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Multi-Stage Filtering of Numerical Solutions With an Application to the Hele-Shaw Problem

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***Abstract*—This work is devoted to improving the accuracy and reliability of computer-generated information in mathematical modeling of physical and technological processes. A method of filtering numerical results for the solutions of different problems is presented for estimating the errors and increasing the accuracy. This method includes a formal two-stage rule for choosing the standard and testing. It is shown that the proposed method allows avoiding the uncertainty and limitations of the Runge and Romberg rules for estimating the errors of numerical data. The proposed method is used to analyze numerical solutions of an electrochemical machining modeling problem. The developed numerical method for solving this problem includes the definition of two conformal mappings. One of them is a conformal mapping of the semicircle region of the parametric plane on a band of the complex potential plane. This function is represented as the sum of a known function with singularities and a function without singularities defined as the Laurent series with known coefficients. The second conformal mapping is used to determine the relationship between the parametric plane and the physical one. We use the Laurent series for this and its coefficients are determined by the method of collocations for the boundary condition fulfillment. The forms of the surface to be machined during its processing with an ellipsoidal cross-section electrode tool are obtained. The carried out investigation of the error dependencies on the numbers of stored coefficients of the series shows that these dependencies are the sums of exponential functions. The error estimates of the studied parameters obtained by filtering are given.**

***Keywords*—error estimation, improving the reliability of calculations, acceleration of convergence**

I. Introduction

There are widely known methods, such as the Runge [1] and Richardson [2] rules, as well as the methods of Aitken [3], Neville [4], Romberg [5] and Winn [6], which allow estimating the errors and accelerating the convergence of a solution on the basis of knowledge about the character of the dependence of the error in the numerical method on the number of grid points n. This is also the subject of investigation of many other authors [7-16]. However, the justification for these methods is based on the representation of the residual term as an infinitely small quantity. Note that an infinitely small quantity is a virtual reality that has no place in real calculations. The presence of a round off error that does not decrease, but usually increases with n, also casts doubt on the convergence and makes it difficult to use these methods. The increasing influence of the round off error on the result of the calculation leads to an increase in the condition number of the matrices for solving the systems of equations with the growth of their dimension n. Therefore, recognizing the great role of these results in the theory of numerical methods, we see the necessity of developing heuristic approaches to analyzing the results of computational experiments that are free from idealizing assumptions.

The Runge and Richardson rules are based on the representation of the error of the numerical method $g_n - g$ (g_n is the result of calculations with the number of grid nodes equal to n, g is the exact value of the desired parameter) in the form

$$g_n - g = a_1 n^{-k} + o\left(n^{-k}\right). \qquad (1)$$

Here it is assumed that the order of accuracy k, is known, and that a_1 does not depend on n. It is assumed that the infinitely small $o\left(n^{-k}\right)$ can be neglected compared to $a_1 n^{-k}$. Then, we use two results of the calculations, for node numbers equal to n and n/P, where P is some integer (usually $P = 2$), and get the formula

$$g \approx g_n + \frac{g_n - g_{n/P}}{P^k - 1} = g_n^*, \qquad (2)$$

which is called the Richardson extrapolation formula. Moving g_n to the right side, we get the error estimate $g_n^* - g_n$ according to the Runge rule.

The Romberg method uses a more complex representation of the error,

$$g_n - g = a_1 n^{-k_1} + a_2 n^{-k_2} + \ldots + a_M n^{-k_M} + o\left(n^{-k_M}\right). \quad (3)$$

For many numerical methods (for example, the one-sided difference derivative, Euler's method, etc.), $k_j=j$, $j=1, 2, 3,\ldots$ For the symmetric difference derivative, the trapezoid method of numerical integration the coefficients $k_j=2j$. Let's assume that for a sequence of numbers of nodes $n_i = n_1 P^{i-1}$ the values $g_{n_i}, i=1,...,M+1$ are calculated. If we write $x = n^{-1}$ for $k_j = j$ or $x = n^{-2}$ for $k_j = 2j$ and discard $o\left(n^{-k_L}\right)$ in (3), then we come to the problem of interpolation of the function given by the table of values by an algebraic polynomial $\Pi_M(x)$. Then the problem of determining the desired value g is reduced to extrapolation of $\Pi_M(x)$ to the value $x = 0$

$$g_{n_{L+1}}^{(M)}\left(g_{n_1},\ldots,g_{n_{M+1}}\right) = \sum_{i=1}^{M+1} g_{n_i} \prod_{\substack{j=1 \\ j\neq i}}^{M+1} \frac{n_j^{-1}}{n_j^{-1} - n_i^{-1}}.$$

One can use two polynomials of degree $M-1$ and the Aitken formula [3] in order to obtain the following polynomial of degree M

$$g_{n_{M-1}}^{(M)}\left(g_{n_1},\ldots,g_{n_{M+1}}\right) =$$

$$= \frac{n_{M+1}^{-1}}{n_{M+1}^{-1} - n_1^{-1}} g_{n_M}^{(M-1)}\left(g_{n_1},\ldots,g_{n_M}\right) - \frac{n_1^{-1}}{n_{M+1}^{-1} - n_1^{-1}} g_{n_{M+1}}^{(M-1)}\left(g_{n_2},\ldots,g_{n_{M+1}}\right).$$

In this case the problem is solved by repeated application of (2) with different values of $k = k_j$. This is the Romberg method [5].

If the number of grid nodes is increased by one instead of a factor of two, we get the Neville extrapolation formula [4]

$$g_n^{(M)} = g_n^{(M-1)} + \frac{n-M}{M}\left(g_n^{(M-1)} - g_{n-1}^{(M-1)}\right).$$

The Aitken extrapolation method [3] is used if the exponents k_j are unknown. In this case, one uses three calculated values g_n, $z_{n/P}$, g_{n/P^2} to get one extrapolated value

$$g \approx g_n + \frac{g_n - g_{n/P}}{\dfrac{g_n - g_{n/P}}{g_{n/P} - g_{n/P^2}} - 1}. \quad (4)$$

We proposed a method for analyzing the results of a computational experiment in [17-19] that does not use the assumption that the residual term is negligibly small.

II. Approach to the Error Estimation Without Assumptions on the Smallness of the Remainder

The proposed method of estimation of the error and its reliability is based on prior information, which is that the dependence of the error on the number of grid nodes may be represented in the form

$$g_n = g + a_1\varphi_1(n) + a_2\varphi_2(n) + \ldots + a_m\varphi_m(n) + r(n). \quad (5)$$

Here, the left side of the equality is a calculated result (real or complex) obtained for n grid nodes, and the right side represents the desired parameter g and the error which is the sum of regular components $\varphi_j(n)$ and an irregular component $r(n)$. The last term $r(n)$ includes round off error, and therefore is not infinitely small, but on the contrary, tends to increase with increasing n. This last term is the main difference of this approach from the theoretical models.

For the methods of numerical differentiation the multicomponent representation follows from the decomposition of the Taylor formula, where $\varphi_j(n) = n^{-k_j}$. Similar dependences are valid for the Newton-Cotes quadrature formulas, which follow from the Euler-Maclaurin formula. For the finite-difference methods for solving differential equations (both ordinary and partial) similar dependences are obtained using computational experiments.

For the simple iterative methods for solving equations and some other problems, $\varphi_j(n) = \lambda_j^n$, for example, for the numerical-analytical method for solving the Hele-Shaw problem considered in this paper.

For these methods, the results of the calculations can be represented as $g_{n_i}^{(0)} = g_{n_i}, i=1,\ldots,M$. In view of (5) these results can be written as expansions

$$\begin{aligned} g_{n_1} &= g + a_1\varphi_1(n_1) + a_2\varphi_2(n_1) + \ldots + a_m\varphi_m(n_1) + r(n_1), \\ g_{n_2} &= g + a_1\varphi_1(n_2) + a_2\varphi_2(n_2) + \ldots + a_m\varphi_m(n_2) + r(n_2), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ g_{n_M} &= g + a_1\varphi_1(n_M) + a_2\varphi_2(n_M) + \ldots + a_m\varphi_m(n_M) + r(n_M). \end{aligned} \quad (6)$$

As much as the $r(n_i)$ are independent on the other parameters, the quantity of unknowns in (6) is greater than M, that is, we are confronted with incorrectness in the problem of the determination of g. The absence of additional information about $r(n_i)$ does not allow applying the known regularization methods [20-22].

We will not solve the problem as a system of equations, that is, we do not define a_j, but we will consistently exclude the components $a_j\varphi_j$. This approach, in contrast to extrapolation, we call filtering. The filtering is performed using the assumption that the value $r(n)$ in (5) does not contain a constant, i.e., the legitimate components tend to zero with increasing n, and the components having a quasi-random or random character have mathematical expectations equal to zero. To test this assumption, we can employ the solution of the same problem by another method, which has its own errors and are estimated independently [23, 24].

III. Description of the Method

The filtering method consists in combination of rows of the matrix (6) similar to the Gaussian method of solving systems of linear algebraic equations. However, unlike to this method we combine the pairs of rows whose numbers differ by one

$$g_{n_i}^{(j)} = (\gamma_j + \eta_j)g + \ldots + (\gamma_j \varphi_j(n_{i-1}) + \eta_j \varphi_j(n_i))a_j + \ldots$$

where $\gamma_j + \eta_j = 1$, $\gamma_j \varphi_j(n_{i-1}) + \eta_j \varphi_j(n_i) = 0$.

The first equation is a condition for maintaining of the desired parameter g, the second equation is the condition for removal of the error component $\varphi_j(n)$.

Then, finding γ_j, η_j we have in the left part of (6)

$$g_{n_i}^{(j)} = g_{n_i}^{(j-1)} + \frac{g_{n_i}^{(j-1)} - g_{n_{i-1}}^{(j-1)}}{P_j - 1}, \; P_j = \frac{\varphi_j(n_{i-1})}{\varphi_j(n_i)}. \quad (7)$$

The expression (7) is the formula for the filter. An essential condition for repeating the filtering to remove the error components is that the P_j be independent of n. In this case, for the other error components $a_l \varphi_l(n_i)$

$$a_l \varphi_l(n_i) + \frac{a_l \varphi_l(n_i) - a_l \varphi_l(n_{i-1})}{P_j - 1} = \frac{P_j - P_l}{P_j - 1} a_l \varphi_l(n_i),$$

i.e., the coefficient in front of $\varphi_j(n_i)$ remains independent on n.

Thus, the filtered values $g_{n_i}^{(j)}$ have a form similar to (5), but do not contain $\varphi_j(n_i)$. This allows eliminating $\varphi_{j+1}(n_i)$ by the formula (7) etc. For exponential dependencies $\varphi_j(n) = \lambda_j^n$ we have $P_j = \text{const}$, if $n_i - n_{i-1} = \beta = \text{const}_1$, $P_j = \lambda_j^{-\beta}$. For power dependencies $\varphi_j(n) = n^{-k_j}$, $n_i / n_{i-1} = R = \text{const}_2$, $P_j = R^{k_j}$. In the last case, (7) coincides with the Richardson formula (2), but it is obtained without the assumption of the smallness of $|r(n)|$.

If one can estimate the dependence of the absolute value of the irregular component to be $|r^{(0)}(n)| \le r^{(0)}$ then the variation of the absolute value of the irregular error contained in the values $g_n^{(j)}$ can be estimated at every step as follows

$$r^{(j)} \le r^{(j-1)} + \frac{r^{(j-1)} + r^{(j-1)}}{P_j - 1} = \frac{P_j + 1}{P_j - 1} r^{(j-1)}.$$

For the cases $P_j = \lambda_j^{-\beta}$ and the infinite multiplication in the estimates using (8) is limited, consequently, the process of repeated filtering is stable. This is not so for the Neville method, therefore, the results of the filtering should be carefully monitored.

These results can be presented in the form of the matrix

$$\begin{array}{cccccc} n & g_{n_i} & g_{n_i}^{(1)} & g_{n_i}^{(2)} & g_{n_i}^{(3)} & \cdots \\ n_1 & g_{n_1} & - & - & - & \\ n_2 & g_{n_2} & g_{n_2}^{(1)} & - & - & \\ n_3 & g_{n_3} & g_{n_3}^{(1)} & g_{n_3}^{(2)} & - & \\ n_4 & g_{n_4} & g_{n_4}^{(1)} & g_{n_4}^{(2)} & g_{n_4}^{(3)} & \\ \ldots & & & & & \end{array} \quad (8)$$

These results can be controlled at this stage. It is necessary to calculate the Aitken ratio $\frac{g_{n_i} - g_{n_{i-1}}}{g_{n_{i-1}} - g_{n_{i-2}}}$ for this. With the correct determination of a component and its removal, this ratio increases from step to step. The absence of a change tells us there has been an incorrect determination of the component.

The next step is the analysis of the results in order to estimate the error, and its confirmation.

IV. Analysis of the results of the filtering

Let us consider the example of the calculation of a central difference derivative

$$\frac{dy}{dx}(x) \approx \frac{y(x+h) - y(x-h)}{2h} = g_n, \; h = \frac{1}{n}$$

of the function $\cos x$ at the point $x = 0.5$. Let us use the function $\lg z = \ln z / \ln 10$. The results in Fig. 1 are presented as a dependence of $-\lg\delta$ on $\lg n$, where $\delta = \left| \Delta_{n_i}^{(j)} / g_{n_i}^{(j)} \right|$. The value $-\lg\delta$ expresses the number of correct significant decimal digits in the result, so it can be called an accuracy. The results of a comparison with the accurate value ($\Delta_{n_i}^{(j)} = g_{n_i}^{(j)} - g$) are designated by the thick lines in Fig. 1, a.

The line corresponding to the results of direct calculation by the difference formula is marked as number *0*. The results from removing of the first, second, etc. components (5) are marked as *1*, *2*, ... The lines are close to straight lines in the zones where the power component dominates. Its angle coefficient is equal to k_j. For the central difference derivative $k_j = 2j$. The lines describe a random variable with the trend $y = 19 - \lg n$ in the zone of the prevalence of round off error that contains the largest part of $r(n)$ in (5).

The number 19 is explained as follows. The calculations are performed with double precision, with which the number of decimal digits of the mantissa is about 16. However, the Fortran compiler for MS DOS makes extensive use of a math coprocessor, which has about 19 decimal digits, i.e., the whole formula $\frac{\cos(x+h) - \cos(x-h)}{2h} + \sin x$ is loaded into the coprocessor.

Therefore, the result of the calculation is determined up to the coprocessor's accuracy. It can be seen that filtering allows getting the value to an accuracy that is a few orders higher than when using the difference formula.

Now we estimate the error without taking into account an accurate solution. The results of applying of the Runge rule presented in Fig. 1, b. Here the calculated values were compared with the filtered ones, the filtered results with results filtered twice, etc. We can see from Fig. 1 that the Runge rule allows to obtain good estimates at the zone of power component prevalence $\varphi_j(n) = n^{-k_1}$.

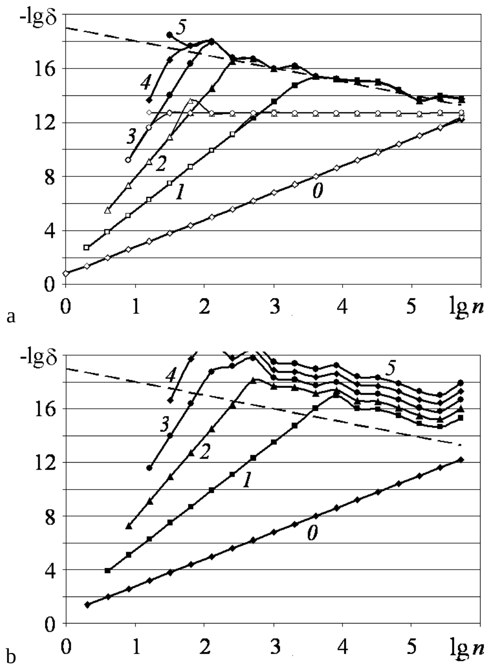

Fig. 1. Results of computation and filtering. a: Comparison with an accurate quantity and with the standard; b: Error estimation with the Runge rule

The Runge rule gives overstated accuracy estimations in the zone of predominance of $r(n)$ ($y \geq 19 - \lg n$).Thus, this rule can be used if the range of its applicability is known.

It is possible to determine this range by comparing the results of calculations and filtering with some standard $\hat{g}$ [18]. An error in the choice of the standard leads to a limitation of the difference between the obtained values and the standard in the value of the standard error. In this case we see the appearance of a "shelf" in the diagram (Fig. 1, a, the thin lines). This error can be eliminated by correcting the standard value in interactive mode (online). However, such a process makes automation rather difficult.

The reason for the incorrectness of the problem of the choice of standard is that the left-hand sides of (6) contain the known values obtained by calculation, and the right contains the unknown accurate value g and the error. However, by pairwise subtraction of equations, the unknown z can be excluded:

$$g^{(0)}_{n_{i-1}} = g_{n_{i-1}} - g_{n_i} . \quad (9)$$

Thus, the right-hand side of the resulting system contains only the error, and zero is the standard for its filtering. The value n is determined and so is the number j for which the error estimation is a minimum. Then the original set of numerical data is filtered and compared with the standard $\hat{g}$, which is the result with the found n and j. Thus, in particular (taking into account the assumption given at the end of Part 1) the incorrectness of the problem is eliminated. The coincidence of the estimates of the original sequence (6) and the sequence of differences (9) allows confirming the estimates and the correctness of the selection of the standard.

Note also that the distance between the lines corresponding to neighboring j (with unit difference) is the logarithm of the ratio of the estimates, which we call the estimate fuzzifying. It is theoretically and experimentally shown in [25, 26] that we can trust the estimates if the fuzzifying is not greater than 0.3. For the scale of graphs (Fig. 1) this means that the distance between the neighboring lines should be at least 0.5. Thus, the upper line confirms the estimate following from the lower one.

V. Statement of the Hele-Shaw problem

The stationary Hele-Shaw problem is considered. It can describe a model of the electrochemical machining (ECM) by an electrode-tool (ET) as a plane with the curvilinear (in particular, semi-elliptical) ledge $A'FCGB'$ (Fig. 2,a). The analogous problem with a semi-circular ledge is solved in [27]. ET is moving downwards with the speed V_{et}. The interelectrode space (IES) is filled with an electrolyte, the current joins and a dissolution of the processed surface (anode) begins. The surfaces of electrodes $A'FCGB'$ and $AMDNB$ are assumed to be equipotential. The processed surface $AMDNB$ assumes a stationary form for a sufficiently long process.

The asymptotic value of the gap S is known. The strength on the left and right at infinity are $E_0=U/S$, where U is the potential difference between the processed surface and ET.

The condition for the process to reach a stationary state is the equality of the electrochemical dissolution velocity $V_{ecm} = k|E|$ (k>0 is a constant determined by the Faraday number) and the projection of the ET velocity on the normal to the processed surface $-V_{et}\sin\theta$, where θ is the angle of inclination of strength vector to the X-axis

$$|E| = -E_0 \sin\theta , \quad E_0 = \frac{V_{et}}{k} . \quad (10)$$

The electric field is to be solenoidal and potential. The problem is solved by the method of the theory of a complex variable and refers to problems with nonlinear conditions on unknown boundaries. Let us consider the complex potential $W = \varphi + i\psi$, where φ is a potential and ψ is a stream function. The function $W(Z)$, where $Z = X + iY$, satisfies the Cauchy-Riemann equations, so it is an analytic function of a complex variable. Because of the equipotentiality of the boundaries (ReW=0 on AB; ReW=–U on $A'B'$) the image of IES on the complex potential plane W is a band (Fig. 2, b).

Let us choose the parametric variable ζ with a semi-annular range of variation $q < \zeta < 1, \operatorname{Im}\zeta > 0$ (Fig. 2, c). The function $W(\zeta)$ is defined by the conformal mapping.

The function $i\ln\frac{\zeta-q}{\zeta+q}+\frac{\pi}{2}$ maps the upper semicircle of the plane ζ on a band. And the semi-annulus is mapped onto the band without the oval part (Fig. 2, b).

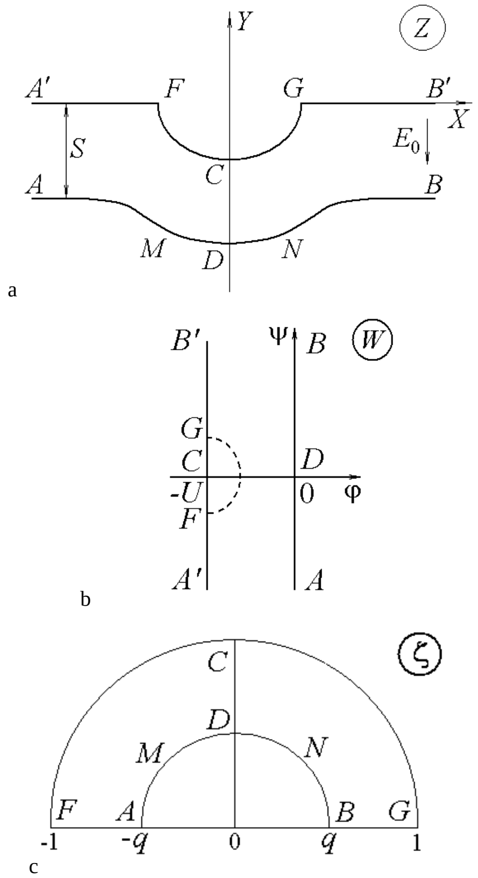

Fig. 2. Forms of the domain corresponding to interelectrode space. a: physical plane; b: complex potential plane; c: parametrical plane

To map the semiring onto the band, the transformation formula is completed by a Laurent series in odd powers ζ with real coefficients D_{2m-1}

$$W=-\frac{2U}{\pi}\left(i\ln\frac{\zeta-q}{\zeta+q}+\frac{\pi}{2}+\right.$$

$$\left.i\sum_{m=1}^{\infty}D_{2m-1}\left(q^{-(2m-1)}\zeta^{(2m-1)}+q^{(2m-1)}\zeta^{-(2m-1)}\right)\right). \quad (11)$$

The function (11) satisfies the conditions of equipotentiality on all boundaries with the exception of *FCG* in the general case. The coefficients D_{2m-1} are chosen to fulfill the condition $\mathrm{Re}W=-U$ on the outer semi-circumference *FCG* $\zeta=e^{i\sigma}=\cos\sigma+i\sin\sigma$ ($0\le\sigma\le\pi$). Expanding the logarithm in the function (11) in powers of ζ and setting equal to zero the real part of every term containing $\sin(2m-1)\sigma$ yields the equality

$$D_{2m-1}=\frac{2q^{2m-1}}{(2m-1)\left(q^{2m-1}-q^{-(2m-1)}\right)},\ m=1,2,\ldots$$

Then the derivative of the function $W(\zeta)$ is

$$\frac{dW}{d\zeta}=-i\frac{4U}{\pi}\left[\frac{q}{\zeta^2-q^2}+\frac{1}{\zeta}\sum_{m=1}^{\infty}\frac{q^{2(2m-1)}}{q^{2(2m-1)}-1}\times\right.$$

$$\left.\left(q^{-(2m-1)}\zeta^{2m-1}-q^{(2m-1)}\zeta^{-(2m-1)}\right)\right]. \quad (12)$$

The image of the ET boundary *FCG* on the strength hodograph plane $\bar{E}=\frac{dW}{dZ}=E_X-iE_Y$ (Fig. 3,a) is a curve corresponding to elliptical shape of ET on the physical plane. A cut along the circular arc with the radius $E_0/2$ with the center at the point $V=iE_0/2$ corresponds to the anode *AMDNB* according to (10).

Let us consider the Levi-Civita function $\omega(\zeta)$ (Fig. 3,b)

$$\omega=i\ln\frac{2}{E_0}\left(\frac{dW}{dZ}-H\right),\quad H=i\frac{E_0}{2}. \quad (13)$$

The function $\omega(\zeta)$ has to satisfy the conditions

– $\mathrm{Re}\,\omega=-\frac{\pi}{2}$ on *CD* and *BH* ;

– $\mathrm{Re}\,\omega=\frac{\pi}{2}$ on *HG* ;

– $\mathrm{Im}\,\omega=0$ on *DNB* ;

– $\left[\mathrm{Re}\,Z\left(e^{i\sigma}\right)\right]^2/R_1^2+\left[\mathrm{Im}\,Z\left(e^{i\sigma}\right)\right]^2/R_2^2=1$ on *CG* , (14)

where R_1 and R_2 are the semi-axes of the ellipse determined by the shape of ET.

The function $\omega(\zeta)$ is sought approximately in the form

$$\omega(\zeta)=\frac{\pi}{2}+i\ln\frac{\zeta^2\left(\zeta^2-h^2\right)}{h^2\zeta^2-q^4}+$$

$$i\sum_{m=1}^{n}C_m q^{2m}\left(q^{-2m}\zeta^{2m}-q^{2m}\zeta^{-2m}\right) \quad (15)$$

with real coefficients C_m . Here *h* is the image of the point *H* ($q<h<1$) in the ζ pane.

The function (15) satisfies all these conditions on the boundaries *CD*, *BH*, *HG* and *DNB*.

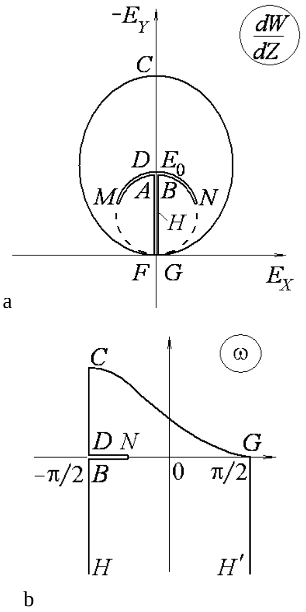

Fig. 3. Forms of the domain corresponding to the interelectrode space. a: hodograph intensity plane; b: ω-plane (13)

So, the function for any finite set of the coefficients $C_m, m=1,\ldots,n$ determines an exact solution of the problem of stationary machining with some curvilinear ET. This form can be approximated by an elliptic one up to some accuracy by adjusting the coefficients C_m.

The parameter h is defined by the condition

$$\omega(1)=\frac{\pi}{2}+i\ln\frac{1-h^2}{h^2-q^4}+i\sum_{m=1}^{n}C_m\left(q^{-2m}-q^{2m}\right)=\frac{\pi}{2},$$

$$h^2=\frac{1+q^4\exp\left[-\sum_{m=1}^{n}C_m\left(q^{-2m}-q^{2m}\right)\right]}{1+\exp\left[-\sum_{m=1}^{n}C_m\left(q^{-2m}-q^{2m}\right)\right]}.$$

Then we use (12), (13) and (15) and find the differential

$$dZ=\frac{2}{E_0}\frac{1}{e^{-i\omega(\zeta)}+i}\frac{dW}{d\zeta}(\zeta)d\zeta .$$

The conformal mapping $Z(\zeta)$ is found by numerical integration of this expression.

The problem is solved numerically by the collocation method. The equation of an ellipse (14) gives the form of the ET. It is satisfied at a finite number of points of the boundary $\zeta=e^{i\sigma_m}, \sigma_m=\pi m/(2n), m=0,\ldots,n$. The obtained system of nonlinear equations is solved with respect to the parameters $C_m, m=1,\ldots,n,\ q$ by Newton's method with the step regulation.

VI. Application of the filtering to the Hele-Shaw problem

The significant difficulty in the numerical solution of this problem occurs for p>0.9, corresponding to r_j>20 ($r_j=R_j/S$). As these solutions are important for applications, the use of solution refinement methods makes sense.

First, we consider the problem of summing the series in (12).

The results of the filtering are presented on the graph with a logarithmic scale (Fig. 4). The base ten logarithms of the relative errors (accuracy of data) $-\lg\delta\quad \delta=\left|\Delta_{n_i}^{(j)}/\hat{g}\right|$ are presented on the ordinate axis, and n is put on the abscissa axis.

The results of direct calculations (the dots on the curve *0*) and the results of the first and second filterings of this dependence (the dots on the lines *1* and *2*) form lines that are nearly straight if the addends in (5) have an exponential form and significantly differ in value. The thick lines in Fig. 4 show the results of pairwise subtraction (9), the thin ones correspond to the results of comparison with chosen standard $\hat{g}$.

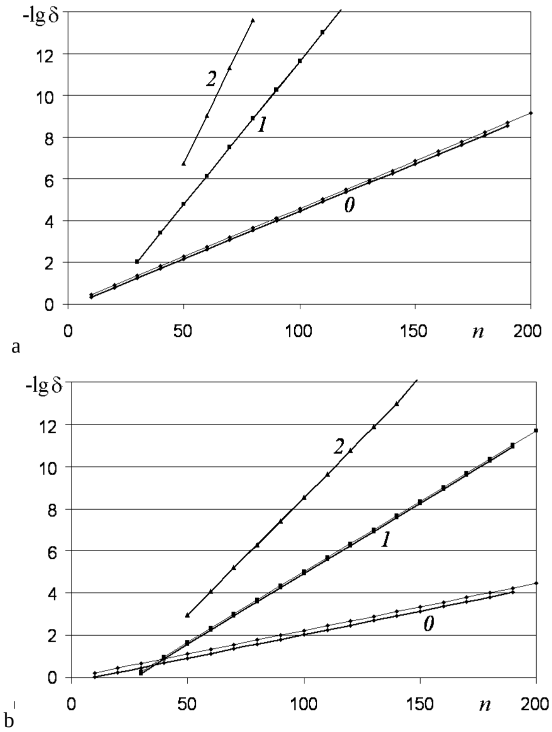

Fig. 4. The error estimate of the calculation of the sums (12). a: for q = 0.95; b: for q = 0.975

The correctness of the choice of standard is verified by superposition of the graphs obtained after the pairwise subtraction and after the comparison with the standard. The absence of significant differences of the two steps, in spite of some changes of the error components, confirms the validity of the chosen standard and confirms the accuracy of the estimates. One can see that with the help of two steps of filtering (the exclusion of the two components of (5)) an accuracy of the order of 14 digits can be achieved for n=75 – 150.

If $n_i=n_{i-1}+10$, formula (7) takes the form

$$g_{n_i}^{(j)} = g_{n_i}^{(j-1)} + \frac{g_{n_i}^{(j-1)} - g_{n_{i-1}}^{(j-1)}}{\lambda_j^{-10} - 1}$$

for exponential functions $\varphi_j(n) = \lambda_j^n$ in (5).

The products of q and ζ with different exponents are taken as the numbers λ_j for the problem of summing the series in (12), so λ_j are complex numbers in general case, and $|\lambda_j| < 1$.

Let us consider the value $s_{\min} = S_{\min}/S$ of the minimal gap calculated for n = 5, 6, 7, ..., 70 (Fig. 5, the symbols correspond to Fig. 2) in order to estimate the relative error of the results of solving the problem by the collocation method.

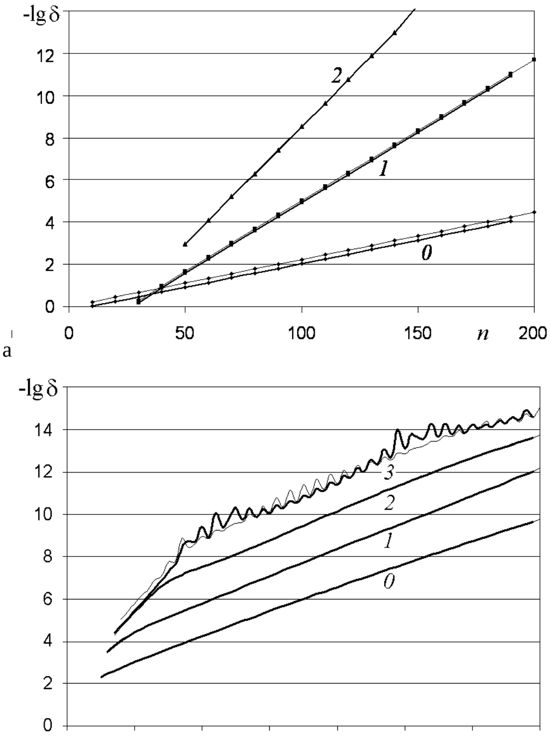

Fig. 5. Error estimation of parameters $S_{\min}/S$. a: for r_1 = 30, r_2 = 60; b: for r_1 = 30, r_2 = 15

In this case the difference between the error components (9) is taken into account by modifying the transformation (7)

$$\zeta_{n_i}^{(0)} = \frac{g_{n_i} - g_{n_{i+1}}}{1-\lambda_1},$$

$$\zeta_{n_i}^{(j)} = \left[\zeta_{n_i}^{(j-1)} + \frac{\zeta_{n_i}^{(j-1)} - \zeta_{n_{i-1}}^{(j-1)}}{\lambda_j^{-1} - 1}\right]\frac{1-\lambda_j}{1-\lambda_{j+1}} = \frac{g_{n_i}^{(j)} - g_{n_{i+1}}^{(j)}}{1-\lambda_{j+1}}, \quad j=1,2,\ldots$$

The values λ_j are approximately defined by the differences as in (4)

$$\lambda_j \approx \frac{g_n^{(j-1)} - g_{n-1}^{(j-1)}}{g_{n-1}^{(j-1)} - g_{n-2}^{(j-1)}}.$$

The difference between the curves corresponding to the two steps of filtering practically disappears as the result of the correction. Note, under conditions of an observable dependence of the absence of a roundoff error (as in Fig. 5), the upper line *3* can be used only as confirmation of the estimates *0-2*. So, the reliable error of standard estimations is obtained in the range of 10^{-12}–10^{-13}. Moreover, this accuracy is achieved only by the filtering. The three exponential components of the dependence (5) are allocated as a result of the filtering. Also the error analysis of the collocation method shows that the values λ_j of these components are negative.

The shapes of the stationary surfaces for different values of r_1 and two ratios: r_2/r_1=2 and r_2/r_1=0.5 are shown in Figs. 6 and 7. The horizontal semi-axis length is chosen as a scale unit.

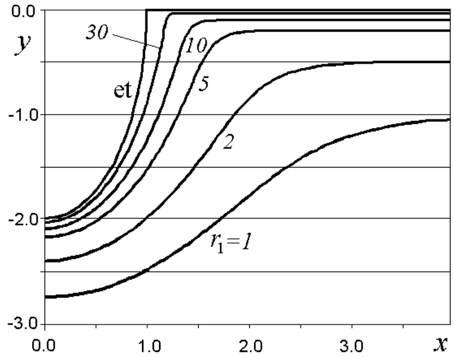

Fig. 6. The shapes of stationary surface for r_2/r_1 = 2

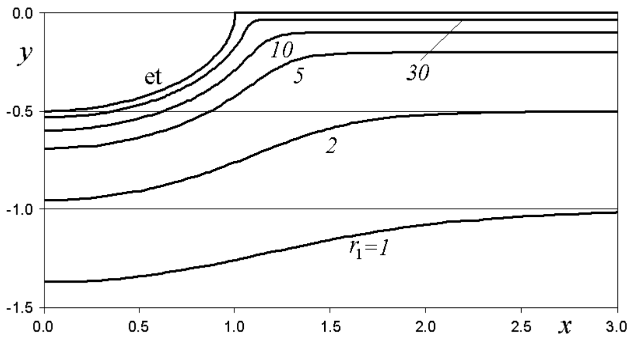

Fig. 7. The shapes of stationary surface for r_2/r_1 = 0.5

The results of the computations of the minimum relative gap for different values of r_1 and for three ratios r_2/r_1 are presented in Table 1 with twelve significant digits.

The increase of the curvature of ET leads to decrease of the gap $S_{\min}$ at the lower point of ET with respect to the asymptotic magnitude S, as well as a decrease of r_1.

The testing of the numerical method and the program was carried out by comparing the determined intervals of uncertainty with the exact solutions. If the image of the boundary of ET is chosen as a circle in the hodograph plane, then the problem is solved in quadratures by conformal mapping. ET in this case has some curved shape. The shapes close to an elliptical one can be chosen from among the two-parameter set of solutions. We verify the reliability of the resulting estimations by comparing these exact solutions with solutions obtained by the numerical method with the corresponding conditions.

TABLE I. THE RELATIVE GAP VALUES S_{min}/S

r_1	r_2/r_1=0.5	r_2/r_1=1	r_2/r_1=2
1	0.871177047008	0.807818278323	0.745845665207
2	0.907630863484	0.860141116105	0.805284431520
5	0.956781025986	0.925719387535	0.881681127922
10	0.977014995175	0.957750456415	0.926732023393
15	0.984289647682	0.970378519843	0.946635477428
20	0.988056998986	0.977174457786	0.957968900568
25	0.990364284304	0.981427734633	0.965310009279
30	0.991923409491	0.984342452243	0.970459233399

VII. CONCLUSION

The existing methods of error estimation and more precise result definition give the opportunity to obtain reliable results for a bounded range of parameters. But it is very difficult to estimate the boundaries of this range a priori.

A method of filtering that does not involve assumptions about the smallness of the residual term has been described. The method is based on excluding those error components that have a regular character. To estimate the error, the results of the filtering are compared with a single standard selected interactively or by filtering the differences of the results, eliminating the uncertainty

A particular problem, the Hele-Shaw problem, was considered as an example. Two steps of filtering complex data with the comparison of two stages of the results has been used directly for the solution of the problem (for the series summation and the numerical integration) and for post-processing data of a numerical experiment. This made it possible to obtain reliable error estimations and to improve significantly the efficiency of the numerical algorithms. These estimations have been confirmed with the help of exact partial solutions. The obtained values can be applied for testing programs of further computing for the formation for other shapes of the electrode-tool.

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