

Inverse Problems of Heat and Mass Transfer and Filtration Theory

Sergey Pyatkov*
Institute of Digital Economics
Yugra State University
Khanty-Mansiysk, Russia
s_pyatkov@ugrasu.ru

Valerii Rotko
Information and Analytical Department
Yugra State University
Khanty-Mansiysk, Russia
v_rotko@ugrasu.ru

Abstract—This is a survey article. We describe results on well-posedness in the Sobolev spaces of inverse problems with pointwise overdetermination for mathematical models of convection-diffusion and filtration described by parabolic equations and systems. The unknowns are time-dependent functions occurring in the source function and the operator itself as coefficients. The overdetermination conditions are the values of a solution at some collection of interior points of the domain. Conditions are imposed that guarantee the local-in-time well-posedness of the problem. It is shown that in linear problems the solvability is global in time. Stability estimates for solutions hold in all cases. Some attention is paid to the case of point sources. We describe the known results and, in particular, those obtained by the authors. The results presented are in many cases sharp. They can serve as a base for creating software packages of recovering pollution sources which can be included into regional intelligent decision support systems.

Keywords—parabolic system, inverse problem, source function, convection-diffusion, heat and mass transfer, filtration

I. INTRODUCTION

We consider the question of determining the source function and the coefficients characterizing the parameters of the medium together with a solution in quasilinear convection-diffusion mathematical models. Let G be a domain in \mathbb{R}^n with boundary Γ of class C^2 and $Q = (0, T) \times G$. The corresponding parabolic system has the form

$$Lu = u_t + A(t, x, D)u - f(t, x, u, \nabla u) = \sum_{i=1}^r f_i(t, x)q_i(t) + f_0(t, x), \quad (t, x) \in Q, \quad (1)$$

where A is a matrix elliptic operator, $A(t, x, D)u = -\sum_{i,j=1}^n a_{ij}(t, x)u_{x_j x_j} + \sum_{i=1}^n a_i u_{x_i} + a_0 u$, a_{ij}, a_i are matrices of dimension $h \times h$, u is a vector of length h and $\vec{q}(t) = (q_1(t), q_2(t), \dots, q_m(t))$ are unknown functions to be determined together with a solution u , which occur in both the right-hand side and the operator A itself as coefficients. System (1) is supplemented by the initial and boundary conditions

$$u|_{t=0} = u_0, \quad Bu|_S = g, \quad (2)$$

where $S = (0, T) \times \Gamma$, $Bu = \sum_{i=1}^n \gamma_i(t, x)u_{x_i} + \sigma(t, x)u$ or $Bu = u$ and $\gamma_i(t, x), \sigma(t, x)$ are matrices of dimension $h \times h$. The overdetermination conditions are as follows:

$$u(x_i, t) = \psi_i(t), \quad i = 1, 2, \dots, s, m = hs. \quad (3)$$

Thus, additional conditions are the data of measurements (for example, the concentration of a transferred substance) at certain points in the domain. The data are employed to determine both the sources themselves (for example, sources of pollution in water or air) and environmental parameters.

Thus, the inverse problem is to find the solution u of equation (1) and the functions $q_i(t)$ ($i = 1, 2, \dots, m$), that appear in the right-hand side or the operator A itself from the data (2), (3). In the heat and mass transfer and filtration problems, the right-hand side in (1) characterizes the distribution of sources and their intensities. In the case of point sources, i.e. $f_i = \delta(x - x_i)$ (δ is the Dirac delta function), q_i is the intensity of the i -th source in the heat and mass transfer problems (see [1]), and in filtration problems, for example, in oil production q_i is the flow rate i -th well, in this case u is the pressure (see [2]). In various practical problems distributed and point sources as well are both employed.

First, we describe some results devoted to problems with spatially distributed sources. A large number of results were obtained in the case of a function f linear in its arguments. We can note the work [3], where a theorem on the existence and uniqueness of solutions to problem (1)-(3) on determining the source in Holder spaces in the case $h = 1$, $r = 1$ was obtained. Similar results in the case of the problem of determining the source function and in the case of more general coefficient problems were obtained in the monograph [4] but in a one-dimensional situation ($n=1$). In [5] the problem of determining the lowest coefficient in a parabolic equation was considered, and in [6], the lowest coefficient and the right-hand side of the form $q(t)f(t, x)$ are determined. In both cases, the well-posedness of the corresponding inverse problems is proven. There are many articles devoted to model equations of the form (1) mainly in the one-dimensional situation (see, for example, [7,8]). The first most significant work devoted to the quasilinear case is the article [9], where the conditions on a nonlinear function were derived that guarantee the global solvability of the problem (1)-(3) in time in the Hölder spaces for case $r=1$. This condition is nothing more than a linear growth of the function f in its arguments $u, \nabla u$. The authors of [10] obtained a similar result already in the case of a parabolic system and in the Sobolev spaces. The problem of local well-posedness of problem (1)-(3) in the Sobolev spaces was further considered in the articles of the authors [11-13]. We can refer to the monograph [14], where inverse problems for quasilinear equations of the form (1) in the abstract case are considered, i.e. for an operator-differential equation of the first order. The results are of interest, but in many cases their applications do not provide optimal conditions for the data and they are not always applicable. Some refinements of these results in the case of the Sobolev spaces were obtained in [15].

In the monograph [16], the questions of well-posedness of more general classes of inverse problems are considered when the overdetermination conditions are given on sections of a spatial domain or on some spatial manifolds [17,18]. Problem (1)-(3) is often a special case in such statements.

A huge amount of articles is devoted to the numerical solution of problems of the form (1)-(3). We can refer, for example, to the articles [19-21]. There is a large number of monographs devoted to numerical methods for solving such inverse problems. Almost all statements and a large number of methods are considered in the monograph [22] in the case $n=1$. The monographs [23,24] are devoted to a more general situation; number of interesting statements and problems (including convective heat transfer problems) are considered in [25,26].

Describe some results in the case of point sources. As already noted, these problems are not well-posed in the classes of finite smoothness, and there are practically no existence and uniqueness theorems for solutions (see [27]).

There is a huge number of articles devoted to the numerical solution of the problem of determining point sources, however, as a rule, these articles do not contain any theoretical justifications and very often both non-existence of solutions and their non-uniqueness can take place in the corresponding problems for certain values of parameters. The articles [28-30] can serve as examples. Let us single out the articles, where there is some theoretical justification and justified algorithms for finding solutions [31-35]. The most interesting idea of constructing point sources is presented in [36]. It was subsequently used in [34]. Note that the problems of determining point sources are nonlinear, in contrast to the case of distributed sources. Here it is necessary to determine the number of sources, their location and intensities.

Now, we describe the content of the article. First, we introduce some notations. Next, the main results concerning with determining distributed sources are exposed. At the end of the article we present some descriptions of the algorithms for finding point sources in the simplest cases. The notation of function spaces is standard (see, for example, [37]).

II. PRELIMINARIES

Let E be a Banach space. By $L_p(G; E)$ (G is a domain in \mathbb{R}^n) we mean the space of strongly measurable functions defined on G with values in E endowed with the norm $\|u\|_{L_p(G; E)}$ [37]. We employ also the Holder spaces $C^\alpha(\bar{G})$. The notations $W_p^s(G; E)$, $W_p^s(Q; E)$ of the Sobolev spaces are standard (see definitions in [37]). If $E = \mathbb{C}$ ($E = \mathbb{R}$) or $E = \mathbb{C}^n$ ($E = \mathbb{R}^n$) then the last space is denoted by $W_p^s(Q)$. Similarly, we use the notation $W_p^s(G)$ or $C^\alpha(\bar{G})$ instead of $W_p^s(G; E)$ or $C^\alpha(\bar{G}; E)$. Thus, the inclusion $u \in W_p^s(G)$ (or $u \in C^\alpha(\bar{G})$) for a given vector function $u = (u_1, u_2, \dots, u_k)$ means that each of its components u_i belongs $W_p^s(G)$ (or $C^\alpha(\bar{G})$). In this case, the norm of the vector is simply the sum of the norms of the coordinates. We accept the same convention for matrix functions. For the interval $J = (0, T)$ we put $W_p^{s,r}(Q) = W_p^r(J; L_p(G)) \cap L_p(J; W_p^s(G))$. Respectively, we have $W_p^{s,r}(S) = W_p^r(J; L_p(\Gamma)) \cap L_p(J; W_p^s(\Gamma))$. The Holder spaces $C^{r,s}(\bar{Q})$ are defined similarly.

Let the symbol $B_\delta(x_i)$ stand for the ball of radius δ centered at x_i . Further we use the following notation: $Q^\tau = (0, \tau) \times G$, $S^\gamma = (0, \gamma) \times \Gamma$. Given a set of points $\{x_j\}$ from

(3), a parameter $\delta > 0$ is called admissible if $\overline{B_\delta(x_i)} \subset G$, $\overline{B_\delta(x_i)} \cap \overline{B_\delta(x_j)} = \emptyset$ for $i \neq j$, $i, j = 1, 2, \dots, r$. Let $G_\delta = \cup_i B_\delta(x_i)$, $Q_\delta = (0, T) \times G_\delta$, $Q_\delta^\tau = (0, \tau) \times G_\delta$.

III. DISTRIBUTED SOURCES

The conditions on the coefficients of L are as follows:

$$a_{ij} \in C(\bar{Q}), a_k \in L_p(Q), p > n + 2; \quad (4)$$

$$a_{ij} \in L_\infty(0, T; W_p^s(G_\delta)),$$

$$a_k \in L_p(0, T; W_p^s(G_\delta)), i, j = 1, 2, \dots, n, \quad (5)$$

$$k = 0, 1, \dots, n,$$

for some admissible $\delta > 0$ and $s \in (n/p, 1]$. Describe the parabolicity condition of the operator L . Consider the matrix $A_0(t, x, \xi) = \sum_{i,j=1}^n a_{ij}(t, x) \xi_i \xi_j$ ($\xi \in \mathbb{R}^n$) and suppose that there exists a constant $\delta_1 > 0$ such that the roots p of the polynomial $\det(A_0(t, x, i\xi) + pE) = 0$ (E is the identity matrix) satisfy the condition

$$\operatorname{Re} p \leq -\delta_1 |\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \quad (6)$$

$$\forall (x, t) \in Q.$$

Let $B_0 u = u$ in the case of the Dirichlet conditions in (2) and $B_0 u = \sum_{j=1}^n \gamma_j \partial_{x_j} u$ otherwise. The Lopatinsky condition can be written in the following form: for any point $(t_0, x_0) \in S$, and the operators $A_0(x, t, D)$ and $B_0(x, t, D)$, written in the local coordinate system y at this point (the axis y_n is directed along the normal to S and the axis y_1, \dots, y_{n-1} lie in the tangent plane at the point (x_0, t_0)), the system

$$(\lambda E + A_0(x_0, t_0, i\xi', \partial_{y_n}))v(z) = 0, \quad (7)$$

$$B_0(x_0, t_0, i\xi', \partial_{y_n})v(0) = h_j,$$

where $\xi' = (\xi_1, \dots, \xi_{n-1})$, $y_n \in \mathbb{R}^+$, has a unique solution from $C(\mathbb{R}^+)$ bounded at infinity for all $\xi' \in \mathbb{R}^{n-1}$, $|\arg \lambda| \leq \pi/2$, and $h_j \in \mathbb{C}$ such that $|\xi'| + |\lambda| \neq 0$. We also assume that

$$u_0(x) \in W_p^{2-\frac{2}{p}}(G), g \in W_p^{2k_0, k_0}(S), \quad (8)$$

$B(x, 0)u_0(x)|_\Gamma = g(x, 0) \quad \forall x \in \Gamma$, where $k_0 = 1 - 1/2p$ in the case of the Dirichlet conditions and $k_0 = 1/2 - 1/2p$ otherwise,

$$u_0(x) \in W_p^{2+s-\frac{2}{p}}(G_\delta), s \in (n/p, 1]. \quad (9)$$

for some admissible $\delta > 0$. The constants $\delta_1 < \delta_2 < \delta$ are given. We construct an auxiliary function $\varphi \in C_0^\infty(G_\delta)$ such that $\varphi \equiv 1$ in the domain G_{δ_1} and $\varphi \equiv 0$ в $G_\delta \setminus G_{\delta_2}$. Assume that

$$f_i \in L_\infty(0, T; L_p(G)) \cap L_\infty(0, T; W_p^s(G_\delta)), \quad (10)$$

$$\text{где } i = 1, \dots, m, m = r, \quad (11)$$

$$\psi_j \in W_p^1(0, T), p > n + 2$$

$$u_0(x_j) = \psi_j(0) \quad (j = 1, 2, \dots, s), \quad (12)$$

$$\gamma_i, \sigma \in C^{\frac{1}{2}-\frac{1}{2p}+\varepsilon_0, 1-\frac{1}{p}+2\varepsilon_0}(\bar{S}), \varepsilon_0 > 0. \quad (13)$$

We can construct a matrix $B(t)$ of dimension $m \times m$, whose rows with numbers from $(j-1)h+1$ to jh are occupied by the column vectors

$$(f_1(x_j, t), f_2(x_j, t), \dots, f_m(x_j, t)). \quad (14)$$

Condition (A). There exists a number $\delta_0 > 0$ such that $|\det B(t)| \geq \delta_0$ a. e. on $(0, T)$.

Theorems 1-3 below were obtained in [11]-[13]. In the following theorem, we assume that the function f in (1) is equal to zero.

Theorem 1. Let condition (A) and conditions (4)-(13) be satisfied for some admissible $\delta > 0$ and $s \in (n/p, 1]$, $f_0 \in L_p(Q)$, $f_0\varphi \in L_p(0, T; W_p^s(G_\delta))$. Then there exists a unique solution $u \in W_p^{2,1}(Q)$, $q_i(t) \in L_p(0, T)$, $i=1, \dots, m$ to the problem (1)-(3) such that $\varphi u \in L_p(0, T; W_p^{2+s}(G_\delta))$, $\varphi u_t \in L_p(0, T; W_p^s(G_\delta))$.

Now we present a theorem on solvability of the problem (1)-(3) in the case when the unknown functions q_i occur in the operator A as the lower order coefficients. As previously, we assume that $f=0$. In this case, the operator A has the form

$$A = L_0 - \sum_{i=r+1}^m q_i(t)l_i(t, x),$$

$$L_0 u = - \sum_{i,j=1}^n a_{ij}(t, x)u_{x_j x_j} + \sum_{i=1}^n a_i u_{x_i} + a_0 u,$$

$$l_k u = \sum_{i=1}^n a_i^k(t, x)u_{x_i} + b_k(t, x)u,$$

where

$$a_i^k, b_k \in L_\infty(0, T; L_p(G)) \cap L_\infty(0, T; W_p^s(G_\delta)). \quad (15)$$

Construct a matrix $B(t)$ of dimension $m \times m$, whose rows with numbers from $(j-1)h+1$ to jh are occupied by the column vectors

$$(f_1(x_j, t), f_2(x_j, t), \dots, f_r(x_j, t), lu_0(x_j), \dots, l_m u_0(x_j)).$$

Condition (B). There exists a number $\delta_0 > 0$ such that $|\det B(t)| \geq \delta_0$ a. e. at $(0, T)$.

Theorem 2. Let condition (B) and conditions (4)-(15) be satisfied for some admissible $\delta > 0$ and $s \in (n/p, 1]$, $f_0 \in L_p(Q)$, $f_0\varphi \in L_p(0, T; W_p^s(G_\delta))$. Then there is a number $\tau_0 \in (0, T]$ such that on the interval $[0, \tau_0]$ there exists a unique solution $(u, q_1, q_2, \dots, q_m)$ problem (1)-(3) such that $u \in L_p(0, \tau_0; W_p^2(G))$, $u_t \in L_p(Q^{\tau_0})$, $q_i(t) \in L_p(0, \tau_0)$, $i = 1, \dots, m$.

Moreover,

$$\varphi u \in L_p(0, \tau_0; W_p^{2+s}(G)), \varphi u_t \in L_p(0, \tau_0; W_p^s(G)).$$

Now we consider problem (1) - (3) on recovering the right-hand side of an equation of the form $\sum_{i=1}^r f_i(x, t)q_i(t) + f_0(x, t)$ and the coefficients, in particular, occurring in the main part of equation (1). Suppose that the operator A is of the form

$$A = L_0 - \sum_{k=r+1}^m q_k(t)L_k,$$

$$L_k u = - \sum_{i,j=1}^n a_{ij}^k(t, x)u_{x_j x_j} + \sum_{i=1}^n a_i^k(t, x)u_{x_i} + a_0^k(t, x)u,$$

where $k = 0, r+1, r+2, \dots, m$. Since the unknowns can enter the main part of the equation, we will search for them in the class $C([0, T])$. We can construct a matrix $B(t)$ of dimension $m \times m$ whose rows with numbers from $(j-1)h+1$ to jh are occupied by column vectors

$$(f_1(x_j, t), f_2(x_j, t), \dots, f_r(x_j, t), \dots, L_{r+1}u_0(x_j), \dots, L_m u_0(x_j)).$$

We assume that

$$a_{ij}^k \in C(\overline{Q}) \cap L_\infty(0, T; W_p^s(G_\delta)), \quad (16)$$

$$a_i^k \in L_p(Q) \cap L_p(0, T; W_p^s(G_\delta)) \quad (17)$$

$$(i, j = 1, \dots, n, \quad l = 0, 1, \dots, n),$$

$$f_i \in L_p(Q) \cap L_p(0, T; W_p^s(G_\delta)) \quad (i = 1, \dots, m), \quad (18)$$

for some admissible $\delta > 0$, $s > (2+n)/p$, and $k = 0, r+1, \dots, m$;

$$a_i^k(x_l, t), f_a(x_l, t) \in C([0, T]) \quad (19)$$

for all possible values of i, j, k, l, a . We also need the following condition.

Condition (C). There exists a number $\delta_0 > 0$ such that

$$|\det B(t)| \geq \delta_0 \text{ on } (0, T).$$

Note that the elements of the matrix B belong to $C([0, T])$. Consider the system

$$\psi_{jt}(0) + L_0 u_0(x_j, 0) - f(0, x_j, u_0(x_j), \nabla u_0(x_j)) =$$

$$= \sum_{k=1}^r q_{0k} f_k(x_j, 0) + \sum_{k=r+1}^m q_{0k} L_k u_0(x_j)$$

relative to the vector $\vec{q}_0 = (q_{01}, q_{02}, \dots, q_{0m})$. Under condition (C), the system has a unique solution. Denote $A_0 = L_0 - \sum_{k=r+1}^m q_{0k} L_k$. Let B_R be the ball of radius R centered at zero in $\mathbb{R}^{(n+1)h}$.

Condition (D). The function $f(t, x, u, p)$ is continuous in variables $(u, p \in \mathbb{R}^{(n+1)h})$, for any $R > 0$, there is a constant $M_0 > 0$ such that

$$|f(t, x, u^1, p^1) - f(t, x, u^2, p^2)| \leq M_0(|u_1 - u_2| + |p_1 - p_2|),$$

for all $(u^1, p^1), (u^2, p^2) \in B_R$, $f(t, x, u, p) \in C([0, T] \times \overline{B_R}; W_p^s(G_\delta))$.

Condition (E). The function $f(t, x, u, p)$ is differentiable with respect to the parameters $(u, p) \in \mathbb{R}^{(n+1)h}$ for a.e. $(t, x) \in Q_\delta$ and

$$f_u(t, x, u, p), f_{p_i}(t, x, u, p) \in C([0, T] \times \overline{B_R}; W_p^s(G_\delta))$$

for any $R > 0$, there are functions $\Phi_1(t, x), \Phi_2(t, x) \in C([0, T]; L_p(G))$ such that

$$\|f_u(t, x, u^1, p^1) - f_u(t, x, u^2, p^2)\|_{L(\mathbb{R}^h)} \leq |\Phi_1(t, x)||u_1 - u_2| + |\Phi_2(t, x)||p^1 - p^2|,$$

$$\|f_{p_i}(t, x, u^1, p^1) - f_{p_i}(t, x, u^2, p^2)\|_{L(\mathbb{R}^h)} \leq |\Phi_1(t, x)||u_1 - u_2| + |\Phi_2(t, x)||p^1 - p^2|$$

for all $(u^1, p^1), (u^2, p^2) \in B_R$ and any $R > 0$. Here the quantities f_u, f_{p_i} ($i = 1, 2, \dots, n$) are the corresponding Jacobi matrices and $\|\cdot\|_{L(\mathbb{R}^h)}$ is the norm in the space of linear continuous mappings from \mathbb{R}^h to \mathbb{R}^h .

As previously, we assume that

$$f_0 \in L_p(Q), f_0\varphi \in L_p(0, T; W_p^s(G_\delta)). \quad (20)$$

Under these conditions, the existence theorem is stated as follows.

Theorem 3. Let the conditions (C)-(E), (8), (9), (11)-(13), (16)-(20) be satisfied. Suppose also that the operator $M_0 = \partial_t + A_0$ is parabolic and the Lopatinsky condition is fulfilled, i.e., the conditions (6), (7) are satisfied. Then there is a number $\tau_0 \in (0, T]$ such that on the interval $[0, \tau_0]$ there exists a unique solution $(u, q_1, q_2, \dots, q_m)$ problem (1)-(3) such that $u \in L_p(0, \tau_0; W_p^2(G))$, $u_t \in L_p(Q^{\tau_0})$, $q_i(t) \in C([0, \tau_0])$, $i = 1, \dots, m$. In addition, $\varphi u \in L_p(0, \tau_0; W_p^{2+s}(G))$, $\varphi u_t \in L_p(0, \tau_0; W_p^s(G))$.

IV. POINT SOURCES

Consider the case of $n=1$.

$$Lu = u_t + A(t, x, D)u = \sum_{i=1}^r \delta(x - x_i) q_i(t) + f_0(t, x), (t, x) \in Q, \quad (21)$$

where A is an elliptic operator of the form $A(t, x, D)u = -a(t, x)u_{xx} + b(t, x)u_x + a_0(t, x)u$ ($x \in (a, b)$).

The initial and boundary conditions have the form $u(0, x) = u_0(x)$, $B_1u(t, a) = \varphi_1(t)$, $B_2u(t, b) = \varphi_2(t)$, where $B_1u = u$ or $B_1u = u_x + \alpha_1u$, $B_2u = u$ or $B_2u = u_x + \alpha_2u$. The overdetermination conditions are written as before, i.e.,

$$u(y_i, t) = \psi_i(t), \quad i = 1, 2, \dots, s. \quad (22)$$

It is required to find a solution to this initial-boundary-value problem, the points x_i , the number s , and the intensity $q_i(t)$, using data (22). The first thing to note is the fact that any number of measurements s does not allow us to determine these unknown quantities uniquely unless some conditions on the mutual locations of the points x_i (see [27]) are satisfied (roughly speaking, points of measurements and unknown points must alternate). A certain condition of this type arises in the multidimensional case (for $n = 2$, any three points do not lie on the same straight line, for $n = 3$, any four points do not lie on the same plane; see [34]). Under the conditions of this type and some additional conditions, the solution of the inverse problem can be written out and we can consider the question of its uniqueness and a presentation of a solution. The most interesting representations are obtained if we use the asymptotic representations of the Green's function of the corresponding elliptic problem with a parameter. In particular, for $r=1$, if the coefficients of the operator A are independent of t , then two measurements (i.e. $s=2$) allow us to determine the point x_1 and the intensity q_1 uniquely, provided that $y_1 < x_1 < y_2$.

To simplify the formula, we take $a = \alpha^2 = \text{const}$, $f_0 = 0$, $u_0 = 0$. In this case, the point x_1 can be determined from equality [35].

$$x_1 = \frac{y_1 + y_2}{2} - \frac{\alpha}{2\sqrt{\lambda}} \ln \frac{\Omega_1(\lambda)}{\Omega_2(\lambda)} - \frac{1}{2\alpha\sqrt{\lambda}} \int_{y_1}^{y_2} b(\eta) d\eta + O\left(\frac{1}{\lambda}\right),$$

$$\text{где } \Omega_i(\lambda) = \int_0^\infty \exp(-\lambda t) \psi_i(t) dt.$$

V. CONCLUSION

We describe some results on well-posedness of inverse problems with pointwise overdetermination for mathematical models of convection-diffusion and filtration described by parabolic equations and systems. The unknowns are time-dependent functions occurring in the source function and the operator itself as coefficients. The results can be applied to the problem of identifying the location and intensity of pollution sources from the measurements at some points. In particular, we prove that the pollution sources are identifiable under certain conditions. The results allow to construct software packages for identifying environment pollution sources and can be applied in some other fields as well. In many cases they are sharp both in terms of the smoothness conditions on the data and in the sense of minimality of the conditions.

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