

Research Article

Multi-folded \mathcal{N} -Structures with Finite Degree and its Application in BCH- Algebras

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ABSTRACT

The generalization of \mathcal{N} -structure is introduced first and then applied to BCH-algebra for research. The concepts of k -folded \mathcal{N} -subalgebra, k -folded \mathcal{N} -closed ideal and (closed) k -folded \mathcal{N} -filter are introduced, and then their relations and several properties are investigated. Conditions for the k -folded \mathcal{N} -subalgebra to be k -folded \mathcal{N} -closed ideal are provided. Characterization of k -folded \mathcal{N} -subalgebra, k -folded \mathcal{N} -closed ideal and (closed) k -folded \mathcal{N} -filter are considered by using the notion of k -folded level sets. A k -folded \mathcal{N} -subalgebra and a k -folded \mathcal{N} -closed ideal are constructed by using the medial part and BCA-part. A k -folded \mathcal{N} -filter is made by using the branch of a BCH-algebra. Conditions for a k -folded \mathcal{N} -closed ideal to be a closed k -folded \mathcal{N} -filter are discussed.

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1. INTRODUCTION

As a generation of BCK/BCI-algebras, Hu and Li introduced the notion of BCH-algebras (see [1,2]), and it is classified by Ahmad (see [3]). Decompositions of BCH-algebras are discussed by Dudek and Thomys (see [4]). Ideals and filters of BCH-algebras are studied by Chaudhry *et al.* (see [5,6]). As a generalization of crisp sets, it is well known that fuzzy sets are widely used in various academic fields. Various generalizations of fuzzy sets have been carried out by many scholars and are being applied in various ways. For example (intuitionistic), fuzzy set theory based on fuzzy points (see [7,8]), bipolar fuzzy set theory based on bipolar fuzzy points (see [9,10]), generalization of intuitionistic fuzzy set theory based on 3-valued logic (see [11,12]) and cubic set theory (see [13–15]). Given that the fuzzy set deals primarily with positive information, we feel that we need tools to deal with negative information. If positive information represents the information of the present world, it may be thought that negative information represents the afterlife. As a tool for dealing with information from the afterlife, Jun *et al.* introduced the so-called \mathcal{N} -structure (see [16]) and applied it to the algebraic structure (see [16–20]).

In this paper, as a generalization of \mathcal{N} -structure, we introduce the multi-folded \mathcal{N} -structure with finite degree and applied it to BCH-algebras. We introduce the notions of k -folded \mathcal{N} -subalgebra, k -folded \mathcal{N} -closed ideal and (closed) k -folded \mathcal{N} -filter, and then we investigate their relations and several properties. We provide conditions for the k -folded \mathcal{N} -subalgebra to be k -folded \mathcal{N} -closed

ideal. Using the notion of k -folded level sets, we consider characterization of k -folded \mathcal{N} -subalgebra, k -folded \mathcal{N} -closed ideal and (closed) k -folded \mathcal{N} -filter. Using the medial part and BCA-part, we make a k -folded \mathcal{N} -subalgebra and a k -folded \mathcal{N} -closed ideal. Using the branch of a BCH-algebra, we make a k -folded \mathcal{N} -filter. We provide conditions for a k -folded \mathcal{N} -closed ideal to be a closed k -folded \mathcal{N} -filter.

2. PRELIMINARIES

An algebra $(X, *, 0)$ is called a *BCH-algebra* (see [1]) if it satisfies the following assertions. If a set X has a special element 0 and a binary operation $*$ satisfying the conditions:

- I. $(\forall u \in X) (u * u = 0)$,
- II. $(\forall u, v \in X) (u * v = 0, v * u = 0 \Rightarrow u = v)$,
- III. $(\forall u, v, w \in X) ((u * v) * w = (u * w) * v)$.

Any BCH-algebra X satisfies the following conditions (see [1,4]):

$$(\forall u \in X) (u * 0 = u), \quad (1)$$

$$(\forall u \in X) (u * 0 = 0 \Rightarrow u = 0), \quad (2)$$

$$(\forall u, v \in X) (0 * (u * v) = (0 * u) * (0 * v)), \quad (3)$$

$$(\forall u \in X) (0 * (0 * (0 * u)) = 0 * u). \quad (4)$$

$$(\forall u, v \in X) (u * v = 0 \Rightarrow 0 * u = 0 * v). \quad (5)$$

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A subset S of a BCH-algebra X is called a *subalgebra* of X if $u * v \in S$ for all $u, v \in S$. A subset I of a BCH-algebra X is called a *closed ideal* of X (see [5]) if it satisfies

$$(\forall u \in X)(u \in I \Rightarrow 0 * u \in I), \quad (6)$$

$$(\forall u, v \in X)(u * v \in I, v \in I \Rightarrow u \in I). \quad (7)$$

Note that every closed ideal is a subalgebra, but the converse is not valid (see [5]).

A subset F of a BCH-algebra X is called a *filter* of X (see [6]) if it satisfies

$$(\forall u, v \in X)(u \in F, u * v = 0 \Rightarrow v \in F), \quad (8)$$

$$(\forall u, v \in X)(u \in F, v \in F \Rightarrow u * (u * v) \in F, v * (v * u) \in F). \quad (9)$$

A filter F of a BCH-algebra X is said to be *closed* (see [6]) if $0 * u \in F$ for all $u \in F$.

Denote by $\mathcal{F}(X, [-1, 0])$ the collection of functions from a set X to $[-1, 0]$. We say that an element of $\mathcal{F}(X, [-1, 0])$ is a *negative-valued function* from X to $[-1, 0]$ (briefly, \mathcal{N} -function on X .) By an \mathcal{N} -structure we mean an ordered pair (X, φ) of X and an \mathcal{N} -function φ on X (see [16]).

3. k -FOLDED \mathcal{N} -IDEALS/SUBALGEBRAS

In what follows, let k be a natural number unless otherwise specified and $[-1, 0]^k$ denote the k -Cartesian product of $[-1, 0]$, that is,

$$[-1, 0]^k = [-1, 0] \times [-1, 0] \times \cdots \times [-1, 0]$$

in which $[-1, 0]$ is repeated k times.

We give orders \lesssim and \gtrsim on $[-1, 0]^k$ as follows:

$$\tilde{t} \lesssim \tilde{s} \Leftrightarrow t_i \leq s_i,$$

$$\tilde{t} \gtrsim \tilde{s} \Leftrightarrow t_i \geq s_i,$$

respectively, for $i = 1, 2, \dots, k$ where $\tilde{t} := (t_1, t_2, \dots, t_k) \in [-1, 0]^k$ and $\tilde{s} := (s_1, s_2, \dots, s_k) \in [-1, 0]^k$.

We define

$$\text{Max}\{\tilde{t}, \tilde{s}\} = (\max\{t_1, s_1\}, \max\{t_2, s_2\}, \dots, \max\{t_k, s_k\}),$$

$$\text{Min}\{\tilde{t}, \tilde{s}\} = (\min\{t_1, s_1\}, \min\{t_2, s_2\}, \dots, \min\{t_k, s_k\}).$$

Definition 3.1. A *multi-folded \mathcal{N} -structure with finite degree k* (briefly, *k -folded \mathcal{N} -structure*) over a universe X is defined to be a pair (\tilde{f}, X) where $\tilde{f} : X \rightarrow [-1, 0]^k$ is a mapping.

For any $x \in X$, the membership value of x is denoted by

$$\tilde{f}(x) = ((\ell_1 \circ \tilde{f})(x), (\ell_2 \circ \tilde{f})(x), \dots, (\ell_k \circ \tilde{f})(x)),$$

where $\ell_i : [-1, 0]^k \rightarrow [-1, 0]$ is the i -th projection for $i = 1, 2, \dots, k$, that is, $\ell_i(\tilde{t}) = t_i$ where $\tilde{t} := (t_1, t_2, \dots, t_k) \in [-1, 0]^k$.

Given a k -folded \mathcal{N} -structure (\tilde{f}, X) over a universe X , we consider the set

$$L(\tilde{f}; \tilde{t}) := \{x \in X \mid \tilde{f}(x) \lesssim \tilde{t}\}, \quad (10)$$

that is,

$$\begin{aligned} L(\tilde{f}; \tilde{t}) &:= \{x \in X \mid (\ell_i \circ \tilde{f})(x) \leq t_i, i = 1, 2, \dots, k\} \\ &= \bigcap_{i=1}^k L(\tilde{f}; \tilde{t})^i, \end{aligned}$$

which is called a *k -folded level set* of (\tilde{f}, X) related to \tilde{t} , where

$$L(\tilde{f}; \tilde{t})^i := \{x \in X \mid (\ell_i \circ \tilde{f})(x) \leq t_i\}$$

for $i = 1, 2, \dots, k$.

Definition 3.2. Let X be a BCH-algebra. A k -folded \mathcal{N} -structure (\tilde{f}, X) over X is called a *k -folded \mathcal{N} -subalgebra* of X if it satisfies

$$(\forall x, y \in X) (\tilde{f}(x * y) \lesssim \text{Max}\{\tilde{f}(x), \tilde{f}(y)\}), \quad (11)$$

that is,

$$(\forall x, y \in X) ((\ell_i \circ \tilde{f})(x * y) \leq \max\{(\ell_i \circ \tilde{f})(x), (\ell_i \circ \tilde{f})(y)\}) \quad (12)$$

for $i = 1, 2, \dots, k$.

Example 3.3. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the binary operation “ $*$ ” which is given in Table 1.

Then $(X, *, 0)$ is a BCH-algebra (see [5]). Let (\tilde{f}, X) be a 3-folded \mathcal{N} -structure over X given by

$$\tilde{f} : X \rightarrow [-1, 0]^3, x \mapsto \begin{cases} (-0.82, -0.45, -0.66) & \text{if } x = 0, \\ (-0.33, -0.25, -0.44) & \text{if } x = 1, \\ (-0.33, -0.25, -0.44) & \text{if } x = 2, \\ (-0.33, -0.25, -0.44) & \text{if } x = 3, \\ (-0.82, -0.45, -0.66) & \text{if } x = 4. \end{cases}$$

It is routine to verify that (\tilde{f}, X) is a 3-folded \mathcal{N} -subalgebra of X .

Proposition 3.4. Every k -folded \mathcal{N} -subalgebra (\tilde{f}, X) of X satisfies the following inequality

$$(\forall x \in X) (\tilde{f}(0 * x) \lesssim \tilde{f}(x)), \quad (13)$$

that is, $(\ell_i \circ \tilde{f})(0 * x) \leq (\ell_i \circ \tilde{f})(x)$ for all $x \in X$ and $i = 1, 2, \dots, k$.

Proof. For any $x \in X$ and $i = 1, 2, \dots, k$, we have

$$\begin{aligned} (\ell_i \circ \tilde{f})(x) &= \max\{\max\{(\ell_i \circ \tilde{f})(x), (\ell_i \circ \tilde{f})(x)\}, (\ell_i \circ \tilde{f})(x)\} \\ &= \max\{(\ell_i \circ \tilde{f})(x * x), (\ell_i \circ \tilde{f})(x)\} \\ &= \max\{(\ell_i \circ \tilde{f})(0), (\ell_i \circ \tilde{f})(x)\} \\ &\geq (\ell_i \circ \tilde{f})(0 * x), \end{aligned}$$

that is, $\tilde{f}(0 * x) \lesssim \tilde{f}(x)$ for all $x \in X$. \square

Table 1 Cayley table for the binary operation “ $*$ ”.

*	0	1	2	3	4
0	0	0	0	0	4
1	1	0	0	1	4
2	2	2	0	0	4
3	3	3	3	0	4
4	4	4	4	4	0

Definition 3.5. Let X be a BCH-algebra. A k -folded \mathcal{N} -structure (\tilde{f}, X) over X is called a k -folded \mathcal{N} -closed ideal of X if it satisfies

$$(\forall x, y \in X) (\tilde{f}(0 * x) \lesssim \tilde{f}(x) \lesssim \text{Max}\{\tilde{f}(x * y), \tilde{f}(y)\}), \quad (14)$$

that is,

$$(\forall x, y \in X) ((\ell_i \circ \tilde{f})(0 * x) \leq (\ell_i \circ \tilde{f})(x) \leq \max(\ell_i \circ \tilde{f}(x * y), (\ell_i \circ \tilde{f})(y))) \quad (15)$$

for $i = 1, 2, \dots, k$.

Example 3.6. (1) Consider the BCH-algebra $(X, *, 0)$ in Example 3.3. Let (\tilde{f}, X) be a 3-folded \mathcal{N} -structure over X given as follows:

$$\tilde{f}: X \rightarrow [-1, 0]^3, y \mapsto \begin{cases} \left(-\frac{1}{4}, -0.45, -\frac{1}{6}\right) & \text{if } x = 4, \\ \left(-\frac{1}{3}, -0.88, -\frac{2}{5}\right) & \text{otherwise.} \end{cases}$$

It is easy to check that (\tilde{f}, X) is a 3-folded \mathcal{N} -closed ideal of X .

(2) Consider a BCH-algebra $X = \{0, 1, 2, 3\}$ with the binary operation “*” which is given in Table 2.

Let (\tilde{f}, X) be a 4-folded \mathcal{N} -structure over X given by

$$\tilde{f}: X \rightarrow [-1, 0]^4, y \mapsto \begin{cases} (-0.82, -0.45, -0.66, -0.23) & \text{if } y \in \{0, 3\} \\ (-0.33, -0.25, -0.44, -0.13) & \text{if } y \in \{1, 2\}. \end{cases}$$

It is routine to prove that (\tilde{f}, X) is a 4-folded \mathcal{N} -closed ideal of X .

It is clear that if a k -folded \mathcal{N} -structure (\tilde{f}, X) over X is a k -folded \mathcal{N} -closed ideal or a k -folded \mathcal{N} -subalgebra of X , then $\tilde{f}(0) \lesssim \tilde{f}(x)$ for all $x \in X$, that is, $(\ell_i \circ \tilde{f})(0) \leq (\ell_i \circ \tilde{f})(x)$ for all $x \in X$ and $i = 1, 2, \dots, k$.

We provide relations between k -folded \mathcal{N} -closed ideal and k -folded \mathcal{N} -subalgebra.

Theorem 3.7. Every k -folded \mathcal{N} -closed ideal is a k -folded \mathcal{N} -subalgebra.

Proof. Let (\tilde{f}, X) be a k -folded \mathcal{N} -closed ideal of a BCH-algebra X . For any $x, y \in X$ and $i = 1, 2, \dots, k$, we have

$$\begin{aligned} (\ell_i \circ \tilde{f})(x * y) &\leq \max\{(\ell_i \circ \tilde{f})((x * y) * x), (\ell_i \circ \tilde{f})(x)\} \\ &= \max\{(\ell_i \circ \tilde{f})(x * x * y), (\ell_i \circ \tilde{f})(x)\} \\ &= \max\{(\ell_i \circ \tilde{f})(0 * y), (\ell_i \circ \tilde{f})(x)\} \\ &\leq \max\{(\ell_i \circ \tilde{f})(y), (\ell_i \circ \tilde{f})(x)\}, \end{aligned}$$

that is, $\tilde{f}(x * y) \lesssim \text{Max}\{\tilde{f}(x), \tilde{f}(y)\}$ for all $x, y \in X$. Hence (\tilde{f}, X) is a k -folded \mathcal{N} -subalgebra of X . \square

Table 2 | Cayley table for the binary operation “*.”

*	0	1	2	3
0	0	3	0	3
1	1	0	3	2
2	2	3	0	1
3	3	0	3	0

The 3-folded \mathcal{N} -subalgebra (\tilde{f}, X) in Example 3.3 is not a 3-folded \mathcal{N} -closed ideal since

$$(\ell_2 \circ \tilde{f})(3) = -0.25 > -0.45 = \max\{(\ell_2 \circ \tilde{f})(3 * 4), (\ell_2 \circ \tilde{f})(4)\}.$$

Hence we know that the converse of Theorem 3.7 is not true in general.

We provide conditions for the converse of Theorem 3.7 to be true.

Theorem 3.8. If a k -folded \mathcal{N} -subalgebra (\tilde{f}, X) of X satisfies

$$(\forall x, y \in X) (\tilde{f}(x) \lesssim \text{Max}\{\tilde{f}(x * y), \tilde{f}(y)\}), \quad (16)$$

that is, $(\ell_i \circ \tilde{f})(x) \leq \max\{(\ell_i \circ \tilde{f})(x * y), (\ell_i \circ \tilde{f})(y)\}$ for all $x, y \in X$ and $i = 1, 2, \dots, k$, then (\tilde{f}, X) is a k -folded \mathcal{N} -closed ideal of X .

Proof. It is straightforward by (13) and (16).

Proposition 3.9. If a k -folded \mathcal{N} -closed ideal (\tilde{f}, X) of X satisfies

$$(\forall x \in X) (\tilde{f}(x) \lesssim \tilde{f}(0 * x)), \quad (17)$$

that is, $(\ell_i \circ \tilde{f})(x) \leq (\ell_i \circ \tilde{f})(0 * x)$ for all $x \in X$ and $i = 1, 2, \dots, k$, then (\tilde{f}, X) satisfies the following inequality:

$$(\forall x, y \in X) (\tilde{f}(y * x) \lesssim \tilde{f}(x * y)), \quad (18)$$

that is, $(\ell_i \circ \tilde{f})(y * x) \leq (\ell_i \circ \tilde{f})(x * y)$ for all $x, y \in X$ and $i = 1, 2, \dots, k$.

Proof. For any $x, y \in X$ and $i = 1, 2, \dots, k$, we have

$$\begin{aligned} (\ell_i \circ \tilde{f})(y * x) &\leq (\ell_i \circ \tilde{f})(0 * (y * x)) \\ &\leq \max\{(\ell_i \circ \tilde{f})((0 * (y * x)) * (x * y)), (\ell_i \circ \tilde{f})(x * y)\} \\ &= \max\{(\ell_i \circ \tilde{f})(((0 * y) * (0 * x)) * (x * y)), (\ell_i \circ \tilde{f})(x * y)\} \\ &= \max\{(\ell_i \circ \tilde{f})(((0 * y) * (x * y)) * (0 * x)), (\ell_i \circ \tilde{f})(x * y)\} \\ &= \max\{(\ell_i \circ \tilde{f})(((0 * (x * y)) * y) * (0 * x)), (\ell_i \circ \tilde{f})(x * y)\} \\ &= \max\{(\ell_i \circ \tilde{f})(((0 * x) * (0 * y)) * (0 * x)) * y), (\ell_i \circ \tilde{f})(x * y)\} \\ &= \max\{(\ell_i \circ \tilde{f})((0 * (0 * y)) * y), (\ell_i \circ \tilde{f})(x * y)\} \\ &= \max\{(\ell_i \circ \tilde{f})(0), (\ell_i \circ \tilde{f})(x * y)\} \\ &= (\ell_i \circ \tilde{f})(x * y) \end{aligned}$$

by (I), (III), (3), (15) and (17). Hence (18) is valid. \square

Theorem 3.10. If a k -folded \mathcal{N} -subalgebra (\tilde{f}, X) of X satisfies the condition (18), then (\tilde{f}, X) is a k -folded \mathcal{N} -closed ideal of X .

Proof. If we put $y = 0$ in (18) and use (1), then $\tilde{f}(0 * x) \lesssim \tilde{f}(x * 0) = \tilde{f}(x)$ for all $x \in X$. Using (I), (III), (1), (12) and (18), we have

$$\begin{aligned} (\ell_i \circ \tilde{f})(x) &= (\ell_i \circ \tilde{f})(x * 0) \leq (\ell_i \circ \tilde{f})(0 * x) \\ &= (\ell_i \circ \tilde{f})((y * y) * x) = (\ell_i \circ \tilde{f})((y * x) * y) \\ &\leq \max\{(\ell_i \circ \tilde{f})(y * x), (\ell_i \circ \tilde{f})(y)\} \\ &\leq \max\{(\ell_i \circ \tilde{f})(x * y), (\ell_i \circ \tilde{f})(y)\} \end{aligned}$$

for all $x, y \in X$ and $i = 1, 2, \dots, k$, that is, $\tilde{f}(x) \lesssim \text{Max}\{\tilde{f}(x * y), \tilde{f}(y)\}$ for all $x, y \in X$. Therefore (\tilde{f}, X) is a k -folded \mathcal{N} -closed ideal of X .

Using the notion of k -folded level sets, we consider characterization of k -folded \mathcal{N} -closed ideal and k -folded \mathcal{N} -subalgebra.

Theorem 3.11. *Given a k -folded \mathcal{N} -structure (\tilde{f}, X) over a BCH-algebra X , the following are equivalent:*

1. (\tilde{f}, X) is a k -folded \mathcal{N} -closed ideal (resp. k -folded \mathcal{N} -subalgebra) of X .
2. The k -folded level set $L(\tilde{f}; \tilde{t})$ of (\tilde{f}, X) is a closed ideal (resp. subalgebra) of X for all $\tilde{t} \in [-1, 0]^k$ with $L(\tilde{f}; \tilde{t}) \neq \emptyset$.

Proof. Assume that (\tilde{f}, X) is a k -folded \mathcal{N} -closed ideal of X and let $\tilde{t} \in [-1, 0]^k$ be such that $L(\tilde{f}; \tilde{t}) \neq \emptyset$. If $x \in L(\tilde{f}; \tilde{t})$, then $x \in L(\tilde{f}; \tilde{t})^i$ for all $i = 1, 2, \dots, k$. Hence $(\ell_i \circ \tilde{f})(0 * x) \leq (\ell_i \circ \tilde{f})(x) \leq t_i$ for all $i = 1, 2, \dots, k$, and so $0 * x \in \bigcap_{i=1}^k L(\tilde{f}; \tilde{t})^i = L(\tilde{f}; \tilde{t})$. Let $x, y \in X$ be such that $x * y \in L(\tilde{f}; \tilde{t})$ and $y \in L(\tilde{f}; \tilde{t})$. Then $x * y \in L(\tilde{f}; \tilde{t})^i$ and $y \in L(\tilde{f}; \tilde{t})^i$ for all $i = 1, 2, \dots, k$. It follows that

$$(\ell_i \circ \tilde{f})(x) \leq \max\{(\ell_i \circ \tilde{f})(x * y), (\ell_i \circ \tilde{f})(y)\} \leq t_i.$$

Hence $x \in L(\tilde{f}; \tilde{t})^i$ for all $i = 1, 2, \dots, k$ and thus $x \in \bigcap_{i=1}^k L(\tilde{f}; \tilde{t})^i = L(\tilde{f}; \tilde{t})$. Therefore $L(\tilde{f}; \tilde{t})$ is a closed ideal of X . Similarly, we can show that if (\tilde{f}, X) is a k -folded \mathcal{N} -subalgebra of X , then $L(\tilde{f}; \tilde{t})$ is a subalgebra of X .

Conversely, suppose that the k -folded level set $L(\tilde{f}; \tilde{t})$ of (\tilde{f}, X) is a closed ideal of X for all $\tilde{t} \in [-1, 0]^k$ with $L(\tilde{f}; \tilde{t}) \neq \emptyset$. If the inequality $\tilde{f}(0 * a) \lesssim \tilde{f}(a)$ is false for some $a \in X$, then there exists $\tilde{t} \in (-1, 0)^k$ such that $\tilde{f}(a) \lesssim \tilde{f}(0 * a)$, and so $a \in L(\tilde{f}; \tilde{t})$ and $0 * a \notin L(\tilde{f}; \tilde{t})$. This is a contradiction, and thus $\tilde{f}(0 * x) \lesssim \tilde{f}(x)$ for all $x \in X$. Now suppose that the inequality $\tilde{f}(a) \lesssim \text{Max}\{\tilde{f}(a * b), \tilde{f}(b)\}$ is not true for some $a, b \in X$. Then $\text{Max}\{\tilde{f}(a * b), \tilde{f}(b)\} \lesssim \tilde{f}(a)$ for some $\tilde{s} \in [-1, 0]^k$, which implies that $a * b \in L(\tilde{f}; \tilde{s})$ and $b \in L(\tilde{f}; \tilde{s})$ but $a \notin L(\tilde{f}; \tilde{s})$. This is impossible, and hence $\tilde{f}(x) \lesssim \text{Max}\{\tilde{f}(x * y), \tilde{f}(y)\}$ for all $x, y \in X$. Therefore (\tilde{f}, X) is a k -folded \mathcal{N} -closed ideal of X . By the similar way, we can show that if the k -folded level set $L(\tilde{f}; \tilde{t})$ of (\tilde{f}, X) is a subalgebra of X for all $\tilde{t} \in [-1, 0]^k$ with $L(\tilde{f}; \tilde{t}) \neq \emptyset$, then (\tilde{f}, X) is a k -folded \mathcal{N} -subalgebra of X .

Let X be a BCH-algebra. Then the BCA-part $X_+ := \{x \in X \mid 0 * x = 0\}$ of X is a closed ideal of X , and the medial part $\text{Med}(X) := \{x \in X \mid 0 * (0 * x) = x\}$ of X is a subalgebra of X (see [5]). Hence the following theorem is a direct result of Theorem 3.11.

Theorem 3.12. *Let (\tilde{f}, X) be a k -folded \mathcal{N} -structure over a BCH-algebra X given by*

$$\tilde{f} : X \rightarrow [-1, 0]^k, x \mapsto \begin{cases} \tilde{t} & \text{if } x \in X_+ \text{ (resp. } \text{Med}(X)) \\ \tilde{s} & \text{otherwise} \end{cases}$$

where $\tilde{t} = (t_1, t_2, \dots, t_k) \lesssim (s_1, s_2, \dots, s_k) = \tilde{s}$ in $[-1, 0]^k$. Then (\tilde{f}, X) is a k -folded \mathcal{N} -closed ideal (resp. k -folded \mathcal{N} -subalgebra) of X .

Theorem 3.13. *If (\tilde{f}, X) is a k -folded \mathcal{N} -closed ideal of a BCH-algebra X , then the set*

$$X_e := \{x \in X \mid \tilde{f}(x) \lesssim \tilde{f}(e)\} \quad (19)$$

is a closed ideal of X for all $e \in X$.

Proof. Note that $X_e = \bigcap_{i=1}^k X_e^i$ where $X_e^i := \{x \in X \mid (\ell_i \circ \tilde{f})(x) \leq (\ell_i \circ \tilde{f})(e)\}$. For any $i = 1, 2, \dots, k$, if $x \in X_e^i$, then $(\ell_i \circ \tilde{f})(0 * x) \leq (\ell_i \circ \tilde{f})(x) \leq (\ell_i \circ \tilde{f})(e)$ and so $0 * x \in X_e^i$. Let $x, y \in X$ be such that $x * y \in X_e^i$ and $y \in X_e^i$. Then $(\ell_i \circ \tilde{f})(x * y) \leq (\ell_i \circ \tilde{f})(e)$ and $(\ell_i \circ \tilde{f})(y) \leq (\ell_i \circ \tilde{f})(e)$. It follows from (15) that

$$(\ell_i \circ \tilde{f})(x) \leq \max\{(\ell_i \circ \tilde{f})(x * y), (\ell_i \circ \tilde{f})(y)\} \leq (\ell_i \circ \tilde{f})(e).$$

Hence $x \in X_e^i$, which shows that X_e^i is a closed ideal of X for all $i = 1, 2, \dots, k$. Therefore $X_e = \bigcap_{i=1}^k X_e^i$ is a closed ideal of X .

Proposition 3.14. *Given a k -folded \mathcal{N} -structure (\tilde{f}, X) over a BCH-algebra X , if the set $X_e^i := \{x \in X \mid (\ell_i \circ \tilde{f})(x) \leq (\ell_i \circ \tilde{f})(e)\}$ is a closed ideal of X for all $e \in X$ and $i = 1, 2, \dots, k$, then (\tilde{f}, X) satisfies the following argument:*

$$(\forall x, y, z \in X)(\text{Max}\{\tilde{f}(y * z), \tilde{f}(z)\} \lesssim \tilde{f}(x) \Rightarrow \tilde{f}(y) \lesssim \tilde{f}(x)), \quad (20)$$

that is, $(\ell_i \circ \tilde{f})(x) \geq \max\{(\ell_i \circ \tilde{f})(y * z), (\ell_i \circ \tilde{f})(z)\}$ implies $(\ell_i \circ \tilde{f})(x) \geq (\ell_i \circ \tilde{f})(y)$ for all $x, y, z \in X$ and $i = 1, 2, \dots, k$.

Proof. Let $x, y, z \in X$ be such that $\text{Max}\{\tilde{f}(y * z), \tilde{f}(z)\} \lesssim \tilde{f}(x)$, that is,

$$(\ell_i \circ \tilde{f})(x) \geq \max\{(\ell_i \circ \tilde{f})(y * z), (\ell_i \circ \tilde{f})(z)\}$$

for $i = 1, 2, \dots, k$. Then $y * z \in X_x^i$ and $z \in X_x^i$. Since X_x^i is a closed ideal of X , we have $y \in X_x^i$, and so $(\ell_i \circ \tilde{f})(x) \geq (\ell_i \circ \tilde{f})(y)$ for $i = 1, 2, \dots, k$. Hence the argument (20) is valid. \square

Theorem 3.15. *If a k -folded \mathcal{N} -structure (\tilde{f}, X) over a BCH-algebra X satisfies the conditions (13) and (20), then the set X_e in (19) is a closed ideal of X for all $e \in X$.*

Proof. If $x \in X_e$, then $\tilde{f}(0 * x) \lesssim \tilde{f}(x) \lesssim \tilde{f}(e)$ by (13) and so $0 * x \in X_e$. Let $x, y \in X$ be such that $x * y \in X_e$ and $y \in X_e$. Then $\tilde{f}(x * y) \lesssim \tilde{f}(e)$ and $\tilde{f}(y) \lesssim \tilde{f}(e)$, which imply $\text{Max}\{\tilde{f}(x * y), \tilde{f}(y)\} \lesssim \tilde{f}(e)$. Using (20), we get $\tilde{f}(x) \lesssim \tilde{f}(e)$, that is, $x \in X_e$. Therefore X_e is a closed ideal of X for all $e \in X$. \square

4. k -FOLDED \mathcal{N} -FILTERS

Definition 4.1. Let X be a BCH-algebra. A k -folded \mathcal{N} -structure (\tilde{f}, X) over X is called a k -folded \mathcal{N} -filter of X if it satisfies

$$(\forall x, y \in X)(x * y = 0 \Rightarrow \tilde{f}(x) \lesssim \tilde{f}(y)), \quad (21)$$

$$(\forall x, y \in X)(\text{Min}\{\tilde{f}(x), \tilde{f}(y)\} \lesssim \text{Min}\{\tilde{f}(x * (x * y)), \tilde{f}(y * (y * x))\}), \quad (22)$$

that is,

$$x * y = 0 \Rightarrow (\ell_i \circ \tilde{f})(x) \leq (\ell_i \circ \tilde{f})(y)$$

and

$$\min\{(\ell_i \circ \tilde{f})(x), (\ell_i \circ \tilde{f})(y)\} \leq \min\{(\ell_i \circ \tilde{f})(x * (x * y)), (\ell_i \circ \tilde{f})(y * (y * x))\}$$

for all $x, y \in X$ and $i = 1, 2, \dots, k$.

A k -folded \mathcal{N} -filter of X is said to be *closed* if $\tilde{f}(x) \lesssim \tilde{f}(0 * x)$ for all $x \in X$, that is, $(\ell_i \circ \tilde{f})(x) \leq (\ell_i \circ \tilde{f})(0 * x)$ for all $x \in X$ and $i = 1, 2, \dots, k$.

Example 4.2. Consider the BCH-algebra $X = \{0, 1, 2, 3, 4\}$ in Example 3.3.

(1) Let (\tilde{f}, X) be a 5-folded \mathcal{N} -structure over X given by

$$\begin{aligned} \tilde{f} : X &\rightarrow [-1, 0]^5, \\ x &\mapsto \begin{cases} (-0.86, -0.77, -0.55, -0.49, -0.33) & \text{if } x = 4 \\ (-0.13, -0.22, -0.28, -0.36, -0.05) & \text{otherwise.} \end{cases} \end{aligned}$$

Then (\tilde{f}, X) is a 5-folded \mathcal{N} -filter of X .

(2) Let (\tilde{f}, X) be a 2-folded \mathcal{N} -structure over X given by

$$\tilde{f} : X \rightarrow [-1, 0]^2, \quad x \mapsto \begin{cases} (-0.7, -0.6) & \text{if } x \in \{0, 1, 2, 3\} \\ (-0.2, -0.3) & \text{if } x = 4 \end{cases}$$

Then (\tilde{f}, X) is a 2-folded \mathcal{N} -filter of X .

Theorem 4.3. A k -folded \mathcal{N} -structure (\tilde{f}, X) over a BCH-algebra X is a (closed) k -folded \mathcal{N} -filter of X if and only if the k -folded level set $L(\tilde{f}, \tilde{t})$ of (\tilde{f}, X) is a (closed) filter of X for all $\tilde{t} \in [-1, 0]^k$ with $L(\tilde{f}, \tilde{t}) \neq \emptyset$.

Proof. It is similar to the proof of Theorem 3.11.

Theorem 4.4. Let (\tilde{f}, X) be a k -folded \mathcal{N} -structure over a BCH-algebra X given as follows:

$$\tilde{f} : X \rightarrow [-1, 0]^k, \quad x \mapsto \begin{cases} \tilde{t} & \text{if } x \in B(x_0), x_0 \in \text{Med}(X), \\ \tilde{s} & \text{otherwise} \end{cases}$$

where $B(x_0)$ is the branch of X , i.e., $B(x_0) = \{x \in X \mid x_0 * x = 0\}$, and $\tilde{t} = (t_1, t_2, \dots, t_k) \lesssim (s_1, s_2, \dots, s_k) = \tilde{s}$ in $[-1, 0]^k$. Then (\tilde{f}, X) is a k -folded \mathcal{N} -filter of X .

Proof. Using Theorem 4.3, it is sufficient to show that $B(x_0)$ is a filter of X for $x_0 \in \text{Med}(X)$. Let $x, y \in X$ be such that $x \in B(x_0)$ and $x * y = 0$. Then $0 * x_0 = 0 * x = 0 * y$ by (5). It follows that $0 * (0 * y) = 0 * (0 * x_0) = x_0$. Hence $x_0 * y = (0 * (0 * y)) * y = 0$, and so $y \in B(x_0)$. Let $x, y \in B(x_0)$. Then $x_0 * x = 0$ and $x_0 * y = 0$, which imply from (5) that $0 * x_0 = 0 * x$ and $0 * x_0 = 0 * y$. It follows from (I), (III), (1) and (3) that

$$\begin{aligned} x_0 * (y * (y * x)) &= (0 * (0 * x_0)) * (y * (y * x)) \\ &= (0 * (y * (y * x))) * (0 * x_0) \\ &= ((0 * y) * ((0 * y) * (0 * x))) * (0 * x_0) \\ &= ((0 * x_0) * ((0 * x_0) * (0 * x_0))) * (0 * x_0) \\ &= 0. \end{aligned}$$

Similarly, we have $x_0 * (x * (x * y)) = 0$. Therefore $x * (x * y) \in B(x_0)$ and $y * (y * x) \in B(x_0)$. This completes the proof. \square

Proposition 4.5. Every closed k -folded \mathcal{N} -filter (\tilde{f}, X) of a BCH-algebra X satisfies

$$(\forall x, y \in X)(\tilde{f}(x * y) \lesssim \tilde{f}(y * x)), \quad (23)$$

that is, $(\ell_i \circ \tilde{f})(x * y) \leq (\ell_i \circ \tilde{f})(y * x)$ for all $x, y \in X$ and $i = 1, 2, \dots, k$.

Proof. Using (I), (III) and (3), we have

$$\begin{aligned} (0 * (x * y)) * (y * x) &= ((0 * x) * (0 * y)) * (y * x) \\ &= ((0 * x) * (y * x)) * (0 * y) \\ &= ((0 * (y * x)) * x) * (0 * y) \\ &= (((0 * y) * (0 * x)) * x) * (0 * y) \\ &= (((0 * y) * (0 * x)) * (0 * y)) * x \\ &= (((0 * y) * (0 * y)) * (0 * x)) * x \\ &= (0 * (0 * x)) * x \\ &= 0 \end{aligned}$$

for all $x, y \in X$. It follows from the closedness of (\tilde{f}, X) and (21) that $\tilde{f}(x * y) \lesssim \tilde{f}(0 * (x * y)) \lesssim \tilde{f}(y * x)$ for all $x, y \in X$. \square

Corollary 4.6. Every closed k -folded \mathcal{N} -filter (\tilde{f}, X) of a BCH-algebra X satisfies

$$(\forall x, y \in X)(\tilde{f}(x) \lesssim \tilde{f}(y * (y * x))), \quad (24)$$

that is, $(\ell_i \circ \tilde{f})(x) \leq (\ell_i \circ \tilde{f})(y * (y * x))$ for all $x, y \in X$ and $i = 1, 2, \dots, k$.

Proof. Using the closedness of (\tilde{f}, X) , (I), (III) and (23), we get

$$\tilde{f}(x) \lesssim \tilde{f}(0 * x) = \tilde{f}(y * y) * x = \tilde{f}(y * x) * y \lesssim \tilde{f}(y * (y * x))$$

for all $x, y \in X$. \square

Corollary 4.7. Every closed k -folded \mathcal{N} -filter (\tilde{f}, X) of a BCH-algebra X satisfies

$$(\forall x, y \in X)(x * y = 0 \Rightarrow \tilde{f}(y) \lesssim \tilde{f}(x)), \quad (25)$$

that is, $(\ell_i \circ \tilde{f})(y) \leq (\ell_i \circ \tilde{f})(x)$ for all $x, y \in X$ with $x * y = 0$ and $i = 1, 2, \dots, k$.

Proof. Let $x, y \in X$ be such that $x * y = 0$. Then $0 * x = 0 * y$ by (5). It follows from (1) and (23) that

$$\tilde{f}(y) = \tilde{f}(y * 0) \lesssim \tilde{f}(0 * y) = \tilde{f}(0 * x) \lesssim \tilde{f}(x * 0) = \tilde{f}(x).$$

This completes the proof.

Theorem 4.8. Given a k -folded \mathcal{N} -structure (\tilde{f}, X) over a BCH-algebra X , the following are equivalent.

1. (\tilde{f}, X) is a k -folded \mathcal{N} -closed ideal of X satisfying the condition (23).
2. The k -folded level set $L(\tilde{f}, \tilde{t})$ of (\tilde{f}, X) is a closed ideal of X for all $\tilde{t} \in [-1, 0]^k$ with $L(\tilde{f}, \tilde{t}) \neq \emptyset$ which satisfies the following condition:

$$(\forall x, y \in X)(y * x \in L(\tilde{f}, \tilde{t}) \Rightarrow x * y \in L(\tilde{f}, \tilde{t})). \quad (26)$$

Proof. Recall that (\tilde{f}, X) is a k -folded \mathcal{N} -closed ideal of X if and only if the k -folded level set $L(\tilde{f}; \tilde{t})$ of (\tilde{f}, X) is a closed ideal of X for all $\tilde{t} \in [-1, 0]^k$ with $L(\tilde{f}; \tilde{t}) \neq \emptyset$ (see Theorem 3.11). Assume that the condition (23) is valid and let $y * x \in L(\tilde{f}; \tilde{t})$ for all $x, y \in X$. Then $\tilde{f}(x * y) \lesssim \tilde{f}(y * x) \lesssim \tilde{t}$, and so $x * y \in L(\tilde{f}; \tilde{t})$. Now suppose the condition (26) is true and let $a, b \in X$ be such that $\tilde{f}(a * b) \lesssim 6 \tilde{f}(b * a)$. Then $\tilde{f}(b * a) \lesssim \tilde{s} \lesssim \tilde{f}(a * b)$ for some $\tilde{s} = (s_1, s_2, \dots, s_k) \in (-1, 0]^k$. Hence $b * a \in L(\tilde{f}; \tilde{s})$ but $a * b \notin L(\tilde{f}; \tilde{s})$ which is a contradiction. Therefore the condition (23) is valid. \square

Theorem 4.9. *If every k -folded \mathcal{N} -closed ideal (\tilde{f}, X) of a BCH-algebra X satisfies the condition (23), then it is a closed k -folded \mathcal{N} -filter of X .*

Proof. Let (\tilde{f}, X) be a k -folded \mathcal{N} -closed ideal of X satisfying the condition (23). Then $L(\tilde{f}; \tilde{t})$ is a closed ideal of X which satisfies (26) (see Theorem 4.8). Let $x, y \in X$ be such that $x \in L(\tilde{f}; \tilde{t})$ and $x * y = 0$. Then $x * y = 0 \in L(\tilde{f}; \tilde{t})$ which implies from (26) that $y * x \in L(\tilde{f}; \tilde{t})$. Hence $0 * x \in L(\tilde{f}; \tilde{t})$ and $y \in L(\tilde{f}; \tilde{t})$ by (6) and (7). It is clear that if $x, y \in L(\tilde{f}; \tilde{t})$, then $x * (x * y) \in L(\tilde{f}; \tilde{t})$ and $y * (y * x) \in L(\tilde{f}; \tilde{t})$ since $L(\tilde{f}; \tilde{t})$ is a subalgebra of X . This shows that $L(\tilde{f}; \tilde{t})$ is a closed filter of X . Therefore (\tilde{f}, X) is a closed k -folded \mathcal{N} -filter of X by Theorem 4.3. \square

5. CONCLUSIONS

In addition to positive information, negative information coexist in the complex and diverse social phenomena. The fuzzy set is a very useful tool for dealing with positive information, but it is not suitable for dealing with negative information. So we feel the need for a scientific tool to deal with negative information. In 2009, Jun *et al.* introduced a new structure called \mathcal{N} -structure, which is suitable for processing negative information. These \mathcal{N} -structures are applied in many ways, including algebra and decision-making problem, and so on. In this paper, we introduced multi-folded \mathcal{N} -structure with white degree in consideration of the generalization of the \mathcal{N} -structure, as if the generalization of the fuzzy set was considered. We applied the \mathcal{N} -structure to an algebraic structure so called BCH-algebra. We introduced the notions of k -folded \mathcal{N} -subalgebra, k -folded \mathcal{N} -closed ideal and (closed) k -folded \mathcal{N} -filter, and then we investigated their relations and several properties. We provided conditions for the k -folded \mathcal{N} -subalgebra to be k -folded \mathcal{N} -closed ideal. Using the notion of k -folded level sets, we discussed characterization of k -folded \mathcal{N} -subalgebra, k -folded \mathcal{N} -closed ideal and (closed) k -folded \mathcal{N} -filter. Using the medial part and BCA-part, we made a k -folded \mathcal{N} -subalgebra and a k -folded \mathcal{N} -closed ideal. Using the branch of a BCH-algebra, we established a k -folded \mathcal{N} -filter. We provided conditions for a k -folded \mathcal{N} -closed ideal to be a closed k -folded \mathcal{N} -filter.

AVAILABILITY OF DATA AND MATERIALS

Not applicable.

CONFLICTS OF INTEREST

The authors declare that they have no competing interests.

AUTHORS' CONTRIBUTIONS

Created and conceptualized ideas, J.-G.L. and Y.B.J.; writing original draft preparation, Y.B.J.; writing review and editing, K.H.; funding acquisition, J.-G.L. All authors have read and agreed to the published version of the manuscript.

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REFERENCES

- [1] Q.P. Hu, X. Li, On BCH-algebras, Math. Semin. Notes Kobe Univ. 11 (1983), 313–320.
- [2] Q.P. Hu, X. Li, On proper BCH-algebras, Math. Japon. 30 (1985), 659–661.
- [3] B. Ahmad, On classification of BCH-algebras, Math. Japon. 35 (1990), 801–804.
- [4] W.A. Dudek, J. Thomys, On decompositions of BCH-algebras, Math. Japon. 35 (1990), 1131–1138.
- [5] M.A. Chaudhry, On BCH-algebras, Math. Japon. 36 (1991), 665–676.
- [6] M.A. Chaudhry, H. Fakhar-Ud-Din, Ideals and filters in BCH-algebras, Math. Japon. 44 (1996), 101–111.
- [7] C. Jana, M. Pal, On (α, β) -US Sets in BCK/BCI-algebras, Mathematics. 7 (2019), 252.
- [8] C. Jana, T. Senapati, M. Pal, $(\in, \in \vee q)$ -intuitionistic fuzzy BCI-subalgebras of a BCI-algebra, J. Intell. Fuzzy Syst. 31 (2016), 613–621.
- [9] C. Jana, M. Pal, A.B. Saeid, $(\in, \in \vee q)$ -bipolar fuzzy BCK/BCI-algebras, Missouri J. Math. Sci. 29 (2017), 139–160.
- [10] C. Jana, T. Senapati, K.P. Shum, M. Pal, Bipolar fuzzy soft subalgebras and ideals of BCK/BCI-algebras based on bipolar fuzzy points, J. Intell. Fuzzy Syst. 37 (2019), 2785–2795.
- [11] C. Jana, M. Pal, Generalized intuitionistic fuzzy ideals of BCK/BCI-algebras based on 3-valued logic and its computational study, Fuzzy Inf. Eng. 9 (2017), 455–478.
- [12] T. Senapati, M. Bhowmik, M. Pal, B. Davvaz, Atanassov's intuitionistic fuzzy translations of intuitionistic fuzzy subalgebras and ideals in BCK/BCI-algebras, Eurasian Math. J. 6 (2015), 96–114. <http://mi.mathnet.ru/eng/emj/v6/i1/p96>
- [13] C. Jana, T. Senapati, M. Pal, A.B. Saeid, Different types of cubic ideals in BCI-algebras based on fuzzy points, Afrika Mat. 31 (2020), 367–381.
- [14] T. Senapati, C. Jana, M. Pal, Y.B. Jun, Cubic intuitionistic q-ideals of BCI-algebras, Symmetry. 10 (2018), 752.
- [15] T. Senapati, C.S. Kim, M. Bhowmik, M. Pal, Cubic subalgebras and cubic closed ideals of B-algebras, Fuzzy Inf. Eng. 7 (2015), 129–149.

- [16] Y.B. Jun, K.J. Lee, S.Z. Song, N -ideals of BCK/BCI-algebras, J. Chungcheong Math. Soc. 22 (2009), 417–437.
- [17] Y.B. Jun, M.A. Ozturk, E.H. Roh, N -structures applied to closed ideals in BCH-algebras, Int. J. Math. Math. Sci. 2010 (2010), 1–9. 943565.
- [18] Y.B. Jun, N.O. Alshehri, K.J. Lee, Soft set theory and N -structures applied to BCH-algebras, J. Comput. Anal. Appl. 16 (2014), 869–886.
- [19] Y.B. Jun, J. Kavikumar, K.S. So, N -ideals of subtraction algebras, Commun. Korean Math. Soc. 25 (2010), 173–184.
- [20] J. Kavikumar, Y.B. Jun, N -subalgebras and N -ideals on a sub-BCK-algebra of a BCI-algebra, Commun. Korean Math. Soc. 27 (2012), 645–651.