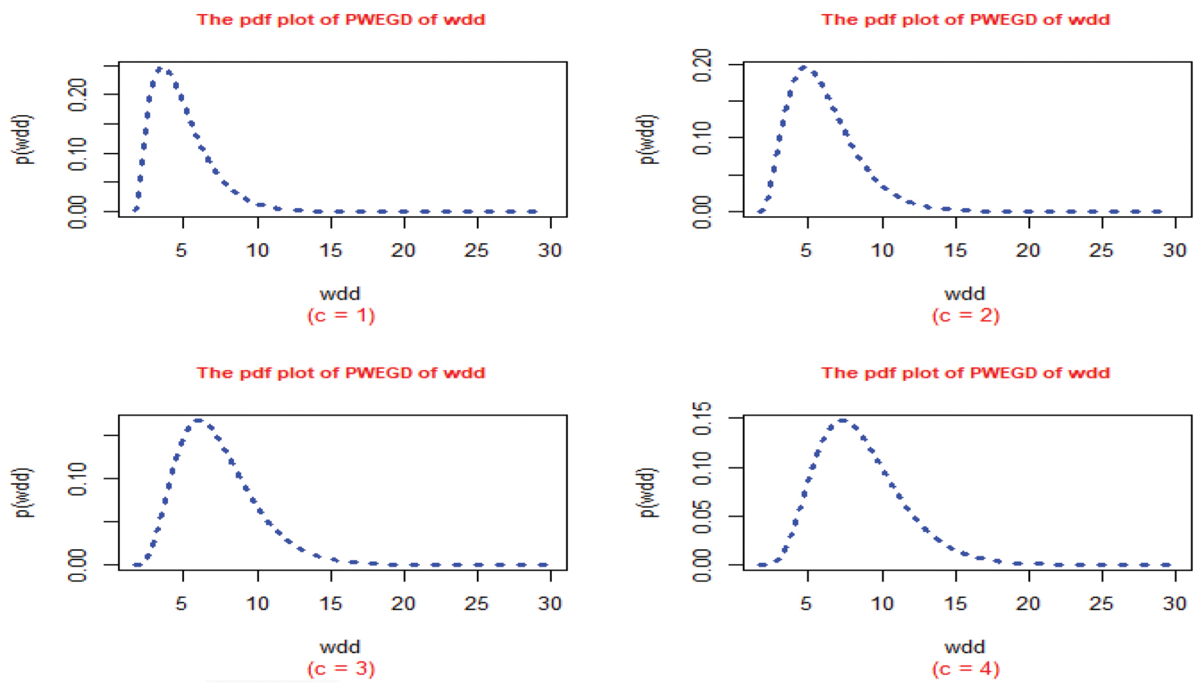

Figure 6 Panel plot of the pdf of the poly-weighted exponentiated gamma distribution of the wind direction in Lagos, Nigeria ($n = 2831$, $a = 1.5$, $b = 2.5$, $\theta = 1.67$).

Table 1 Estimates of the probabilities of the PWEGD.

Data Used **P-WEGD**	**Statistics ($p(x)$)**	**EGD**				
	***c* = 0**	***c* = 1**	***c* = 2**	***c* = 3**	***c* = 4**	
Monte Carlo simulated data ($n = 1491$, $a = 1.5$, $b = 2.5$, $\theta = 0.8$)	Minimum	0	0.0002	0	0	0
	1st quartile	0.0005	0.0087	0.0301	0.0415	0.0399
	Median	0.0073	0.0466	0.0818	0.0885	0.1003
	Mean	0.0704	0.0840	0.0924	0.0895	0.0872
	3rd quartile	0.0877	0.1565	0.1561	0.1418	0.1340
	Maximum	0.3871	0.2467	0.1953	0.1666	0.1477
Wind direction (degrees) ($n = 2831$, $a = 2.5$, $b = 3.5$, $\theta = 1.67$)	Minimum	0	0.0000	0.0000	0.0000	0.0000
	1st quartile	0.0001	0.0003	0.0008	0.0014	0.0025
	Median	0.0021	0.0043	0.0063	0.0093	0.0122
	Mean	0.0416	0.0357	0.0318	0.0290	0.0272
	3rd quartile	0.0755	0.0687	0.0673	0.0473	0.0438
	Maximum	0.1762	0.1395	0.1190	0.1055	0.0957
Wind direction (degrees) ($n = 2831$, $a = 1.5$, $b = 2.5$, $\theta = 1.67$)	Minimum	0	0	0	0	0
	1st quartile	0	0	0	0	0.0001
	Median	0	0.0002	0.0007	0.0013	0.0020
	Mean	0.0594	0.0469	0.0402	0.0364	0.0334
	3rd quartile	0.0370	0.0700	0.0545	0.0618	0.0676
	Maximum	0.3871	0.2467	0.1953	0.1666	0.1477

P-WEGD, Poly-Weighted Exponentiated Gamma Distribution; EGD, Exponentiated Gamma Distribution.

Table 2 Statistical properties of the estimates ($x - \theta$).

Data Used Poly-Weighted EGD	**Statistics Properties**	**EGD**				
		$c = 0$	$c = 1$	$c = 2$	$c = 3$	$c = 4$
Monte Carlo simulated data ($n = 1491$, $a = 1.5$, $b = 2.5$, $\theta = 0.8$)	Mean	1.875	3.125	4.375	5.625	6.875
	Variance	2.661	26.043	282.8989	3321.503	46174.18
	Std error	0.0422	0.1321	0.4256	1.4925	5.5649
	Skewness	1.6701	0.3589	0.0322	0.0016	5.27e-05
	Kurtosis	3.780	0.4123	0.0151	0.0003	2.60e-06
	Mode	1.2495	1.2491	1.2488	1.2486	1.2484
	Median	2.9495	3.1244	4.3741	5.8023	7.0124
	Q1	2.4120	3.1242	4.3739	5.7130	6.9429
	Q3	3.4870	3.1245	4.3742	5.8917	7.0820
	AIC	16739.53	1860.002	2187.992	2616.924	3121.249
	BIC	16760.76	1872.045	2200.034	2628.966	3133.292
	AICC	16739.56	1860.278	2188.268	2617.199	3121.525
Wind direction (degrees) ($n = 2831$, $a = 2.5$, $b = 3.5$, $\theta = 1.67$)	Mean	4.375	6.125	7.875	9.625	11.375
	Variance	51.0447	554.4819	6510.161	90501.39	1492577
	Std error	0.1343	0.4425	1.5164	5.6540	22.9614
	Skewness	0.3589	0.0322	0.0016	5.2699e-05	1.218e-06
	Kurtosis	0.4123	0.0151	0.0003	2.6044e-06	1.657e-08
	Mode	1.7494	1.7493	1.7491	1.7490	1.7489
	Median	5.2761	7.0153	8.7605	10.5079	12.2562
	Q1	4.8253	6.5698	8.3173	10.0659	11.8150
	Q3	5.7269	7.4608	9.2038	10.9498	12.6973
	AIC	53.562.82	61898.20	72138.85	83802.37	96602.08
	BIC	53586.62	61922.08	72162.65	83826.16	96625.87
	AICC	53562.84	61898.30	72138.87	83802.38	96602.09
Wind direction (degrees) ($n = 2831$, $a = 1.5$, $b = 2.5$, $\theta = 1.67$)	Mean	1.875	3.125	4.375	5.625	6.875
	Variance	2.6607	26.0432	282.8989	3321.503	46174.18
	Std error	0.0306	0.0959	0.3161	1.0832	4.0386
	Skewness	1.6701	0.3589	0.0322	0.0016	5.270e-05
	Kurtosis	3.780	0.4123	0.0151	0.0003	2.604e-06
	Mode	1.2497	1.2495	1.2494	1.2493	1.2492
	Median	2.0797	3.6095	4.9154	6.1893	7.4525
	Q1	1.9772	3.3670	4.6449	5.9068	7.1633
	Q3	2.1822	3.8520	5.1859	6.4718	7.7417
	AIC	62883.30	66421.36	72851.72	81187.18	90945.59
	BIC	62907.09	66445.16	72875.51	81210.98	90969.38
	AICC	62883.31	66421.38	72851.74	81187.20	90945.60

AIC, Akaike Information Criteria; BIC, Bayesian Information Criteria; AICC, Corrected Akaike Information Criteria; EGD, Exponentiated Gamma Distribution.

3.3. INFERENCES

The parameters θ and b were estimated using the maximum likelihood methods, the 95% confidence interval of θ was also obtained and tested for significance at 5% level of significance.

Table 3 shows the MLE inferences of the parameters of PWEGD using MC simulated data compared also with the base distribution (the EGD, $c = 0$). The value of θ estimated increase with increased in c while the MSE of the estimate decreased with increase in c. The 95% confidence interval of θ was also obtained and the t-test show that the parameter θ is significant from zero (0). However, the value of b estimated decreased with the increase in c.

Also, shown is the MLE inferences of the parameters of PWEGD of the wind direction (degrees) in Lagos, Nigeria. The value of θ estimated increased with increase in c while the MSE of the estimate decreased with increase in c and was not estimable when $c = 0$. The 95% confidence interval of θ obtained and the t-test shows that the parameter θ is significant from zero (0).

3.4. DISCUSSION

The results agreed with existing literatures that the MSE of the parameters decrease with increased value of the poly parameter [20,21]. The decrease in the MSE also agrees with the submission of Scott [22] that increasing information (parameters) of the distribution reduces the uncertainty in the distribution. The results also show that the proposed distribution is more flexible than its other counterparts and will be appropriate for modelling real-life failure data with thresholds [23]. However, its suppleness is at an increased cost of intricacy in the computations but with available programming applications, this complexity is minimized. The results also show that the model performs

Table 3 Inferences of the MLE estimates of the parameters of PWEGD.

Data Used	Parameter	Statistics	EGD	PWEGD			
			***c* = 0**	***c* = 1**	***c* = 2**	***c* = 3**	***c* = 4**
MC simulated data (n = 1491, a = 1.5, b = 2.5, θ = 0.8)		Value (MSE)	1.874 (0.0267)	3.12 (0.0014)	4.3674 (0.0001)	5.624 (3.4e-05)	6.873 (2.167e-05)
	Θ	CI (LCL, UCL)	(1.822, 1.927)	(3.10, 3.15)	(4.358, 4.389)	(5.612 5.635)	(6.864, 6.882)
		t (p-value)	70.081 (<0.0001)	261.175 (< 0.001)	558.545 (<0.001)	963.492 (<0.001)	1476.229 (<0.001)
	B	–	4.3000	0.0012	0.0009	0.7150	0.5562
Wind direction (degrees) (n = 2831, a = 2.5, b = 3.5, θ = 1.67)		Value (MSE)	4.3744 (7.34e-03)	6.1243 (4.80e-03)	7.8741 (3.58e-03)	9.6240 (2.85e-03)	11.3739 (2.376e-03)
	Θ	CI (LCL, (UCL)	(4.360, 4.389)	(6.115, 6.134)	(7.867, 7.881)	(9.6184, 9.6296)	(11.369, 11.378)
		t (p-value)	596.246 (<0.001)	1275.107 (<0.001)	2199.538 (<0.001)	3370.021 (<0.001)	4786.704 (<0.001)
	B	–	3.607	3.5640	3.5457	3.5355	3.5291
Wind direction (degrees) (n = 2831, a = 1.5, b = 2.5, θ = 1.67)		Value (MSE)	1.8747 (0.0194)	3.124 (8.68e-03)	4.3744 (0.0057)	5.6243 (4.23e-03)	6.8742 (0.0034)
	Θ	CI (LCL, (UCL)	(1.837, 1.913)	(3.107, 3.141)	(4.363, 4.385)	(5.616, 5.632)	(6.867, 6.881)
		t (p-value)	96.5811 (<0.001)	359.934 (<0.001)	769.742 (<0.001)	1327.794 (<0.001)	2034.381 (<0.001)
	B	–	0.8200	1.9400	2.1640	2.2600)	2.3133

CI, Confidence Interval; LCL, Lower Confidence Limit; UCL, Upper Confidence Limit; MSE, Mean Square Error; PWEGD, Poly - Weighted Exponentiated Gamma Distribution; EGD, Exponentiated Gamma Distribution; MLE, Maximum Likelihood Equation.

better at higher values of c with lower scale (b) and shape (a) parameters concurring with Munkhammar *et al.* [24] that polynomializing are useful approximations. Hence, this study has shown that weighting or biasing the EGD with a polynomial parameter is more flexible than lower values and that the proposed distribution is applicable in modelling real-life data in production, manufacturing, engineering, sciences and in finances.

4. CONCLUSIONS

The parameters of the fitted PWEGD like the threshold (θ) and the shape parameter (b) were estimated using maximum likelihood method; exhibiting all the features of a real distribution function. The distribution approached symmetry as the poly value (c) increased. In addition, the constraints of the distribution were estimated and θ showed increase with increasing c, its mean square error (MSE) decreased with increasing c and the parameter was significant. Furthermore, the estimate of b (the shape parameter) decreased with increasing c. The statistical properties of the fitted PWEGD like its expected value, variance, standard error, median, mode, hazard and survival functions, CDF, moments, skewness and kurtosis were obtained while the properties estimated applied to the MC simulated data and the wind direction (degrees) in Lagos, Nigeria, was equally tested. Furthermore, the study recommends that the proposed PWEGD with $c \geq 3$ could be explored with lower shape (b) and scale (a) parameters in attempts to deal with real-life conditions in finance, manufacturing, biological sciences, medical statistics, environmental studies, etc.

CONFLICTS OF INTEREST

The authors declare no conflict of interest.

AUTHORS' CONTRIBUTIONS

EMI conceived, designed and developed the study, EBN supervised the study, EMI analysed the data and drafted the manuscript while both authors reviewed and approved the manuscript.

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