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On Some Classes of Inverse Problems on Determining the Source Function

Safonov Egor*
Institute of digital economics
Yugra state university
Khanty-Mansiysk, Russia
dc.gerz.hd@gmail.com

Pyatkov Sergei
Institute of digital economics
Yugra state university
Khanty-Mansiysk, Russia
s_pyatkov@ugrasu.ru

***Abstract*—The inverse problem of determining the solution, the location, and intensity of a point source in the multidimensional advection-dispersion-reaction equation is considered. The equation is supplemented with the initial and boundary conditions of the Neumann or Dirichlet type. Methods for a numerical solution of similar inverse problems in the multidimensional case are considered in many articles. However, most of them are based on reducing the problem to an optimal control problem and minimizing the corresponding functional, which as a rule requires large computational capabilities and does not always lead to the desired result. Our numerical algorithm for determining the location of the source is justified using an explicit asymptotic formula for the Green function of the corresponding elliptic problem with a parameter. The intensity is determined by the Duhamel formula and the Tikhonov regularization. The numerical implementation is based on the finite element method and the finite difference method for the corresponding system of ordinary differential equations. The results of numerical experiments of recovering the location and intensity of a source are presented. Numerical experiments demonstrate good convergence. The corresponding software packages of recovering pollution sources were created which can be included into intelligent decision support system for sustainable regional development.**

Keywords—parabolic equation, inverse problem, finite element method, source function

I. Introduction

Consider the problem of determining locations and intensities of point sources in the advection-dispersion-reaction equation

$$Lu = u_t - L_0 u = \sum_{i=1}^{r} N_i(t)\delta(x - x_i) + f(x,t), \quad (x,t) \in G \times (0,T), \tag{1}$$

where $L_0 u = \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j} - \sum_{i=1}^{n} a_i(x,t) u_{x_i} - a_0(x,t)u$ and δ is the Dirac delta-function. Here $u(x,t)$ is an unknown concentration of a pollutant in water or air, the functions $N_i(t)$ are the intensities of the pollution sources, $x_i \in G$ are their coordinates, and r is the number of these sources (a description of the models is given in [1]).

To determine unknown sources, the equation (1) is supplemented with boundary and initial conditions

$$Bu|_\Gamma = \varphi(x,t) \ (\Gamma = \partial G), \ u(x,0) = u_0(x), \tag{2}$$

where $Bu = u$ or $Bu = \sum_{i=1}^{n} \gamma_i u_{x_i} + \sigma(x,t)u$ (γ is a non-tangent vector field on Γ). Generally speaking, different boundary conditions can be specified on various connected components of Γ. As overdetermination conditions, we take conditions of the form

$$u(y_j,t) = \psi_j(t), \ j = 1,2,\dots,s. \tag{3}$$

In the literature, various statements of inverse problems on determining sources are considered (see [2-11]). The conditions of the form (3) are often replaced with additional conditions on the boundary (for example, the Cauchy data are specified on Γ). In some cases the location of the sources is considered to be known and the functions $N_i(t)$ are the unknowns. Sometimes, in the statements of the problems the intensities $N_i(t)$ are constants or known functions multiplied by unknown constants, and this simplifies the problem (see [2-5]). Articles [6, 10-13], are devoted to recovering the source function in (1) with the same or similar overdetermination conditions, but the location of the sources is assumed to be known. A theoretical study of the problem (1)-(3) in the multi-dimensional case is carried out in [14-19]. In the most general statement, we need to recover a solution u, the functions $N_i(t)$, the coordinates of points x_i ($i = 1,2,\dots,m$), and the number m. We present some theoretical results which serve as justification of the numerical algorithm below based on an explicit asymptotic formula for the Green function of the corresponding elliptic problem with a parameter. The intensity is determined by the Duhamel formula and the Tikhonov regularization. The numerical implementation is based on the finite element method and the finite difference method for the corresponding system of ordinary differential equations. The results of numerical experiments of recovering the location and intensity of a source are presented.

II. Preliminaries

Let E be a Banach space. Denote by $L_p(G;E)$ (G is a domain in $\mathbb{R}^n$) the space of strongly measurable functions defined on G with values in E and the finite norm $\| \|u(x)\|_E \|_{L_p(G)}$ [20]. We also use the spaces $C^k(\overline{G};E)$ comprising functions with values in E having in G all derivatives up to the order k continuous in G and admitting continuous extensions to the closure $\overline{G}$. The definition of the Sobolev spaces $W_p^s(G;E)$ is conventional (see [20-22]). Given an interval $J = (0,T)$ and the cylinder $Q = G \times J$, assign

$$W_p^{s,r}(Q) = W_p^s(J; L_p(G)) \cap L_p(J; W_p^r(G)),$$

and

$$W_p^{s,r}(S) = W_p^s(J; L_p(\Gamma)) \cap L_p(J; W_p^r(\Gamma)).$$

We also employ the Hölder spaces $C^{\alpha/2,\alpha}(\overline{Q})$, $C^{\alpha/2,\alpha}(\overline{S})$ (see the definitions in [23]). Define also the Laplace transform $L(u)(p) = \int_0^\infty e^{-pt}\, u(t)\, dt$.

We consider the equation

$$Lu = u_t - \Delta u + q(x)u = f = f_0 + \sum_{i=1}^{m} N_i\,(t)\delta(x - x_i), \quad (4)$$

where $(x,t) \in Q = G \times (0,T)$, δ is the Dirac δ-function and $G \subset \mathbb{R}^n$ is a bounded domain with boundary $\Gamma \in C^2$, $G = \mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_n > 0\}$, or $G = \mathbb{R}^n$. Note that in the case of if a potential vector-field $\omega(x)$, i.e. $\omega(x) = \nabla\varphi(x)$ and $\varphi \in W_\infty^2(G)$) then the change of the variables $u = e^{-\varphi/2}v$ reduces the equation

$$u_t - \Delta u + \omega(x)\cdot\nabla u + k(x)u = f_0 + \sum_{i=1}^{m} N_i\,(t)\delta(x - x_i),$$

to the form

$$v_t - \Delta v + q(x)u = \sum_{i=1}^{m} N_i\,(t)e^{\frac{\varphi(x_i)}{2}}\delta(x - x_i) + f_0 e^{\frac{\varphi(x)}{2}},$$

where $q(x) = k(x) + \frac{\Delta\varphi}{2} - \frac{|\nabla\varphi|^2}{4}$. The equation (4) is furnished with the initial and boundary conditions

$$u(x,0) = u_0(x),\, Bu|_S = g(x,t), S = \Gamma \times (0,T), \quad (5)$$

where $Bu = u$ or $Bu = \frac{\partial u}{\partial \nu} + \sigma(x)u$ (ν is the unit outward normal to Γ) and the overdetermination conditions

$$u(x_i,t) = \psi_i(t),\, i = 1,2,\ldots,s. \quad (6)$$

In what follows we assume that $\sigma \in C^{1/2+\varepsilon/2,1+\varepsilon}(\overline{S})$ for some $\varepsilon > 0$. The following theorem is the solvability theorem for the direct problem (4), (5).

Theorem 1. Let $T = \infty$. The there exists $\gamma_0 > 0$ such that, for $\gamma \geq \gamma_0$, if $u_0(x) \in W_2^1(G) \cap W_p^{2-2/p}(G)$ $\left(p < \frac{n}{n-1}\right)$, $e^{-\gamma t}N_i(t) \in L_2(0,T)$, $e^{-\gamma t}f_0 \in L_2(Q) \cap L_p(Q)$, $e^{-\gamma t}g(x,t) \in W_2^{1/2,3/2}(S) \cap W_p^{1-1/2p,2-2/p}(S)$ in the case of the Dirichlet boundary conditions and $e^{-\gamma t}g(x,t) \in W_2^{1/4,1/2}(S) \cap W_p^{1/2-1/2p,1-2/p}(S)$ otherwise, and $q(x) \in L_\infty(Q)$ then there exists a unique solution to the problem (4), (5) such that $e^{-\gamma_1 t}u \in L_p\left(0,\infty; W_p^1(G)\right)$, $e^{-\gamma_1 t}u_t \in L_p(0,\infty; W_p^{-1}(G))$ for all $\gamma_1 > \gamma$, $e^{-\gamma t}u \in W_2^{2,1}(Q_\varepsilon)$ ($Q_\varepsilon = \{(x,t) \in Q : |x - x_i| > \varepsilon\ \forall i \leq r\}$) for all $\varepsilon > 0$.

Proof. Consider the auxiliary problem

$$Lv = f_0,\, v(x,0) = u_0(x),\, Bv|_S = g(x,t). \quad (7)$$

As a consequence of Theorem 5.7 in the case of $G = \mathbb{R}^n$, Theorem 7.11 and Theorem 8.2 in the remaining cases there exists a unique solution v to the problem (7) such that, for $\gamma \geq \gamma_0$ with some $\gamma_0 > 0$, $e^{-\gamma t}v \in W_p^{2,1}(Q) \cap W_2^{2,1}(Q)$ whenever the data satisfy the conditions of the theorem. The change $u = v + w$ reduces the problem (4), (5) to the problem

$$Lw = \sum_{i=1}^{m} N_i\,(t)\delta(x - x_i) = \tilde{f},\quad w(x,0) = 0,\, Bw|_S = 0. \quad (8)$$

Note that $\tilde{f} \in L_p(0,T; W_p^{-1}(G))$ and we have the estimate

$$\|e^{-\gamma t}\tilde{f}\|_{L_p(0,T;W_p^{-1}(G))} \leq c\|e^{-\gamma t}N\|_{L_p(0,T)}\ (p < 2). \quad (9)$$

In the case of $T < \infty$ we can refer to Theorem 1 in [24] or Theorem 14.2 in [25]. In this case we can conclude that there exists a unique solution to the problem (9) such that $w \in L_p(0,T; W_p^1(G))$, $w_t \in L_p(0,T; W_p^{-1}(G))$. In the case of $T = \infty$, we did not find a direct references but the result can be proven with the use of the arguments those of Theorem 14.2 in [25] and the corresponding results in [22]. Next, using the conventional results devoted to the smoothness of solutions to parabolic problems (see, for instance, [23]) we can additionally prove that $w \in W_2^{2,1}(Q_\varepsilon)$. The function $u = v + w$ meets the conditions of the theorem.

Proceed with some properties of solutions to elliptic problems with parameter. We examine the problem

$$Lu = \lambda u - \Delta u + q(x)u = \delta(x - x_0),\quad (x,t) \in Q = G \times (0,T), \quad (10)$$

where δ is the Dirac δ-function and $G = \mathbb{R}^2$ or $G = \mathbb{R}_+^2$, $\lambda \in \mathbb{R}$ is a parameter, and $q \in L_\infty(G)$.

We do not consider the case of a bounded domain here. The results in this case are close to those below and the asymptotic representation are the same. But this case requires a separate consideration. Nevertheless, the results presented here can serve as some justification of the numerical algorithm described in the next section.

In the case of $G = \mathbb{R}_+^2$, the equation (10) is furnished with the boundary conditions

$$Bu|_{x_2} = 0, \quad (11)$$

where $Bu = u$ or $Bu = \frac{\partial u}{\partial x_2}$.

Write out the asymptotics of a solution to the equation (10) decreasing at infinity.

As is known (see, for instance, sect. 3.1 in [27] or Chapters. 4, 8 in [28]) a solution to the Helmgoltz equation

$$L_0 u = \lambda u - \Delta u = \delta(x - x_0) \quad (12)$$

is written as

$$E_2(x) = \frac{i}{4} H_0^1(i\sqrt{\lambda}|x - x_0|). \quad (13)$$

Here H_0^1 is the Hankel function, $\sqrt{\lambda} = |\lambda|^{1/2} e^{iarg\,\lambda/2}$ is a branch of the root analytic on the plane with the cut $arg\,\lambda = \pi$. We have the asymptotic representations (see (2.57), p. 82 [28])

$$E_2(x) = \frac{e^{-\sqrt{\lambda}|x-x_0|}}{2\sqrt{2\pi|x-x_0|}\lambda^{1/4}}(1 + O(\frac{1}{\sqrt{|\lambda|}|x-x_0|})), \quad (14)$$

which is valid for $\sqrt{|\lambda||x - x_0|} >> 1$ and $|arg\lambda| \le \pi - \delta$ ($\delta > 0$). The function $E_2(x)$ has a logarithmic singularity as $x \to x_0$. We have $H_0^1(ix) \approx \frac{2}{\pi i} \ln \frac{2}{x}$ (see Sect. 11 [28]) and the relation $K_0(z) = \frac{i\pi}{2} H_0^1(iz)$, where K_0 is the Macdonald function. For $|\arg\ z| < \pi/2$ we have the representation (p. 201, Sect. 6.22 in [29])

$$K_0(z) = \int_0^\infty e^{-zt}\, dt.$$

Hence, $E_2(x)$ is positive for real $x > 0$ and decreases as $x \to \infty$.

Theorem 2. Let $G = \mathbb{R}^2$ or $G = \mathbb{R}^2_+$. Then for $\lambda > \lambda_0$ with some $\lambda_0 \ge 0$ there exists a unique solution u to the equation (10) in $G = \mathbb{R}^2$ or to the problem (10)-(11) in the case $G = \mathbb{R}^2_+$ decreasing at infinity and such that $u \in W_p^1(G) \cap W_2^2(G_\varepsilon)$ for all $p \in (1, n/(n-1))$ and > 0 ($G_\varepsilon = \{x \in G: |x - x_0| > \varepsilon\}$), and, for every domain $0 < \epsilon \le |x - x_0| \le R < \infty$, we have the representation

$$u(x) = \frac{e^{-\sqrt{\lambda}|x-x_0|}}{2\sqrt{2\pi|x-x_0|}\lambda^{1/4}}(1 + O(\frac{1}{\sqrt{|\lambda|}})). \quad (15)$$

We can assume that the constants c in the inequality $|O(\frac{1}{\sqrt{|\lambda|}})| \le \frac{c}{\sqrt{|\lambda|}}$ are independent of λ and x from some compact set not containing x_0 (but they depend on the size of this compact set).

The proof of this theorem relies on the weak maximum principle and constructing upper and lower solutions.

Now we describe an algorithm of defining a point source and its intensity using the data (3). We consider the case of $r = 1$.

The main condition is as follows:

(A) any three points in (3) do not lie on the same straight line.

Assume that the conditions of Theorem 1 hold. In this case there exists $\gamma_0 \ge 0$ such that the claim of Theorem 1 holds and the problem (7) has a solution v. After the change $u = v + w$, we arrive at the problem

$$Lw = N_1(t)\delta(x - x_1), \quad w(x,0) = 0,\ Bw|_S = 0. \quad (16)$$

$$w(y_j, t) = \psi_j - v(y_j, t) = \tilde{\psi}_j(t), \quad (17)$$

where $e^{-\gamma t}\tilde{\psi}_j(t) \in W_2^{1-n/4,2-n/2}(0,T)$ ($\gamma \ge \gamma_0$), (see the trace theorems in Sect. 6.5-6.7 in [26]). We assume that

(B) there exists $\gamma_0 \ge 0$ such that

$$\sup_{\gamma > \gamma_0} \int_{-\infty}^{\infty} |L(\tilde{\psi}_i)(\gamma + i\sigma)|^2 |\gamma + i\sigma| e^{\delta_0 Re\sqrt{\gamma + i\sigma}}\, d\sigma < \infty,$$

where δ_0 is some estimate from below for the quantity $\max_j \alpha_j$, $\alpha_j = |x_1 - y_j|$, $j = 1,2,3$. We can assume that we have some estimate of this kind.

Theorem 3. Assume that $G = \mathbb{R}^2$ or $G = \mathbb{R}^2_+$, and in the latter case $Bu = u$ or $Bu = \frac{\partial u}{\partial x_n}$ and the conditions (A), (B) hold. Assume also that $q(x) \in L_\infty(Q)$, $u_0(x) \in W_2^1(G) \cap W_p^{2-2/p}(G)$ ($p < 2$), for some $\gamma \ge \gamma_0$, $e^{-\gamma t} f_0 \in L_2(Q) \cap L_p(Q)$, $e^{-\gamma t} g(x,t) \in W_2^{1/2,3/2}(S) \cap W_p^{1-1/2p,2-2/p}(S)$ in the case of the Dirichlet boundary conditions and $e^{-\gamma t} g(x,t) \in W_2^{1/4,1/2}(S) \cap W_p^{1/2-1/2p,1-2/p}(S)$ otherwise. Then a solution $u, N_1(t), x_1$ to the problem (1)-(3), where $r = 1$ and $s = 3$ is unique and such that $N_1(t)e^{-\gamma_0 t} \in L_2(0,\infty)$ and $e^{-\gamma_1 t} u \in L_p(0,\infty; W_p^1(G))$, $e^{-\gamma_1 t} u_t \in L_p(0,\infty; W_p^{-1}(G))$ for all $\gamma_1 > \gamma$, $e^{-\gamma t} u \in W_2^{2,1}(Q_\varepsilon)$ for all $\varepsilon > 0$.

Proof. Applying the Laplace transform, we obtain that the function $\hat{w} = L(w) = \int_0^\infty e^{-\lambda t} w(x,t)\, dt$ is a solution to the problem

$$\lambda\hat{w} - \Delta\hat{w} = \hat{N}_1(\lambda)\delta(x - x_1),\ Bw|_S = 0. \quad (18)$$

$$\hat{w}(y_j) = L(\tilde{\psi}_j)(\lambda). \quad (19)$$

The definitions yield

$$L(\tilde{\psi}_j)(\lambda) = \hat{N}_1(\lambda) v_0(y_j, \lambda), \quad (20)$$

where v_0 is a solution to the equation (10) in the case of $G = \mathbb{R}^2$ or to the problem (10)-(11) in the case of $G = \mathbb{R}^2_+$ decreasing at infinity such that $v_0 \in W_p^1(G) \cap W_2^2(G_\varepsilon)$ for all $p \in (1,2)$ and $\varepsilon > 0$ ($G_\varepsilon = \{x \in G: |x - x_1| > \varepsilon\}$). The asymptotics of Theorem 1 implies that

$$\frac{L(\tilde{\psi}_j)(\lambda)}{L(\tilde{\psi}_i)(\lambda)} = \frac{e^{-\sqrt{\lambda}|x_1-y_j|}\sqrt{|x_1-y_i|}}{e^{-\sqrt{\lambda}|x_1-y_i|}\sqrt{|x_1-y_j|(1+O(\frac{1}{\sqrt{\lambda}}))}},\ i,j = 1,2,3. \quad (21)$$

Denote $G_{ij} = L(\tilde{\psi}_j)(\lambda)/L(\tilde{\psi}_i)(\lambda)$. Without loss of generality, we can assume that $|O(\frac{1}{\sqrt{\lambda}})| \le 1/2$ for all i, j and

$\lambda \geq \lambda_0$ ($\lambda_0 > 0$). Take $\alpha \geq \sqrt{R_0}$. In this case, we infer from (21) that

$$\frac{G_{ij}(\alpha^2)}{G_{ij}((\alpha+1)^2)} = e^{|x_1-y_j|-|x_1-y_i|}(1 + O(\tfrac{1}{\alpha})). \quad (22)$$

Thus, we have

$$\ln \frac{G_{ij}(\alpha^2)}{G_{ij}((\alpha+1)^2)} = \alpha_j - \alpha_i + O(\tfrac{1}{\alpha}). \quad (23)$$

This equality ensures that there exists the limit

$$\lim_{\alpha\to+\infty} \ln \frac{G_{ij}(\alpha^2)}{G_{ij}((\alpha+1)^2)} = \alpha_j - \alpha_i. \quad (24)$$

It is possible that $\alpha_1 = \alpha_2 = \alpha_3$ and thereby all limits in (24) are equal to zero. In this case the point x_1 is the center of the circle passing through the point y_1, y_2, y_3 which is uniquely determined. In this case $\widehat{N}_1 = \frac{L(\tilde{\psi}_j)(\lambda)}{v_0(y_j,\lambda)}$, $N_1 = L^{-1}(\frac{L(\tilde{\psi}_j)(\lambda)}{v_0(y_j,\lambda)})$ and $e^{-\gamma t}N_1(t) \in L_2(0,\infty)$ for all $\gamma \geq \max(R_1, R_0)$ due to (B). Assume that there are distinct limits in (24), for example, $\alpha_{j_0} - \alpha_{i_0} \neq 0$. We have

$$\ln \frac{G_{ij}(\alpha^2)}{G_{ij}((\alpha+1)^2)} = \alpha_{j_0} - \alpha_{i_0}, \quad (25)$$

and (21) can be rewritten in the form

$$\sqrt{\frac{\alpha_{i_0}}{\alpha_{j_0}}} = \frac{L(\tilde{\psi}_{j_0})(\lambda)}{L(\tilde{\psi}_{i_0})(\lambda)} e^{\sqrt{\lambda}(\alpha_{j_0}-\alpha_{i_0})} = G(\lambda). \quad (26)$$

Since the right-hand side is not identically equal to 1 and is known, there exists $\lambda_0 \in \mathbb{R}$, $\lambda_0 \geq \max(R_1, R_0)$ such that $G(\lambda_0) \neq 1$ (it is true for suffciently large $|\lambda|$). We obtain approximate equations (25) and (26) for the quantities $\alpha_{j_0}, \alpha_{i_0}$, where we choose and fix a sufficiently large $\alpha \in \mathbb{R}$ and the number λ_0; the determinant of this system does not vanish. Solving the system, we find some approximations of $\tilde{\alpha}_{j_0}, \tilde{\alpha}_{i_0}$. It is easy to see that the module of the determinant of the system is bounded form below not by some positive constant uniformly in λ lying in some half-plane $Re\,\lambda \geq \lambda_1$. Therefore, we obtain the representation $\alpha_{i_0} = \tilde{\alpha}_{j_0} + O(\frac{1}{\alpha}) + O(\frac{1}{\sqrt{\lambda_0}})$. Choosing nonzero limits $\alpha_i - \alpha_j$ and using the equalities $\alpha_i = \alpha_j$ in the case of the zero limit, we determine the quantities α_i, $i = 1,2,3$. Then the point x_1 is defined as the intersection point of the spheres centered at y_i and of radii α_i. The function N_1 is defined with the use of the inverse Laplace transform, and the function $w(x,t)$ as a solution of the direct problem.

III. Description of the algorithm

The algorithm relies on the finite element method. We take the Neumann boundary conditions. The case of other boundary can be considered by the same scheme. First, define a triangulation grid $\{y_i\}$ with the number of nodes m in the domain under consideration and the time step $\tau = T/M_0$, where M_0 is a positive integer.

The algorithm is carried out in 3 consecutive steps.

Step 1. Recovering the coordinates of the source x_1.

We solve the auxiliary direct problem (7). The approximation of the function v has the form $v^m(x,t) = \sum_{i=1}^m C_i(t)\varphi_i(x)$, where the functions C_i satisfy the following system of ordinary differential equations:

$$(Lv^m, \varphi_i)_0 = (f, \varphi_i)_0 = \int_0^1 f(x,t)\varphi_i(x)dx, \quad i = 1,2,\ldots,m. \quad (27)$$

The system (27) can be rewritten as follows:

$$M\vec{C}_t + K\vec{C} = \vec{F} = ((f,\varphi_1)_0, (f,\varphi_2)_0, \ldots, (f,\varphi_m)_0)^T, \quad (28)$$

where M, K are the mass and stiffness matrices, respectively, and F is the vector of the right-hand side. Approximations of a solution to the system (28) are determined with the use of the finite difference method (FEM). We define $\vec{C}_0 = (u_0(y_1), u_0(y_2), \ldots, u_0(y_m))^T$. The vectors $\vec{C}_n$ for $n \geq 1$ are calculated as solutions to the system

$$M\frac{\vec{C}_n - \vec{C}_{n-1}}{\tau} + K\vec{C}_n = \vec{F}_n = \vec{F}(t_n), \; n = 1,2,\ldots,M_0. \quad (29)$$

Thus, we have

$$\vec{C}_n = (M + \tau K)^{-1}(M\vec{C}_{n-1} + \tau\vec{F}_n). \quad (30)$$

We calculate x_1 as the intersection point of three circles with radii α_1, α_2, α_3 (or as the center of a circle passing through the measurement points) (see (24)). The radii of the three circles are calculated using the formulas (22)-(26). The parameter α in these formulas is determined by increasing it from 0.0001 to 50 up to the first intersection of three circles which is the desired point x_1.

Step 2. Recovering the intensity N_1.

Let ω be a solution to the problem (16)-(17), and ω_0 be a solution to the same problem with $N(t) \equiv 1$. We have the equality (the Duhamel formula):

$$\omega(x,t) = \int_0^t N(\tau)\omega_{0t}(x, t-\tau)\,d\tau, \; t \leq T.$$

Using the overdetermination conditions, we obtain

$$\tilde{\psi}_j(t) = \int_0^t N(\tau)\omega_{0t}(y_j, t-\tau)\,d\tau, \; t \leq T. \quad (31)$$

We have already determined x_1. Now we can find the approximation of a solution to the problem (16)-(17) with $N(t) \equiv 1$ as follows: $\omega_0^m = \sum_{i=1}^m \xi_i(t)\varphi_i(x)$, where functions $\xi_i(t)$ of the solution of the system

$$(L\omega_0^m, \varphi_i)_0 = \varphi_i(x_1),\ i = 1,2,\ldots,m. \quad (32)$$

Note that the vector of the right-hand side has at most three non-zero elements with indices j_1, j_2 and j_3, which are the vertices of the triangle containing the point x_1. The system (11) can be written as

$$M\vec{\xi}_t + K\vec{\xi} = \vec{F}_1 = (\varphi_1(x_1), \varphi_2(x_1), \ldots, \varphi_m(x_1))^T, \quad (33)$$

We construct an approximate solution to this system. Let $\vec{\xi}^0 = 0$. The vectors $\vec{\xi}^n$ for $n \geq 1$ are calculated as follows:

$$\vec{\xi}^n = (A + \tau B)^{-1}(A\vec{\xi}^{n-1} + \tau\vec{F}_1). \quad (34)$$

We denote the i-th coordinates of the vector $\vec{\xi}^n$ as $\xi_{i,n}$. Let i_k be the numbers of nodes corresponding to the measurement points y_k, we assume that they are also grid nodes. Find the vector β with coordinates

$$\beta_j^k = \xi_{i_k,j},\ j = 0,1,\ldots,M_0,\ k = 1,2,3. \quad (35)$$

We also find the vector $\tilde{\beta}^k$ with coordinates $\tilde{\beta}_j^k = \beta_j^k - \beta_{j-1}^k$, $j = 1,2,\ldots,M_0$, $\tilde{\beta}_0 = 0$, $k = 1,2,3$. We look for the approximation $\widetilde{N}(t)$ of the function $N(t)$ in the form of a piecewise constant function equal to N_i on the interval $(t_{i-1}, t_i]$. In view of (31), replacing the integrals with integral sums, we obtain the systems

$$\tilde{\psi}_i(\tau j) = \sum_{k=1}^{j} N_k\, \beta_{j+1-k}^i,\ j = 1,2,\ldots,M_0,,\ i = 1,2,3. \quad (36)$$

The right-hand side is the approximation of the integral $\int_0^t N(t)\omega_{0t}(y_i, t-\tau)d\tau$. To solve the system (1), we use the Tikhonov regularization. We rewrite the system (36) as $\vec{\psi}_i = B\vec{N}$. Fix some small constant $\epsilon_0 > 0$ and write out the approximate solution of the system as follows:

$$\vec{N}_i = (B^*B + \epsilon_0 E)^{-1}B^*\vec{\psi}_i,\ i = 1,2,3. \quad (37)$$

Here B^* is conjugate to the matrix B, E is the identity matrix. As an approximation of $\widetilde{N}$, we can take one of the piecewise constant functions defined by the vectors $\vec{N}_1 = (N_{11}, N_{12}, \ldots, N_{1M_0})$, $\vec{N}_2 = (N_{21}, N_{22}, \ldots, N_{2M_0})$, $\vec{N}_3 = (N_{31}, N_{32}, \ldots, N_{3M_0})$ or $\vec{N}_0 = (\vec{N}_1 + \vec{N}_2 + \vec{N}_3)/3$.

Step 3. Recovery of the function u

Assume that N_1 is a known function. Solve the auxiliary problems (7) and (8). Summing their solutions, we obtain the approximation of the function u.

IV. Results of Computational Experiments

In this section, we analyze the results of numerical experiments for several groups of input data. To simplify the exposition, we present the results of calculations of the point x_1 and the function $N_1(t)$ only.

As a result of the calculations, the approximations of the quantities (u, N_1, x_1) are determined at the grid nodes and the moments of time $t_1, t_2, \ldots, t_N$. Our domain is the unit circle centered at $(0; 0)$. We consider two grids with a different number of nodes: $Z_1 = 2813$ and $Z_2 = 725$. Three measurement points are close to the boundary of the domain (in the figure 1 they are indicated by the symbol " $*$ "). Their coordinates are $(-0.85; -0.475)$, $(0; 0.95)$, $(0.85; -0.475)$ and these points are the vertices of a regular triangle. We consider several locations of the source. Different locations of the source are displayed on the Fig. 1 (the locations are denoted by the symbol "o").

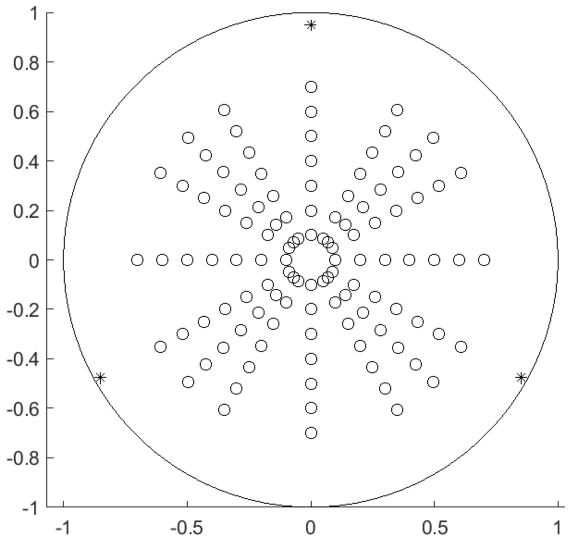

Fig. 1. Location of source and measurement points

Different locations of the source lie on seven concentric circles, each circle contains 16 locations. We designate them as $c_1, c_2, \ldots, c_7$ (total 112 points in all circles). The first circle is of radius $r = 0.1$ centered at the origin.

For experiments, we write out a solution to the problem (1)-(3), as well as the initial and boundary conditions. We consider the equation

$$Lu = u_t - \Delta u + a(x,t)u = f = f_0 + N_1(t)\delta(x - x_1),$$

where $x = (x_1, x_2)$, $x_1 = (x_{11}, x_{12})$, $N_1(t) = -(t - 0.5)^2 + 1$, $a(x,t) = 1/(t+1)$. The initial and boundary data are as follows:

$$u_0 = \varphi(x,0) - \frac{N_1(0)}{2\pi} ln|x - x_1|,$$

$$\varphi(x,t) = (x_1^2 + x_2^2 + 1)(t+1),$$

$$\frac{\partial u}{\partial n}|_\Gamma = g(x,t),$$

where the functions f, g are given by the equalities $f = u_t - \Delta u + a(x,t)u$, $g = x_1 u_{x_1} + x_2 u_{x_2}$. Here $\varphi(x,t)$ is an arbitrary smooth function and a solution u has the form $u(x,t) = \varphi(x,t) - \frac{N(t)}{2\pi} ln|x - x_1|$.

We divide the numerical experiments into groups depending on the unknown functions $u(x,t)$, the intensity $N_1(t)$, coordinates of the point x_1, coefficients of the equation and the right-hand side f. Several grids with different time steps $\tau = T/M_0$ are considered. First, we take $\tau = 0.01$.

Introduce several quantities that describe the calculation errors: the quantity $\varepsilon_{x1} = \sqrt{(x_{11} - x^1)^2 + (x_{12} - x^2)^2}$ is the error of calculating the location of the source (here (x^1, x^2) is the result of calculation); $\varepsilon_{N_1} = \max_i |N_1(i\tau) - N_{1i}|$ is the error in calculating the intensity of the source, where the numbers $N_{1i} \approx N_1(\tau i)$ are the results of calculations, $\varepsilon_u = \max_{i,j} |u_{i,j} - u(y_i, \tau j)|$ is the error in calculating the concentration of a pollutant, where $i = 1,2,\dots,m$ and $j = 1,2,\dots,M_0$. Let τ_s be the calculation time in seconds.

First, we determine the average calculation errors on each of the circles c_i $(i = 1,2,\dots,7)$. The average values of the quantities $\varepsilon_{x1}, \varepsilon_{N_1}, \varepsilon_u$ on each circle are presented for the grids Z_1 in the table II and Z_2 in table I.

TABLE I. ARITHMETIC MEAN VALUES OF COMPUTATIONAL ERRORS FOR THE GRID Z_2

c_n	ε_{x1}	ε_N	ε_u
1	0.0142	0.0299	0.1349
2	0.0153	0.0309	0.1161
3	0.0171	0.0333	0.1095
4	0.02	0.0421	0.1669
5	0.0249	0.0555	0.1299
6	0.0327	0.0631	0.1542
7	0.0457	0.0743	0.1996

The average experiment takes 915.866 seconds.

TABLE II. ARITHMETIC MEAN VALUES OF COMPUTATIONAL ERRORS FOR THE GRID Z_1

c_n	ε_{x1}	ε_N	ε_u
1	0.0141	0.0757	0.0853
2	0.015	0.0856	0.1358
3	0.0172	0.0846	0.1364
4	0.0205	0.0865	0.1228
5	0.0264	0.0881	0.0989
6	0.0351	0.1096	0.1446
7	0.0491	0.125	0.1612

The average experiment takes 43.9 seconds.

As you can see from the tables, the largest errors ε_{x1}, ε_N, ε_u are achieved if the point x_1 located on the circle c_7.

It can be noted that an increase in the number of grid nodes in the considered domain weakly affects the accuracy of calculating the coordinate x_1, but increases the accuracy of calculating the intensity N_1. In the case of the grid Z_1, the execution time is 21 times longer than that in the case of the grid Z_2. Next, we present the results in the latter case of the grid Z_2. The results of calculations for points located on a circle c_7 with the number of grid nodes Z_2 are presented in the table III.

TABLE III. RESULTS OF CALCULATIONS FOR POINTS LOCATED ON A CIRCLE c_7

No	x_1	ε_{x1}	ε_N	ε_u	τ_s
1	[0.7;0]	0.0464	0.1371	0.1244	908
2	[0.6062;0.35]	0.0465	0.0864	0.094	869
3	[0.495;0.495]	0.0503	0.1177	0.1243	852
4	[0.35;0.6062]	0.0614	0.1105	0.2109	806
5	[0;0.7]	0.0771	0.2236	0.2773	809
6	[-0.35;0.6062]	0.0611	0.1046	0.036	868
7	[-0.495;0.495]	0.0455	0.0803	0.144	869
8	[-0.6062;0.35]	0.0426	0.0863	0.1598	887
9	[-0.7;0]	0.0451	0.0819	0.125	881
10	[-0.6062;-0.35]	0.0573	0.1854	0.1873	906
11	[-0.495;-0.495]	0.0487	0.0912	0.0936	969
12	[-0.35;-0.6062]	0.0436	0.1169	0.1179	1015
13	[0;-0.7]	0.0032	0.0879	0.3364	968
14	[0.35;-0.6062]	0.0479	0.107	0.0878	953
15	[0.495;-0.495]	0.0531	0.196	0.2234	982
16	[0.6062;-0.35]	0.0567	0.1867	0.2003	987

Let us compare the graphs of the initial and recovered functions N_1 in the case of $x_1 = [0; 0.7]$, Fig. 2.

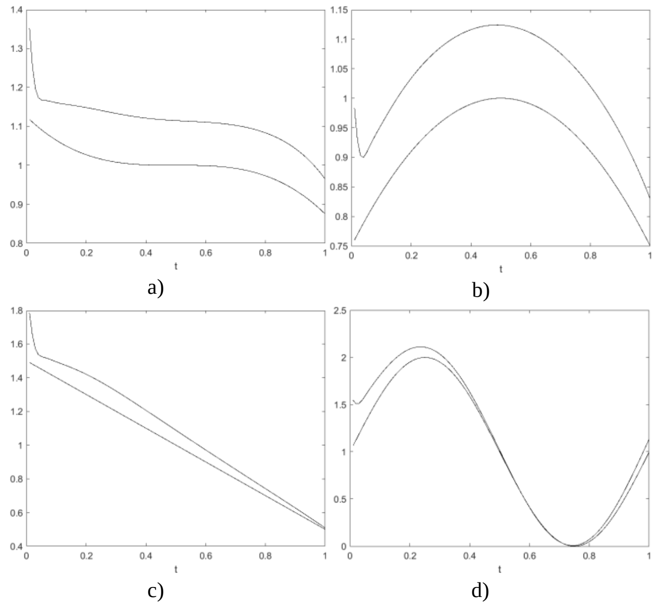

Fig. 2. Graphs of source and restored functions $N_1(t)$: a) $N_1 = -(t-0.5)^3 + 1$, $\varepsilon_{x1} = 0.0796$, $\varepsilon_N = 0.2337$, $\varepsilon_u = 0.3132$; b) $N_1 = -(t-0.5)^2 + 1$, $\varepsilon_{x1} = 0.0771$, $\varepsilon_N = 0.2236$, $\varepsilon_u = 0.2773$; c) $N_1 = -(t-0.5) + 1$, $\varepsilon_{x1} = 0.0813$, $\varepsilon_N = 0.2938$, $\varepsilon_u = 0.5938$; d) $N_1 = \sin(2\pi \cdot t) + 1$, $\varepsilon_{x1} = 0.0806$, $\varepsilon_N = 0.4838$, $\varepsilon_u = 0.4204$.

As you can see, the graphs of the recovered functions are quite close to the initial graphs 2.a-2.d.

As we have previously determined, the largest calculation error appears when the point $x_1$is located on the circle c_7. Thus, in a subsequent experiment, we will consider points located in this circle.

Next, we present the results of computational experiments for functions $\varphi(x,y,t) = x^2 + y^2 + t^2$, $N_1(t) = -(t - 0.5)^2 + 1$, $a(x,y,t) = (x+y)/(t+1)$. Consider the dependence of the result on the time step. To this aim, we take $\tau = 0.01$ and $\tau = 0.0025$, (in table IV, the data are written through "/").

TABLE IV. RESULTS OF CALCULATIONS FOR POINTS LOCATED ON A CIRCLE c_7 WITH DIFFERENT τ

x_1	ε_{x1}	ε_N	ε_u
[0.7;0]	0.1742/0.0814	0.1025/0.0846	0.236/0.09
[0.6062;0.35]	0.0653/0.0465	0.1404/0.2341	0.2046/0.1157
[0.495;0.495]	0.1528/0.0493	0.1755/0.1319	0.1132/0.0638
[0.35;0.6062]	0.1657/0.0817	0.061/0.1355	0.3096/0.197
[0;0.7]	0.0759/0.096	0.1276/0.119	0.3159/0.2925

[-0.35;0.6062]	0.1647/0.0589	0.0473/0.0862	0.2627/0.1616
[-0.495;0.495]	0.1499/0.034	0.0603/0.0344	0.4820/0.1619
[-0.6062;0.35]	0.0762/0.0245	0.127/0.2088	0.1687/0.1993
[-0.7;0]	0.1837/0.0871	0.1858/0.0383	0.6006/0.1126
[-0.6062;-0.35]	0.0502/0.0653	0.0956/0.1053	0.1679/0.1484
[-0.495;-0.495]	0.1033/0.0535	0.1475/0.0619	0.2705/0.1254
[-0.35;-0.6062]	0.2185/0.1041	0.3434/0.0621	0.4868/0.2387
[0;-0.7]	0.1854/0.0697	0.1157/0.0894	0.3518/0.2965
[0.35;-0.6062]	0.2051/0.0747	0.0441/0.1371	0.2295/0.0417
[0.495;-0.495]	0.1088/0.0355	0.0511/0.1203	0.2583/0.0895
[0.6062;-0.35]	0.0715/0.091	0.1329/0.2357	0.1689/0.2319

V. Conclusion

A numerical algorithm for determining the coordinates and intensity of a point source for a two-dimensional advection-dispersion-reaction equation using three measurement points is presented. The results of numerical experiments show good accuracy in reconstructing the coordinates and the intensities of the source. The time step τ strongly affects on the accuracy of calculations whose decrease leads to an increase in the accuracy of calculating of the point x_1.

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