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A Numerical Algorithm of Determining the Coefficients and Functions of Sources in the Filtration Equation

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***Abstract*—The inverse problems of recovering the right-hand side and coefficients in a pseudoparabolic equations of filtration with the use of the pointwise overdetermination are studied. We expose some existence and uniqueness theorems which are the base of a numerical algorithm of recovering the right-hand side (the source function), left-hand side (coefficient problem) and a solution. The problem is well-posed and the stability estimates hold. It can be reduced to a Volterra-type integral equation, where the operator has a small norm for small time segments. The finite element method is used to reduce the problem to a system of ordinary differential equations which is solved by the finite difference method. The idea of the predictor-corrector method is employed in the algorithm. The results of numerical experiments are presented. They show a good convergence of an approximate solutions to a solution. Also this article can develop models and algorithms for modeling situations in a decision support system. For problems arising in determining the parameters of the reservoir where oil is produced or determining the flow rates of wells.**

Keywords—inverse problem, pseudoparabolic equation, filtration, fissured rock, numerical solution

I. INTRODUCTION

In the present article we consider an inverse problem of recovering the right-hand side in a Sobolev-type equation of the third order. These equations belong to the class of equation unsolvable with respect to higher derivatives. The systematical study of equations of this class began with the S.L. Sobolev articles (see [1]). Afterwards, S.L. Sobolev's results were generalized by many authors. We can refer to the known results by S.A. Galpern, S.G. Krein, M.I. Vishik, R. Schovalter, T.I. Zelenyak, G.V. Demidenko, and S.I. Uspenskii, and many other authors (see the bibliography in [2]). The most known third order Sobolev-type models are the equation of Rossby waves [3] proposed by Rossby C.G. in 1939 and the filtration theory equations derived by Barenblatt G.I., Zheltov Iu.P. and Kochina I.N. [4] in 1960. The latter model is written as

$$u_{1t} - \eta\Delta u_{1t} - k\Delta u_1 = 0 \tag{1}$$

where the parameter k corresponds to the piezo-conductivity of fissured rock and u is the pressure. The dimensionless coefficient characterizes the intensity of the liquid transfer between the blocks and fissures. More general models can include nonlinearities arising from fluid type (a liquid or a gas), concentration (porosity, absorption or saturation) and the exchange rate [5].

General equations of the form (1) can be written as follows:

$$L(t,x,D)u_t - L_1(t,x,D)u = f,$$
$$(x,t) \in Q = G \times (0,T), \tag{2}$$

where L, L_1 are second order operators and G is a bounded domain in R^n. The equation (2) is furnished with initial and boundary conditions of the form

$$u(0,x) = u_0(x), Ru|_s = g(t,x), \tag{3}$$

with $Ru=u$ or $Ru = \sum_{i=0}^{n} \gamma_i(t,x)u_{x_i} + \sigma(t,x)u$ (other boundary conditions are also possible).

Sobolev-type equations of the form (2) with various differential operators L_1 and L_2 of even order in the spacial variables also arise in the mathematical models of the heat conduction, wave processes, quasistationary processes in semiconductors and magnetics, in the models for filtration of the two-phase flow in porous media with the dynamic capillary pressure (see [6-7] and the bibliography therein). Detailed bibliography and the results concerning the solvability of direct problems for Sobolev-type equations and their abstract analogs can be found, for instance, in [8-10]. The first results devoted to inverse problems for pseudoparabolic equations were obtained in [11], where an inverse problems of recovering an unknown source f of a special form in (2) is considered. Large number of results is exposed in the monographs [12-13]).

The problems of recovering coefficients, in particular, the coefficients *k(t)* and are studied in [14, 15], where integral overdetermination conditions are used. The problem (2), (3) is considered in previous work and it is proven that this problem is uniquely solvable under natural conditions for the data. Closed results on recovering the right-hand side of the form *f(t)g(x)* (the function *f(t)* is not known) are exposed in [16, 17] even for more general classes of the equations. Exposition of numerical methods for solving inverse problem can be found, for instance, in [18, 19]. At the same time, the number of articles devoted to numerical solving inverse problems for Sobolev-type equations is rather limited (see, for instance, [21, 22]. Most of the articles are devoted to different model problems. Some numerical methods for solving filtration problems of the form (2)(3) but for simpler models are presented in [20]. Here the Sobolev-type equation for the pressure is replaced with a parabolic one.

We use the theoretical results exposed in our previous works, where the existence and uniqueness theory as well as the stability estimates for solutions can be found, describe numerical methods applicable to a wide class of inverse problems with the pointwise ovedetermination of the form (2), (3), and present the results of numerical experiments.

II. PRELIMINARIES

We consider a general inverse problem on recovering functions occurring into the right-hand side and left-hand side of the equation. We look for the right-hand side f of the form

$$f=\sum_{i=1}^{r} q_i(t)f_i(t,x)+f_0(t,x), \quad f_i\in L_\infty(0,T,L_p(G)) \quad (4)$$

Where the functions f_i are given. Also we assume that

$$L=\sum_{i,j=1}^{n} a_{ij}(t,x)\partial_{x_ix_j}+\sum_{i=1}^{n} a_i(t,x)\partial_{x_i}+a_0(t,x)$$

and the operator L_1 is respensentable as

$$L_1u=L_0u+\sum_{k=r+1}^{m} q_k(t)L_1^k u$$

$$L_1^k u=\sum_{i,j=1}^{n} b_{ij}^k(x,t)u_{x_ix_j}+\sum_{i=1}^{n} b_i^k(x,t)u_{x_i}+b_0^k(x,t)u$$

Our problem is stated as follows: find the functions$\{q_i(t)\}_{i=1}^m$ and a solution u to the problem (1)-(2). such that

$$u(t,y_i)=\Psi_i(t), \quad (i=1,2,..,m), \quad (5)$$

where y_i are arbitrary points lying in G.

Put $(u,v)=\int_G u(x)v(x)dx$. All function spaces as well as coefficients of the equations are assumed to be real.

We employ the Sobolev spaces $W_p^s(G)$and Holder spaces $C^\alpha(G)$ (see the definitions in [23]). The symbol $L_p(0,T;H)$ (H is a Banach space) stands for the space of strongly measurable functions defined on with values in H. Given an interval $J=(0;T)$ and a domain $G\in R^n$, put $Q=(0,T)\times G$ and $W_p^{r,s}(Q)=W_p^r(J;L_p(G))\cap L_p(J;W_p^s(G))$. Respectively, $W_p^{r,s}(S)=W_p^r(J;L_p(\Gamma))\cap L_p\left(J;W_p^s(\Gamma)\right)$ $(S=(0,T)\times\partial G)$. Similarly, we can define the Holder spaces $C^{r,s}(\bar{Q})$.

Next, we describe the condition on the data of the problem. We assume that the operator L is elliptic, i. e., there exists a constant $\delta>0$ such that

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j\geq\delta_0|\xi|^2 \quad \forall\xi\in R^n,\forall(t,x)\in\bar{Q} \quad (6)$$

Fix a parameter $p>n$ and assume that

$$b_{ij}^k\in L_\infty(Q),(i=1,2\ldots,m), b_i^k,b_0^k\in L_\infty\left(0,T;L_p(G)\right) \quad (7)$$

$$a_{ij}\in C(\bar{Q}), a_i,a_0\in C\left([0,T];L_p(G)\right) \quad (i,j=1,2,..,n), \quad (8)$$

$a_0(t,x)\leqslant 0$ a.e. (almost everywhere) in Q in the case of the Dirichlet boundary conditions and $a_0^0\leq 0$ a.e. in Q and $a_0^0<0$ a.e. in some neighborhood about S and $\sigma\geq 0$ in S in the case of the oblique derivative problem.

$$\gamma_i,\gamma_{it},\sigma,\sigma_t\in C^{1/2,1}(S), \quad i=1,2,\ldots n \quad (9)$$

$$\psi_i(0)=u_0(y_i) \quad (i=1,2,..,m), \quad R(0,x,D)\,u_0|_\Gamma=g(0,x) \quad (10)$$

Let $s_0=2-1/p$ in the case of the Dirichlet boundary condtions and $s_0=1-1/p$ otherwise. We can determine a function $\Phi\in C([0,T];W_p^2(G))$ $(p>n)$ such that $\Phi\in L_p(0,T;W_p^2(G))$ $\Phi|_{t=0}=u_0(x)$ and $R\Phi|_t=g$. Construct a matrix B with the rows

$$L^{-1}f_1(t,y_j),..,L^{-1}f_r(t,y_j),-L^{-1}L_{1,r+1}\Phi(t,y_j),..,-L^{-1}L_{1,m}\Phi(t,y_j)$$

where $j=1,2,\ldots,m$ and assume that the there exists a constant $\delta_0>0$ such that

$$|\det B|\geq\delta_0 \ \forall t\in[0,T]. \quad (11)$$

Here $L^{-1}f_i$ is a solution U_i to the problem $LU_i=f_i$, $U_i|_{t=0}=0$, $RU_i|_S=0$.

The following theorem follows from the results in our previous works.

Theorem 1. Let the conditions (6) – (11) be fulfilled and let

$$f_0\in L_p(Q),\ f_i\in L_\infty(0,T;L_p(G)),\ u_0(x)\in W_p^2(G),$$

$$g_t\in L_p(0,T;W_p^{s_0}(G)),\ \psi_i\in W_p^1(0,T), i=1,2,\ldots,m,\ p>n.$$

Then there exists a unique solution to $(u,q_1,\ldots,q_m)$ the problem (2) – (4) such that

$$u\in W_p^1(0,\gamma_0;W_p^2(G)),\quad q_i(t)\in L_p(0,\gamma_0)\ (i=1,2,\ldots,m).$$

A solution satisfies the estimate

$$||u||_{W_p^1\left(0,T;W_p^2(G)\right)}+\sum_{i=1}^{m}||q_i(t)||_{L_p(0,T)}\leq$$

$$c(||f_0||_{L_p(Q)}+||g_t||_{L_p\left(0,T;W_p^{s_0}(G)\right)}+\sum_{i=1}^{m}\|\Psi_i\|_{W_p^1(0,T)})$$

This theorem actually justifies the numerical algorithm presented below and the scheme of the algorithm is taken from its proof.

III. DESCRIPTION OF THE ALGORITHM

To simplify the presentation, we describe the idea of the algorithm in the model case. We rely on some integral identities. Consider the problem

$$L_0u_t+k(t)L_1u=f=f_0+\sum_{i=1}^{m}q_i(t)f_i(x,t) \quad (12)$$

$$\frac{\partial u}{\partial n}|_\Gamma=g \quad u(0,x)=u_0(x), \quad (13)$$

$$u(y_i,t)=\psi_i(t),\quad i=1,2,\ldots,m, \quad (14)$$

where

$$L_0u=-div(a_0(x,t)\nabla u_t)+b_0(x,t)\cdot\nabla u+c_0(x,t)u,$$

$$L_1u=-div(a_1(x,t)\nabla u)+b_1(x,t)\cdot\nabla u+c_1(x,t)u,$$

and a_0,a_1,c_0,c_1 are scalar functions and b_0,b_1 are vector-function of length n. The functions u and $q_i(t)$ are unknown.

We assume that all conditions of Theorem 1 for the data are fulfilled. Let $\varphi \in L_q\left(0,T;W_q^1(G)\right)$ $(\frac{1}{q}+\frac{1}{p}=1)$ be a test function and let a function u be a solution to the problem (12), (13) from the class pointed out in Theorem 1. Integrating by parts in the identity

$$(L_0u_t,\varphi)+k(t)(L_1u,\varphi)=(f,\varphi), \quad (15)$$
$$\varphi \in L_q(0,T;W_q^1(G))$$

we arrive at the equality

$$a(u_t,\phi)+k(t)b(u,\varphi)=l(\varphi)+ \quad (16)$$
$$k(t)l_1(\varphi)+\sum_{k=1}^{r} q_k(t)(f_k,\varphi)$$

where $a(u_t,\varphi)=(a_0\nabla u_t,\nabla\varphi)+(b_0\cdot\nabla u_t+c_0u_t,\varphi)$, $b(u,\varphi)=(a_1\nabla u,\nabla\varphi)+(b_1\cdot\nabla u_t+c_1u,\varphi)$,

$$l(\varphi)=(f_0,\varphi)+l_0(\varphi),$$
$$l_0(\varphi)=\int_\Gamma a_0g_t\,\varphi d\Gamma, l_1(\varphi)=\int_\Gamma a_1g\varphi dt.$$

Next, we look for a solution $\varphi_j(x,t)$ $(j=1,2,\dots,m)$ to the problem

$$L_0^*\varphi_j=\delta(x-y_j), a_0\frac{\partial\varphi_j}{\partial n}+b\cdot n\varphi_j|_\Gamma=0 \quad (17)$$

where L_0^* is a formally adjoint to L_0 and δ is Dirac delta-function. Inserting φ_j in (16), we obtain that

$$\psi_{jt}+k(t)\left(b(u,\varphi_j)-l_1(\varphi_j)\right)=$$
$$l(\varphi_j)\ \textstyle\sum_{k=1}^{m} q_k(t)(f_k,\varphi_j) \quad (18)$$

Hence, we conclude that

$$\sum_{k=1}^{rr} q_k(t)(f_k,\varphi_j)=\psi_{jt}$$
$$+k(t)\left(b(u,\varphi_j)-l_1(\varphi_j)\right)-l(\varphi_j) \quad (19)$$

Let $k^0=(l_0(\varphi_j)-\psi_{jt})/\left(b(u_0,\varphi_0)-l_1(\varphi_0)\right)|_{t=0}$. We can to determine thr next iteration k^{i+1} from one of the equalities

$$k^{i+1}(t)=(l(\varphi_0)-\psi_t)/\left(b(u^{i+1},\varphi_0)-l_1(\varphi_0)\right) \quad (20)$$
$$k^{i+1}=(l(\varphi_0)-\psi_t$$
$$-k^ib(u^{i+1}-u_0,\varphi_0))/\left(b(u_0,\varphi_0)-l_1(\varphi_0)\right) \quad (21)$$

Note that the definition of φ_j implies that

$$\sum_{k=1}^{m} q_k(t)(f_k,\varphi_j)=\sum_{k=1}^{m} q_k(t)L^{-1}f_k(y_j,t)$$

Thus, this expression can be written as $R\vec{q}$ ˙ and in view of the condition (11) the determinant of the matrix R does not vanish. The above integral identities allow us to construct the iteration procedure realized in the proof of Theorem 1 (see in our previous works). Let $\vec{q}_0=R^{-1}F_0$, with $F_{0j}=\psi_{jt}+k(t)\left(b(u_0,\varphi_j)-l_1(\varphi_j)\right)-l(\varphi_j)$. Given a vector function $\vec{q}_i$, we can construct u^{i+1} as solution to the problem (12), (13) with $\vec{q}=\vec{q}_i$ and to determine the next iteration $\vec{q}_{i+1}$ from the equalities

$$\vec{q}_{i+1}=R^{-1}F_i,\ F_i=(F_{i1},\dots,F_{im}), \quad (22)$$
$$F_{ij}=\psi_{jt}+k(t)\left(b(u_{i+1},\varphi_j)-l_1(\varphi_j)\right)-l(\varphi_j).$$

The latter formula almost corresponds to the iteration procedure in the proof of the fixed point theorem for the operator S constructed in the proof of Theorem 1 in our previous works where it is proven that the process converges.

IV. Numerical algorithm

The algorithm is iterative and relies on the finite element method. We define a triangulation of G, the mesh nodes, x_1, x_2, ..., x_N, and the corresponding piecewise linear functions $\{\varphi_i(x)\}$ (thus, $\varphi_i(x_j)=\delta_{ij}$, where δ_{ij} is Kronecker symbol. Without loss of generality, we can assume that the observation points y_j are mesh node x_{m_j} $(j=1,2,\dots,m)$. An approximate solution to (12), (13) is sought in the form $u^N=\sum_{i=1}^{n} q_i(t)\varphi_i(x)$. Assume that

$$a(u,\varphi)=(a_0\nabla u,\nabla\varphi)+(b_0\cdot\nabla u+c_0u,\varphi),$$
$$b(u,\varphi)=(a_1\nabla u,\nabla\varphi)+(b_1\cdot\nabla u+c_1u,\varphi),$$
$$l(\varphi)=\left(f_{0,}\varphi\right)+l_0(\varphi),$$
$$l_0(\varphi)=\int_\Gamma a_0\,g_t\varphi d\Gamma, l_1(\phi)=\int_\Gamma a_1\,g\,\varphi dt$$

The vector-function $C(t)=(c_1(t), c_2(t), \dots, c_N(t))^T$ is solution to the system of ordinary differential equations

$$AC_t+k(t)BC=F_0+F_1, \quad (23)$$
$$C(0)=(u_0(x_1),u_0(x_2),\dots,u_0(x_N))^T.$$

where A, B are matrices with the entries $a_{ij}=a(\varphi_j,\varphi_i)$, $b_{ij}=b(\varphi_j,\varphi_i)$, and $F_0=(F_{01},\dots,F_{0N})^T$, $F_{0i}(t)=l(\varphi_j)+k(t)l_1(\varphi_i)$, $F_1=(F_{11},\dots,F_{1N})^T$, where $F_{1j}=\sum_{k=1}^{m} q_k(t)\varphi_j(x)$. Thus, the vector F_1 is representable as $F_1=R\vec{q}$, where the matrix $R=\{r_{ij}\}$ has the entries $r_{ij}=(f_j,\varphi_i)$. We thus have that

$$AC_t+k(t)BC=F_0+R\vec{q}, \quad (24)$$
$$C(0)=(u_0(x_1),u_0(x_2),\dots,u_0(x_N))^T.$$

To solve (24) we involve the finite difference method (FDM) (the implicit scheme) and replace (24) with the finite difference equation

$$A_n\frac{C_n-C_{n-1}}{\tau}+k_nB_nC_n=F_{0,n}+R_n\vec{q}_n, C_0=C(0) \quad (25)$$

Where $n=1,2,\dots,M$, $\tau=T/M$, and F_{0n}, A_n, B_n, R_n are the values of the right-hand side in (22), and the matrices A, B at $n\tau$. We assume here that the approximation $\vec{q}_n$ is a piecewise constant vector-function taking the value $\vec{q}_n$ on $\left((n-1)\tau,n\tau\right]$. Respectively, a piecewise constant approximation of a solution $C(t)$ to (21) is a piecewise constant function equal to the vector C_n on set $\left((n-1)\tau,n\tau\right]$. An analog of the overdetermination condition is as follows:

$$\frac{(C_n)_{m_j}-(C_{n-1})_{m_j}}{\tau}=\psi_{jt}((n-1)\tau), \quad (26)$$

$$j = 1,2,\dots,m,$$

where $(C_n)_{m_j}$ is the $m_j - th$ coordinate of the vector C_n. From (26) we have that

$$\psi_{jt}\big((n-1)\tau\big) + k_n((A_n)^{-1}B_nC_n)_{m_j} \quad (27)$$
$$-\big((A_n)^{-1}F_{0,n}\big)_{m_j} = ((A_n)^{-1}R_n\vec{q}_n)_{m_j}$$

where $\mathrm{i} = 1,2,\dots,m$. Denote by a_{ij}^n the enties of the matrix $(A_n)^{-1}R_n$. We have that right-hand side in (27) can be written as $S_n\vec{q}_n$ and S_n is the matrix with entries $b_{j,l}^n = a_{m_j,l}^n$. The left-hand side is the vector G_n with the coordinates $G_j^n = \psi_{jt}\big((n-1)\tau\big) + k_n((A_n)^{-1}B_nC_n)_{m_j}$ $-\big((A_n)^{-1}F_{0,n}\big)_{m_j}$. Thus, we can consider the equation

$$G_n = S_n\vec{q}_n. \quad (28)$$

Next, describe the numerical algorithm. First, we find the vector $\vec{q}_0$ from the equality

$$G_0 = S_0\vec{q}_0. \quad (29)$$

where the vector G_0 the coordinates $G_j^0 = \psi_{jt}(0) + k_0((A_0)^{-1}B_0C_0)_{m_j} - \big((A_0)^{-1}F_{0,0}\big)_{m_j}$. Next, we put $\vec{q}_1^1 = \vec{q}_0$ and find the vector C_1^1 from the equality

$$A_n\frac{C_n^i - C_{n-1}}{\tau} + k_nB_nC_n = F_{0,n} + R_n\vec{q}_n^i, C_0 = C(0) \quad (30)$$

where $i = 1$ and $n = 1$. Let G_n^i be the vector with the coordinates $G_i = \psi_{jt}\big((n-1)\tau\big) + k_n((A_n)^{-1}B_nC_n)_{m_j} - \big((A_n)^{-1}F_{0,n}\big)_{m_j}$, and next we find the vector $\vec{q}_1^2$ from the equality

$$G_n^i = S_n\vec{q}_n^{i+1}, \quad (31)$$

where i = 1 and n = 1. Using the vector $\vec{q}_1^2$ in (30) with n = 1 and i = 2 we can find the vector C_1^2 and so on. The process is going on until $\|\vec{q}_1^{i+1} - \vec{q}_1^i\| < \varepsilon$, with $\varepsilon > 0$ is given number. Next, we take $C_1 = C_1^{i+1}$, $\vec{q}_1 = \vec{q}_1^{i+1}$. Assume that we have found the vectors $C_{n-1}, \vec{q}_{n-1}$. We take $\vec{q}_n^1 = \vec{q}_{n-1}$ and calculate the vector C_n^1 from (30) with i = 1. Define the vector G_n^1 and find the vector $\vec{q}_n^2$ from (31) with i = 1. We repeat the arguments until $\|\vec{q}_1^{i+1} - \vec{q}_1^i\| < \varepsilon$. In this case we put $\vec{q}_n = \vec{q}_n^{i+1}$, $C_n = C_n^{i+1}$. Repeating the arguments we can calculate all quantities $\vec{q}_i$ and C_i for $i = 1,2,\dots,m$.

V. THE RESULTS OF NUMERICAL EXPERIMENTS

In this section we analyze the results of numerical experiments. The described numerical algorithm was implemented in the Matlab. It is one of the most suitable choices, since it has a fairly convenient environment for programming and implementing various calculations, as well as the ability to visualize the results. As a result of calculations, we obtain approximate values of a solution $(u(x,y,t), \vec{q}(t), k(t))$ of the problem (12)-(14) at points $(t_1, t_2, \dots, t_N)$. To establish the stability of the solution definition under random data perturbations, we carry out numerical experiments by adding random perturbations to the redefinition data $\widetilde{\psi}_t(x_0,t) = \psi_t(x_0,t)(1 + \delta(2\sigma - 1))$, where δ defined by user and $\sigma \in [0,1]$ determined by Matlab random number generator. Here the point (x,y) belongs to the unit circle centered at $(0,0)$, $t \in (0,1)$.If $\delta \neq 0$ then the perturbations of the overdetermination data at the moments of time Δtk, $k = 1,2,\dots,N$ (Δt is a step in time).

As an example, consider the mathematical model of Barenblatt–Zheltov–Kochina type and problem

$$u_t - \xi\Delta u_t - k(t)\Delta u = f \quad (32)$$
$$= f_0 + \sum_{i=1}^{2} q_i(t)f_i(x,t)$$
$$\frac{\partial u}{\partial n}|_\Gamma = g(x,y,t), \quad u(x,y,0) = u_0(x,y)$$

We will use the Neumann boundary conditions from (13) which are represented as

$$\frac{\partial u}{\partial n}|_\Gamma = \big(u_x x + u_y y\big) = g.$$

First, we describe the input data, which have the following form:

the initial data: $u|_{t=0} = (x^2+1)\cdot(y^2+1)$;

the Neumann boundary condition: $g = 2(t+1)\cdot(y(x^2+1) + x(y^2+1))$;

the additional information: $\psi_i\big(x_{j_0}, t\big) = (x_0^2+1)\cdot(y_0^2+1)\cdot(1+t)$;

the coefficients: $a_0 = (t+1)(x^2+1), a_1 = \frac{(t^2+y+4)}{(x^2+1)}$,

$$b_{0,1} = xt,\ b_{0,1} = yt,\ b_{1,1} = \frac{y^2}{t},\ b_{1,2} = \frac{xt}{1+y},$$
$$c_0 = \frac{ty}{(x^2+1)},\ c_1 = \frac{x+t}{2+y};$$

the right-hand side: $f = \frac{\left(2(x^2+1)^2\right)}{t+1} + \frac{(t-0.5)^3(x^2+1)(y^2+1)}{(t+1)^2} - \frac{2y+2t^2+8}{t+1} - \frac{2y}{t+1} - \frac{2(y^2+1)(y+t^2+4)}{(x^2+1)(t+1)} + \frac{2xy^2(y^2+1)}{(t+1)^2} + \frac{4x^2(y^2+1)(y+t^2+4)}{(x^2+1)(t+1)} + \frac{2txy(x^2+1)}{(t+1)(y+1)} + \frac{2(x^2+1)(y^2+1)}{t+1} - \frac{2tx^4(y^2+1)}{(t+1)^2} - \frac{2ty(x^2+1)(x+y)}{(t+1)^2} - \frac{(x^2+1)(y^2+1)}{(t^2+1)(x^2+y^2+1)}$

Graph of the sought function $u(x,y,t) = (x^2+y^2+1)(1+t)$ at point $t = 1$ present in Figure 1.

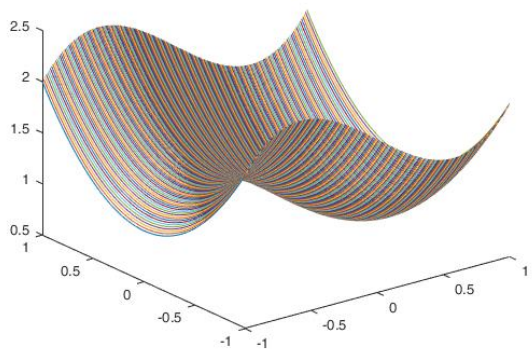

Fig. 1. Solution *u(x,y,1)*

As a result of numerical calculations we get approximations of the function $u(x,y,t)$. The obtained graphs of approximation of functions with perturbations are presented in Figure 2.

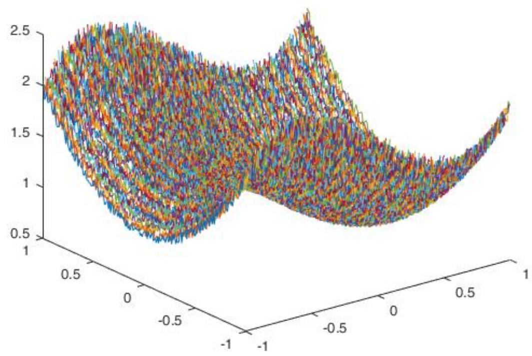

Fig. 2. Approximation of *u(x,y,1)* with perturbations 5%

Also, as a result of numerical calculations, together with the solution $u(x,y,t)$, we obtained approximations of the functions q_1, q_2 and k(t). Consider the segment $[0,T], T = 1, \Delta t = \frac{T}{N}, N = 600$. The additional information (14) is given at the observation point $x_{m_1} = (x_1, y_1) = (0.3, -0.3), x_{m_2} = (x_2, y_2) = (0.5, -0.5)$. The obtained graphs of the initial functions $q_1 = t^2 + 1, q_2 = (t-2)^3$ and its approximation are presented in Figure 3. The obtained graphs of the initial function and its approximation after adding perturbations are presented in Figure 4.

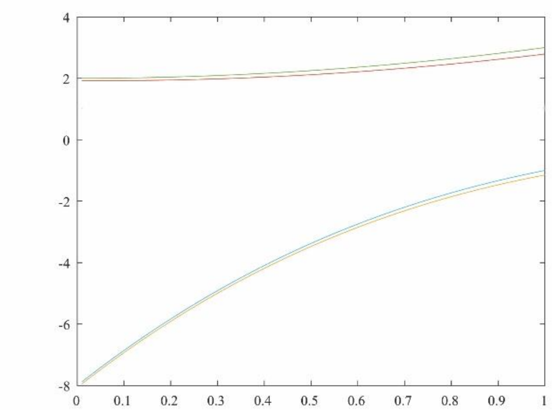

Fig. 3. Approximation of functions q_1, q_2

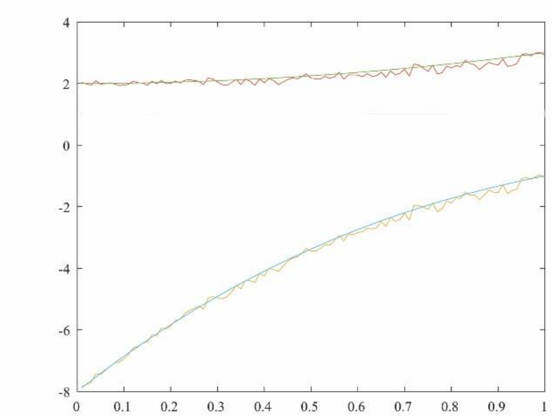

Fig. 4. Approximation of functions q_1, q_2 with perturbations 5%

The additional information (14) is given at the observation point $x_3 = (x_3, y_3) = (0.2, -0.2)$.The obtained graphs of the initial function $k(t) = t^3 - t^2$ and its approximation are presented in Figure 5.

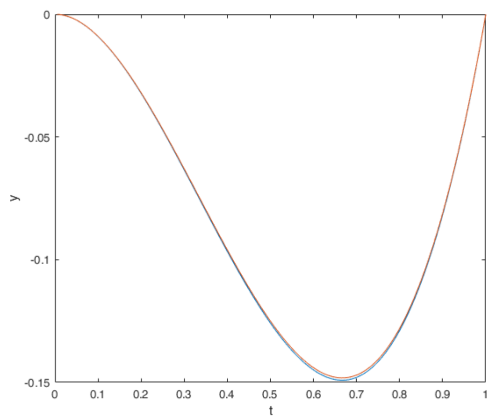

Fig. 5. Approximation of function *k(t)*

The obtained graphs of the initial function and its approximation after adding perturbations are presented in Figure 6.

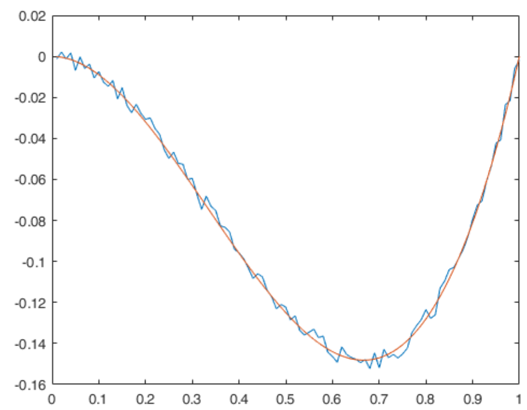

Fig. 6. Approximation of function *k(t)* with perturbations 5%

The results shows that the obtained graphs of the initial functions and their approximation pass next to each other and, regardless of the level of random perturbations, the calculation results repeat the desired functions or are located next to them. So, numerical results coincide with theoretical. Using some parameters, such as the number of layers, the error level, the accuracy of the calculation, and others, we can correct and direct the calculation process, as well as adjust the time of the calculations depending on the need.

VI. CONCLUSIONS

Under consideration is an inverse problem of recovering the right-hand side and coefficients in a pseudoparabolic equation. Some theoretical results and stability estimates for solution are exposed. We propose a numerical algorithm of recovering the right-hand side and left-hand side with the use of the pointwise overdetermination conditions. A numerical algorithms is based on the finite element method combined with the finite difference schemes. The results of numerical experiments show a sufficiently good convergence of the algorithm.

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