Research Article

α-consensus Value of Cooperative Game with Intuitionistic Fuzzy Payment

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ABSTRACT

This paper studies the α-consensus value of a cooperative game with payoffs of triangular intuitionistic fuzzy numbers and gives the formation mechanism of the α-consensus value, as well as some properties. Using the extended Hukuhara difference of triangular intuitionistic fuzzy numbers, the α-consensus value of the triangular intuitionistic fuzzy cooperative game is obtained. Furthermore, we study the condition of the α-consensus value satisfying the individual rationality, which makes the α-consensus value of the cooperative game more practical in real problems. Finally, a numerical example is given to illustrate the validity and applicability of the α-consensus value of the triangular intuitionistic fuzzy cooperative game proposed in this paper.

1. INTRODUCTION

Cooperative games mainly study how to distribute the payoff value of a grand coalition fairly and reasonably, which has become one of the important scientific methods for analyzing people's decision-making behavior. Moreover, cooperative games have been widely used in economic management, humanities and engineering issues [1–3]. The allocation scheme of the cooperative game is called the solution of cooperative game. Since each solution of cooperative game has certain deficiencies, researchers have perfected the problem by convexly combining the existing solutions of cooperative games.

In recent years, there exist some investigations on the combinatorial solutions of cooperative games. Brink *et al*. [4] studied two classes of equal surplus shared solutions. The first class consists of all convex combinations of the equal division (ED) solution and the center-of-gravity of the imputation-set (CIS) value. The second class is the dual class consisting of all convex combinations the (ED) solution and the egalitarian non-separable contribution (ENSC) value, respectively. The two combinatorial solutions make the payoffs of all players of equal surplus shared solutions more fair and reasonable than the payoff obtained only in the CIS value or the ENSC value. But the two solutions of CIS value and ENSC value do not consider the case of sub-coalition. Wang *et al*. [5], Brink *et al*. [6], Casajus and Huettner [7] studied that the Shapley value and ED are convexly combined by the social selfish coefficient $\alpha \in [0, 1]$, denoted as α average Shapley value, and some properties of the α average Shapley value are studied. Xu *et al*. [8] studied the α-CIS

value that is convex combinations of the ED value and the CIS value, and they proposed some properties of the α-CIS value.

In addition, Ju *et al*. [9–11] obtained a consensus value through negotiating recursive method, which is average of the Shapley value with the CIS value, and α-consensus value is also proposed. However, the formation mechanism of the α-consensus value is not given. Moreover, individual rationality is one of the important properties of cooperative game, individual rationality is defined as: for a cooperative game $v \in G^N$, let $x = (x_1, x_2, \cdots, x_n)$ be an n-dimensional vector, satisfy $x_i \geqslant v\{i\}$, $i = 1, 2, \cdots, n$. That is to say, each player will get more payoff value in the grand coalition than that of the player alone. The ranges of α, which makes the combinatorial solutions of cooperative games satisfy the individual rationality, have not been considered in the existing researches. Therefore, in this paper, according to Wang *et al*. [5] studied the formation mechanisms of α average Shapley value, we give the formation mechanism of the α-consensus value of cooperative game. Furthermore, we study the ranges of α, which make the α-consensus value satisfy the individual rationality. Thus, α-consensus value can make up for the shortcoming that Shapley value not satisfy the individual rationality by adjusting the value of α. So α-consensus value is more reasonable than Shapley value.

Furthermore, the existing researches about the combinatorial solutions are only for crisp cooperative games. As we all known, due to the influence of economic, political and other social factors, in some scenarios, there are some uncertainties in the game process. So fuzzy cooperative games have been extensively studied [11–14]. Fuzzy sets can represent the uncertain information of cooperative games, but they cannot reflect the hesitation degrees

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of the players. Thus, using the intuitionistic fuzzy sets (IFS) given Atanassov [15,16] to represent the uncertain information of cooperative games, intuitionistic fuzzy cooperative game has been studied. Nan *et al.* [17] studied the Shapley function of intuitionistic fuzzy cooperative games. Based on the extended Hukuhara difference of intuitionistic fuzzy numbers, the specific expression of the Shapley value of intuitionistic fuzzy cooperative games with multilinear extension form is obtained, and its existence and uniqueness are discussed. Liu and Zhao [18] studied the least squares pre-nucleolus of cooperative game with characteristic function of trapezoidal intuitionistic fuzzy numbers. As far as we know, most of the studies on fuzzy cooperative games and intuitionistic fuzzy cooperative games are focused on the single value solutions. However, there exists no investigation on the combination solutions of fuzzy cooperative games and intuitionistic fuzzy cooperative games. As the intuitionistic fuzzy cooperative games are expansions of fuzzy cooperative games, in this paper, the α-consensus value of intuitionistic fuzzy cooperative game is studied, and its some properties are proved.

The rest part of this paper is organized as follows: in Section 2, some basic concepts of intuitionistic fuzzy sets are given and Hukuhara difference of intuitionistic fuzzy numbers is introduced as well as the ranking order relation of intuitionistic fuzzy numbers. In Section 3, some definitions of the α-consensus value of cooperative game are reviewed. Then, the formation mechanism and some important conclusions of the α-consensus value of cooperative game are given. Section 4 studies the definition and property of the α-consensus value of the intuitionistic fuzzy cooperative game. In Section 5, a numerical example is offered to illustrate the validity and applicability of the α-consensus proposed in this paper. Section 6 is the summary of this paper.

2. PRELIMINARIES

2.1. Intuitionistic Fuzzy Set

The concept of an IFS was firstly introduced by Atanassov [15].

Definition 1. [15,16] Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite universe set. An IFS $\tilde{A}$ in X may be mathematically expressed as $\tilde{A} = \{\langle x_l, \mu_{\tilde{A}}(x_l), \upsilon_{\tilde{A}}(x_l)\rangle | x_l \in X\}$, where $\mu_{\tilde{A}} \colon X \mapsto [0,1]$ and $\upsilon_{\tilde{A}} \colon X \mapsto [0,1]$ are the membership degree and the non-membership degree of an element $x_l \in X$ to the set $\tilde{A} \subseteq X$, respectively, such that they satisfy the following condition: $0 \leqslant \mu_{\tilde{A}}(x_l) + \upsilon_{\tilde{A}}(x_l) \leqslant 1$ for all $x_l \in X$.

Let $\pi_{\tilde{A}}(x_l) = 1 - \mu_{\tilde{A}}(x_l) - \upsilon_{\tilde{A}}(x_l)$, which is called the intuitionistic index (or hesitancy degree), which is the degree of indeterminacy membership of the element x_l to the set $\tilde{A}$. Obviously, $0 \leqslant \pi(x_l) \leqslant 1$.

2.2. Triangle Intuitionistic Fuzzy Numbers and Related Operations

Definition 2. [19,20] Suppose that $\tilde{a} = \langle (a_1^-, a, a_1^+); (a_2^-, a, a_2^+)\rangle$,

is the triangular intuitionistic fuzzy number (TIFN) on the real number set R, then the membership function and the non-membership function of $\tilde{a}$ are defined:

$$\mu_{\tilde{a}}(x) = \begin{cases} 0, & (x < a_1^-, x > a_1^+), \\ (x - a_1^-)/(a - a_1^-), & (a_1^- \leqslant x < a), \\ 1, & (x = a), \\ (a_1^+ - x)/(a_1^+ - a), & (a \leqslant x < a_1^+), \end{cases}$$

$$\upsilon_{\tilde{a}}(x) = \begin{cases} 1, & (x < a_2^-, x > a_2^+), \\ (a - x)/(a - a_2^-), & (a_2^- \leqslant x < a), \\ 0, & (x = a), \\ (x - a)/(a_2^+ - a), & (a \leqslant x < a_2^+). \end{cases}$$

The set of all TIFNs is denoted by $\tilde{\mathfrak{R}}$.

Definition 3. [18,19] Let $\tilde{a} = \langle (a_1^-, a, a_1^+); (a_2^-, a, a_2^+)\rangle$ and $\tilde{b} = \langle (b_1^-, b, b_1^+); (b_2^-, b, b_2^+)\rangle$ be two TIFNs on the set $\tilde{\mathfrak{R}}$ and $r \in R$ is any real number. TIFNs arithmetic operations are given as follows:

$(1)\ \tilde{a} + \tilde{b} = \langle (a_1^- + b_1^-, a + b, a_1^+ + b_1^+)\,; (a_2^- + b_2^-, a + b, a_2^+ + b_2^+)\rangle,$

$$(2)\ r\tilde{a} = \begin{cases} \langle (ra_1^-, ra, ra_1^+); (ra_2^-, ra, ra_2^+)\rangle, & r \geqslant 0, \\ \langle (ra_1^+, ra, ra_1^-); (ra_2^+, ra, ra_2^-)\rangle, & r < 0. \end{cases}$$

According to extended Hukuhara difference of interval numbers given Meng *et al.* [21], the extended Hukuhara difference of TIFNs are defined as follows.

Definition 4. [17] Let $\tilde{a} = \langle (a_1^-, a, a_1^+); (a_2^-, a, a_2^+)\rangle$, $\tilde{b} = \langle (b_1^-, b, b_1^+); (b_2^-, b, b_2^+)\rangle$ and $\tilde{c} = \langle (c_1^-, c, c_1^+); (c_2^-, c, c_2^+)\rangle$ be TIFNs. If $b_i^- - a_i^- > b_i^+ - a_i^+$, $\tilde{c} = \tilde{a} -_H \tilde{b}$ is said to the "imaginary" Hukuhara difference. The extended Hukuhara difference of TIFNs is defined $\tilde{a} -_H \tilde{b} = \langle (a_1^- - b_1^-, a - b, a_1^+ - b_1^+); (a_2^- - b_2^-, a - b, a_2^+ - b_2^+)\rangle$.

The ranking order relation of intuitionistic fuzzy number is an important problem for intuitionistic fuzzy cooperative game. Based on the λ weighted mean areas of intuitionistic fuzzy numbers, the ranking order relation of TIFNs is defined as follows.

Definition 5. Let $\tilde{a} = \langle (a_1^-, a, a_1^+); (a_2^-, a, a_2^+)\rangle$ and $\tilde{b} = \langle (b_1^-, b, b_1^+); (b_2^-, b, b_2^+)\rangle$ be TIFNs, $S_\lambda(\tilde{a})$ and $S_\lambda(\tilde{b})$ are the index values of λ weighted mean areas of $\tilde{a}$ and $\tilde{b}$ respectively, $\lambda \in [0, 1]$,

where

$S_\lambda(\tilde{a}) = \lambda(a_1^- + 2a + a_1^+)/4 + (1 - \lambda)(a_2^- + 2a + a_2^+)/4$, $S_\lambda(\tilde{b}) = \lambda(b_1^- + 2b + b_1^+)/4 + (1 - \lambda)(b_2^- + 2b + b_2^+)/4$,

then

1. if $S_\lambda(\tilde{a}) > S_\lambda(\tilde{b})$, then $\tilde{a} > \tilde{b}$;

2. if $S_\lambda(\tilde{a}) < S_\lambda(\tilde{b})$, then $\tilde{a} < \tilde{b}$;

3. if $S_\lambda(\tilde{a}) = S_\lambda(\tilde{b})$, then $\tilde{a} = \tilde{b}$.

3. THE α-CONSENSUS VALUE OF COOPERATIVE GAME

3.1. The Solutions of Cooperative Game

An n-person cooperative game is a pair (N, v) where $N = \{1, 2, \cdots, n\}$ is a finite set of players with $|N| \geqslant 2$ and $v : 2^N \to$

R is a characteristic function on N such as $v(\phi) = 0$. For each coalition $S \subseteq N$, $v(S)$ is called the payoff of coalition S. which is the value of the members of coalition S can obtain by agreeing to cooperate. The set of all n-person cooperative games are denoted by G^N.

Shapley value and CIS value are two important solutions of cooperative game. For a cooperative game $v \in G^N$, let Shapley value of a cooperative game be expressed as

$$Sh_i(v) = \rho[v(S) - v(S\backslash\{i\})], \tag{1}$$

where $\rho = \sum_{S \subseteq N} \frac{(n - |S|)! \, (|S| - 1)!}{n!}, S \subseteq N, v \in G^N$.

For a cooperative game $v \in G^N$, let CIS value of a cooperative game be expressed as

$$CIS_i(v) = v\{i\} + \frac{1}{n}\left[v(N) - \sum_{j \in N} v\{j\}\right]. \tag{2}$$

Ju *et al.* [9,10] studied the $\alpha-$consensus value, which is the combinatorial solution of Shapley value and CIS value of cooperative games.

Definition 6. For a cooperative game $v \in G^N$, let $\alpha-$consensus value of the cooperative game be expressed as

$$\gamma(v) = \alpha Sh(v) + (1 - \alpha)CIS(v),$$

where $Sh(v) = \rho[v(S) - v(S\backslash\{i\})]$, and $CIS(v) = v\{i\} + \frac{1}{n}\left[v(N) - \sum_{j \in N} v\{j\}\right], \alpha \in [0, 1]$.

3.2. Formation Mechanism of the $\alpha-$Consensus Value

In this subsection, the formation mechanism of the α-consensus value of the cooperative game is studied. For a cooperative game $(N, v) \in G^N$, the payoff of player i participating in the grand coalition is x_i consisting of two parts $x_i = \alpha x_{i1} + (1 - \alpha) x_{i2}$. The first part of the value x_{i1} is obtained when player joins a grand coalition, and the second part x_{i2} is obtained when player leaves the grand coalition. The parameter $\alpha \in [0, 1]$ represents a proportion of the profit when player i joins the grand coalition, $1 - \alpha$ is a proportion of the payoff when player i leaves the grand coalition. Let $\pi(N)$ be set for all possible permutations on N. For player $i \in N$ and any permutation $\pi \in \pi(N)$, $S_i^{\pi(k)} = \{\pi(k) \in N | k \leqslant \pi^{-1}(i)\}$ represents that player i joins into the coalition of his predecessors to form the new coalition, where $\pi^{-1}(i)$ stands for player i in the order π. $P_i^{\pi(k)} = \{\pi(k) \in N | k \geqslant \pi^{-1}(i)\}$, represents that player i departs from the coalition of his predecessors to form the new coalition.

The formation process of $\alpha-$consensus value of the cooperative game is given as follows.

Step 1: The players arrive in or depart from a random order π, and all orders in $\pi(N)$ have the same probability.

Step 2: Player $i \in N$ joins a coalition and forms a new coalition $S_i^{\pi(k)}$. Before leaving, the coalition is $P_i^{\pi(k)}$. The first player who joins or leaves always obtains a payoff value of $\alpha v\{i\}$.

Step 3: The joining player $i \in N$ obtains his payoff value $\alpha v(S_i^{\pi(k)})$, and his marginal contribution surplus $v(S_i^{\pi(k)}) - v(S_i^{\pi(k)}\backslash\{i\}) - \alpha v(S_i^{\pi(k)})$ is evenly distributed to the subsequent players. Moreover, the leaving player $i \in N$ obtains a payoff value $\alpha v\{i\}$ and his marginal contribution surplus is equally distributed to the remaining player in the coalition.

Step 4: The last player $\pi(n)$, joining into grand coalition N can only get his marginal contribution $v(N) - v(N\backslash\pi(n))$, while the last player who leaves the coalition obtains payoff value $v(\pi(n))$. Note that the whole quantity $v(N)$, for every permutation π, is distributed among all players.

Step 5: The payoff value $\gamma_i^\pi(N, v)$ of each player i is composed of two parts in the grand coalition according to the order π, one is from participating in the coalition and the other is from departure the coalition. Assuming that α is the allocation ratio of player i joining in coalition payoff value Sh_i^π in order π, and $1 - \alpha$ is the allocation ratio of player i leaving the coalition payoff value $CIS_i^\pi(v)$, we have $\gamma_i^\pi(v) = \alpha Sh_i^\pi(v) + (1 - \alpha)CIS_i^\pi(v)$.

Step 6: The final payoff value $\gamma_i(N, v)$ of the player i in grand coalition is the expected value of the payoff value $\gamma_i^\pi(N, v)$ for all orders. Then, for crisp cooperative game $(N, v) \in G^N$, and all order $\pi \in \pi(N)$, the payoff value $\gamma_i(N, v)$ of player $i \in N$ is determined from Steps 1-6 as follows:

$$\gamma_i^\pi(v) = \begin{cases} \alpha v\{i\}, & \pi^{-1}(i) = 1, \\[2mm] \left[\alpha v(S_i^{\pi(k)}) + \lambda_i^\pi\right] + \left[(1 - \alpha)v\{i\} + \mu_i^\pi\right], \\[2mm] \qquad\qquad 1 < \pi^{-1}(i) < n, \\[2mm] \alpha\left[v(N) - \tilde{v}(N\backslash\{i\}) + \lambda_i^\pi\right] + (1 - \alpha)\left[v\{i\} + \mu_i^\pi\right], \\[2mm] \qquad\qquad \pi^{-1}(i) = n, \end{cases}$$

where $\lambda_i^\pi = \sum_{k=1}^{\pi^{-1}(i)-1} \dfrac{v\left(S_i^{\pi(k)}\right) - v\left(S_i^{\pi(k)}\backslash\pi(k)\right) - \alpha v\left(S_i^{\pi(k)}\right)}{n - k}$ and

$$\mu_i^\pi = \sum_{k=1}^{\pi^{-1}(i)-1} \frac{v\left(P_i^{\pi(k)}\right) - v\left(P_i^{\pi(k)}\backslash\pi(k)\right) - \alpha v(\pi(k))}{n - k}.$$

Thus, we can get

$$\gamma_i(v) = \frac{1}{n!} \sum_{\Pi(N)} \gamma_i^\pi(v), i \in N, \alpha \in [0, 1].$$

Furthermore,

$$\begin{aligned} \gamma_i(v) &= \frac{1}{n!} \sum_{\Pi(N)} \gamma_i^\pi(v) \\ &= \frac{1}{n!} \sum_{\Pi(N)} \left[\alpha Sh_i^\pi(v) + (1 - \alpha) CIS_i^\pi(v)\right] \\ &= \alpha \frac{1}{n!} \sum_{\Pi(N)} Sh_i^\pi(v) + (1 - \beta) \frac{1}{n!} \sum_{\Pi(N)} CIS_i^\pi(v) \\ &= \alpha Sh_i(v) + (1 - \alpha) CIS_i(v). \end{aligned}$$

So, for $v \in G^N$, the $\alpha-$consensus value of cooperative game is to be expressed as

$$\gamma_i(v) = \alpha Sh_i(v) + (1 - \alpha)CIS_i(v). \quad (3)$$

for all $i \in N$, where $\alpha \in [0, 1]$.

3.3. The Range of α for the $\alpha-$Consensus Value of the Cooperative Games

The individual rationality is an important condition for evaluating solutions of cooperative games. The coefficient $\alpha \in [0, 1]$ of existing of convex combinatorial solutions, such as $\alpha-$consensus value, α average Shapley value and $\alpha-$CIS value, is given prior, that is not to make these combinatorial solutions satisfy the individual rationality. Thus, it is crucial to find the range of the coefficient α such that the $\alpha-$consensus value satisfies the individual rationality. For cooperative game $v \in G^N$, we have

$$\gamma_i(v) = \alpha Sh_i(v) + (1 - \alpha)CIS_i(v)$$

$$= \alpha Sh_i(v) + CIS_i(v) - \alpha CIS_i(v)$$

$$= \alpha(Sh_i(v) - CIS_i(v)) + CIS_i(v)$$

$$\geqslant v\{i\}.$$

By simple calculating, we get

$$\alpha \geqslant \frac{v\{i\} - CIS_i(v)}{(Sh_i(v) - CIS_i(v))}.$$

i. If $Sh_i(v) > CIS_i(v)$, then we can get

$$\frac{v\{i\} - CIS_i(v)}{Sh_i(v) - CIS_i(v)} \leqslant 0.$$

Hence, for any $\alpha \in [0, 1]$, the $\alpha-$consensus value of cooperative game satisfies the individual rationality.

ii. If $Sh_i(v) < CIS_i(v)$, then we have

$$0 \leqslant \alpha \leqslant \frac{v\{i\} - CIS_i(v)}{Sh_i(v) - CIS_i(v)}.$$

If $Sh_i(v) < v\{i\}$, we obtain

$$0 \leqslant \alpha \leqslant \frac{v\{i\} - CIS_i(v)}{Sh_i(v) - CIS_i(v)} < 1.$$

Thus, for $\alpha \in \left[0, \frac{v\{i\}-CIS_i(v)}{Sh_i(v)-CIS_i(v)}\right]$, the $\alpha-$consensus value of cooperative game satisfies the individual rationality.

Obviously, when $Sh_i(v) \geqslant v\{i\}$, for any $\alpha \in [0, 1]$, $\alpha-$consensus value satisfies individual rationality.

3.4. Some Conclusions of Procedural Values

Malawski [22] gives the concept of procedural values "a procedural value is determined by an underlying procedure of sharing marginal contributions to coalitions formed by players joining in random order." Obviously, the $\alpha-$consensus value is the procedural value, so $\alpha-$consensus value has the following conclusions.

Lemma 1. Every linear efficient value $\gamma_i(v)$ having the equal treatment property is of the form:

$$\gamma_i(v) = \sum_{S \subseteq N, i \in S} \frac{p_t v(S)}{t} - \sum_{S \subset N, i \notin S} \frac{p_t v(S)}{n - t},$$

for every v and i, where $p_1, p_2, \cdots, p_n \in R, p_n = 1$.

Thus, the value of each player is a weighted sum of all coalitions with weights depending only on cardinalities of coalitions. We call $p_1, p_2, \cdots, p_n$ coefficients of the value $\gamma_i(v)$.

Lemma 2. For the value $\gamma_i(v) \in G^N$ determined by a procedure $q = (q_1, q_2, \cdots, q_n)$, the coefficients $p_1, p_2, \cdots, p_n$ are expressed as follow:

$$p_n = 1, p_t = \frac{q_{t+1}}{\binom{n}{t}}, \quad for \quad t < n.$$

Corollary 3. Every linear efficient value with equal treatment property on $v \in G^N$ with coefficients $p_1, p_2, \cdots, p_n$ satisfying

$$p_n = 1, 0 \leqslant p_t \leqslant \frac{1}{\binom{n}{t}} \quad for \quad t = 1, 2, \cdots, n - 1.$$

is procedural, and the coefficients of its procedure are

$$q_1 = 1, q_k = p_{k-1} \cdot \binom{n}{k-1} \quad for \quad k = 2, 3, \cdots, n.$$

Lemma 1 indicates that when the procedure value satisfies the validity and linearity, then it can be expressed as a linear combination of characteristic functions of coalitions. Lemma 2 indicates by the definition of procedure, a procedural value of a player is always a linear combination of characteristic functions of coalitions in the game, and we only need to find the coefficients of this combination. Corollary 3 is the generalization of Lemma 2. The proof of Lemma 2 is similar to Lemma 2 of [21], we omit it for unnecessary repetition. Since consensus value is the procedure value, it satisfies Lemmas 1 and 2 and Corollary 3.

4. THE $\alpha-$CONSENSUS VALUE OF INTUITIONISTIC FUZZY COOPERATIVE GAMES AND PROPERTIES

4.1. The $\alpha-$Consensus Value of Intuitionistic Fuzzy Cooperative Game

Let $(N, \tilde{v})$ be n-person intuitionistic fuzzy cooperative game with characteristic functions of TIFNs, where $N = \{1, 2, \cdots, n\}$ is a finite set of players with $|N| \geqslant 2$, and $\tilde{v} : 2^N \to \tilde{\mathfrak{R}}$ is a characteristic function on N such as $\tilde{v}(\phi) = \langle 0, 0, 0 \rangle$. For each coalition $S \subseteq N$, TIFN $\tilde{v}(S) = \langle (v_1^-(S), v(S), v_1^+(S)); v_2^-(S), v(S), v_2^+(S) \rangle$ is called the payoff of coalition S. This is what the members of coalition S can obtain by agreeing to cooperate. The set of all n-person intuitionistic fuzzy cooperative games $(N, \tilde{v})$ is denoted by $\tilde{G}^N$.

Nan *et al.* [17] studied the Shapley function of intuitionistic fuzzy cooperative games. Based on the extended Hukuhara difference of intuitionistic fuzzy numbers, the specific expression of the Shapley value of intuitionistic fuzzy cooperative games is defined as follows:

Definition 7. [17] For an intuitionistic fuzzy cooperative game $\tilde{v} \in \tilde{G}^N$, let Shapley value be expressed as

$$Sh_i(\tilde{v}) = \rho[\tilde{v}(S) -_H \tilde{v}(S\backslash\{i\})]. \tag{4}$$

where $\rho = \sum_{S \subseteq N} \dfrac{(n - |S|)! \, (|S| - 1)!}{n!}, S \subseteq N, \tilde{v} \in \tilde{G}^N$.

Based on the extended Hukuhara difference of intuitionistic fuzzy numbers, the CIS value of intuitionistic fuzzy cooperative games $(N, \tilde{v})$ is defined as follows:

Definition 8. For an intuitionistic fuzzy cooperative game $\tilde{v} \in \tilde{G}^N$, let CIS value of the cooperative game be expressed as

$$CIS_i(\tilde{v}) = \tilde{v}\{i\} + \frac{1}{n}\left[\tilde{v}(N) -_H \sum_{j \in N} \tilde{v}\{j\}\right]. \tag{5}$$

Thus, extending the consensus value of cooperative game, the α−consensus value of intuitionistic fuzzy cooperative game is obtained as follows:

$$\gamma_i(\tilde{v}) = \alpha Sh_i(\tilde{v}) + (1 - \alpha)CIS_i(\tilde{v}). \tag{6}$$

4.2. Properties of the α−Consensus Values of Intuitionistic Fuzzy Cooperative Game

Form Eq. (6), it is easily seen that the consensus value of intuitionistic fuzzy cooperative game is calculated using the Hukuhara difference of TIFNs, and satisfies some properties similar to those of cooperative game.

Let $\gamma : \tilde{G}^N \to \tilde{R}^n$, we consider the following properties.

Efficiency : For any intuitionistic fuzzy cooperative game $\tilde{v} \in \tilde{G}^N$, $\sum_{i \in N} \gamma_i(\tilde{v}) = \tilde{v}(N)$.

Symmetry : For any intuitionistic fuzzy cooperative game $\tilde{v} \in \tilde{G}^N$, two players $i, j \in N$ are symmetric if for every coalition $S \subseteq N\backslash\{i, j\}$, $\tilde{v}(S \bigcup\{i\}) = \tilde{v}(S \bigcup\{j\})$, then $\gamma_i(\tilde{v}) = \gamma_j(\tilde{v})$.

Linearity : For any intuitionistic fuzzy cooperative game $\tilde{v}, \tilde{w} \in \tilde{G}^N$ and $b, c \in R$, $\gamma_i(b\tilde{v} + c\tilde{w}) = b\gamma_i(\tilde{v}) + c\gamma_i(\tilde{w})$, where $b\tilde{v} + c\tilde{w}$ is given by $(b\tilde{v} + c\tilde{w})(S) = b\tilde{v}(S) + c\tilde{w}(S)$, for all $S \subseteq N$.

Variability : For any intuitionistic fuzzy cooperative game $\tilde{v}, \tilde{w} \in \tilde{G}^N$, $a > 0$ and $d \in R^n$, where $\tilde{w}$ is given by $\tilde{w}(S) = a\tilde{v}(S) + \sum_{i \in S} d_i$ for all $S \subseteq N$, then $\gamma_i(\tilde{w}) = a\gamma_i(\tilde{v}) + d$.

Dummy : For any intuitionistic fuzzy cooperative game $\tilde{v} \in \tilde{G}^N$, if player $i \in N$ is a dummy player, for each coalition $S \subseteq N\backslash i$, $\tilde{v}(S \bigcup\{i\}) -_H \tilde{v}(S) = \tilde{v}\{i\}$, then $\gamma_i(\tilde{v}) = \tilde{v}\{i\}$.

Theorem 4. *For any intuitionistic fuzzy cooperative game $\tilde{v} \in \tilde{G}^N$, the α−consensus value satisfies the following properties: Efficiency, Linearity, Variability and Dummy.*

Proof. *Efficiency*: Based on the effectiveness of Shapley values, CIS values and combine Eq. (6), one obtains that

$$\sum_{i=1}^{n} \gamma_i(\tilde{v}) = \sum_{i=1}^{n} \alpha Sh_i(\tilde{v}) + (1 - \alpha)CIS_i(\tilde{v}).$$

By Eqs. (1) and (2) of Definition 3, we can get

$$\begin{aligned}
\sum_{i=1}^{n} \gamma_i(\tilde{v}) &= \sum_{i=1}^{n}(\alpha\rho[\tilde{v}(S) -_H \tilde{v}(S\backslash i)]) + \sum_{i=1}^{n}((1 - \alpha)[\tilde{v}\{i\} \\
&\quad + \frac{1}{n}\left(\tilde{v}(N) -_H \sum_{j \in N} \tilde{v}\{j\}\right)]) \\
&= \alpha \sum_{i=1}^{n}(\rho[\tilde{v}(S) -_H \tilde{v}(S\backslash i)]) + (1 - \alpha)\sum_{i=1}^{n}[\tilde{v}\{i\} \\
&\quad + \frac{1}{n}\left(\tilde{v}(N) -_H \sum_{j \in N} \tilde{v}\{j\}\right)] \\
&= \alpha\tilde{v}(N) + (1 - \alpha)\tilde{v}(N) \\
&= \tilde{v}(N).
\end{aligned}$$

Symmetry:

Similarly, it follows from Eq. (6)

$$\begin{aligned}
\gamma_i(\tilde{v}) &= (\alpha\rho[\tilde{v}(S \bigcup\{i\}) -_H \tilde{v}(S)]) + (1 - \alpha)[\tilde{v}\{i\} \\
&\quad + \frac{1}{n}\left(\tilde{v}(N) -_H \sum_{j \in N} \tilde{v}\{j\}\right)].
\end{aligned}$$

and

$$\begin{aligned}
\gamma_j(\tilde{v}) &= \left(\alpha\rho\left[\tilde{v}(S \bigcup\{j\}) -_H \tilde{v}(S)\right]\right) + (1 - \alpha)[\tilde{v}\{j\} \\
&\quad + \frac{1}{n}\left(\tilde{v}(N) -_H \sum_{i \in N} \tilde{v}\{i\}\right)].
\end{aligned}$$

According to the characteristic function $\tilde{v}(S \bigcup\{i\}) = \tilde{v}(S \bigcup\{j\})$, it can be obtained that

$$\gamma_i(\tilde{v}) = \gamma_j(\tilde{v}).$$

Linearity:

By Eqs. (1) and (2) of Definition 3 and Eq. (4), we have

$$\begin{aligned}
&\gamma_i(b\tilde{v} + c\tilde{w}) \\
&= (\alpha\rho[b(\tilde{v}(S) -_H \tilde{v}(S\backslash\{i\})) + c(\tilde{w}(S) \\
&\quad -_H \tilde{w}(S\backslash\{i\}))]) + (1 - \alpha)\left[b(\tilde{v}\{i\} + c\tilde{w}\{i\}) \right. \\
&\quad \left. + \frac{1}{n}\left(b\left(\tilde{v}(N) -_H \sum_{j \in N} \tilde{v}\{j\}\right) + c\left(\tilde{w}(N) -_H \sum_{j \in N} \tilde{w}\{j\}\right)\right)\right].
\end{aligned}$$

It follows that

$$\begin{aligned}
&\gamma_i(b\tilde{v} + c\tilde{w}) \\
&= \alpha\rho b(\tilde{v}(S) -_H \tilde{v}(S\backslash\{i\})) + \alpha\rho c(\tilde{w}(S) -_H \tilde{w}(S\backslash\{i\})) \\
&\quad + (1 - \alpha) b\left[\left(\tilde{v}\{i\} + \frac{1}{n}\left(\tilde{v}(N) -_H \sum_{j \in N} \tilde{v}\{j\}\right)\right)\right]
\end{aligned}$$

$$+ (1 - \alpha) c \left[\tilde{w}\{i\} + \frac{1}{n} \left(\tilde{w}(N) -_H \sum_{j \in N} \tilde{w}\{j\} \right) \right]$$

$$= \left(\alpha \rho b \left(\tilde{v}(S) -_H \tilde{v}(S \backslash \{i\}) \right) + (1 - \alpha) b \left[\left(\tilde{v}\{i\} + \frac{1}{n} \left(\tilde{v}(N) \right. \right. \right. \right.$$

$$\left. \left. \left. -_H \sum_{j \in N} \tilde{v}\{j\} \right) \right] \right) + \left(\alpha \rho c \left(\tilde{w}(S) -_H \tilde{w}(S \backslash \{i\}) \right) \right.$$

$$\left. + (1 - \alpha) c \left[\tilde{w}\{i\} + \frac{1}{n} \left(\tilde{w}(N) -_H \sum_{j \in N} \tilde{w}\{j\} \right) \right] \right)$$

$$= b \gamma_i(\tilde{v}) + c \gamma_i(\tilde{w}).$$

Variability:

Similarly, it follows from Eq. (6)

$$\gamma_i(\tilde{w}) = \alpha Sh_i(\tilde{w}) + (1 - \alpha) CIS_i(\tilde{w})$$

$$= \alpha \rho [\tilde{w}(S) -_H \tilde{w}(S \backslash \{i\})] + (1 - \alpha)(\tilde{w}\{i\}$$

$$+ \frac{1}{n} \left[\tilde{w}(N) -_H \sum_{j \in N} \tilde{w}\{j\} \right] \right).$$

According to $\tilde{w}(S) = a\tilde{v}(S) + \sum_{i \in S} d_i$, $\gamma_i(\tilde{w})$ can be calculated as follows:

$$\gamma_i(\tilde{w}) = \alpha \rho \left[\left(a\tilde{v}(S) + \sum_{i \in S} d_i \right) - \left(a\tilde{v}(S \backslash \{i\}) + \sum_{S \backslash \{i\}} d \right) \right]$$

$$+ (1 - \alpha) \left((a\tilde{v}\{i\} + d_i) + \frac{1}{n} \left[\left(a\tilde{v}(N) + \sum_{i \in N} d_i \right) \right. \right.$$

$$\left. \left. -_H \sum_{j \in N} a\tilde{v}\{j\} + d_j \right] \right)$$

$$= \alpha \rho \left[(a\tilde{v}(S) -_H a\tilde{v}(S \backslash \{i\})) + \left(\sum_{i \in S} d_i - \sum_{S \backslash \{i\}} d \right) \right]$$

$$+ (1 - \alpha)((a\tilde{v}\{i\} + d_i) + \frac{1}{n} \left(\left[\left(a\tilde{v}(N) -_H \sum_{j \in N} a\tilde{v}\{j\} \right] \right. \right.$$

$$\left. \left. + \left(\sum_{i \in N} d_i - \sum_{j \in N} d_j \right) \right) \right).$$

Thus, one has

$$\gamma_i(\tilde{w}) = a\alpha\rho [(\tilde{v}(S) -_H \tilde{v}(S \backslash \{i\})] + \alpha \rho d_i$$

$$+ a(1 - \alpha)(\tilde{v}\{i\} + \frac{1}{n} \left[\tilde{v}(N) -_H \sum_{j \in N} \tilde{v}\{j\} \right]) + (1 - \alpha)d_i$$

$$= a[\alpha\rho [(\tilde{v}(S) -_H \tilde{v}(S \backslash \{i\})] + (1 - \alpha)(\tilde{v}\{i\} +$$

$$\frac{1}{n} \left[\tilde{v}(N) -_H \sum_{j \in N} \tilde{v}\{j\} \right]) \right] + (\alpha\rho d_i + (1 - \alpha)d_i)$$

$$= a\gamma_i(\tilde{v}) + d.$$

Dummy:

From Eq. (6), we have

$$\gamma_i(\tilde{v}) = \alpha Sh_i(\tilde{v}) + (1 - \alpha) CIS_i(\tilde{v})$$

$$= \alpha \rho [\tilde{v}(S) -_H \tilde{v}(S \backslash \{i\})] + (1 - \alpha)(\tilde{v}\{i\}$$

$$+ \frac{1}{n} \left[\tilde{v}(N) -_H \sum_{j \in N} \tilde{v}\{j\} \right] \right).$$

According to the characteristic function
$\tilde{v}(S \bigcup \{i\}) -_H \tilde{v}(S) = \tilde{v}(\{i\})$, it follows that

$$\gamma_i(\tilde{v}) = \alpha \tilde{v}\{i\} + (1 - \alpha)\tilde{v}\{i\}$$
$$= \tilde{v}\{i\}.$$

4.3. The Range of α for the α−Consensus Value of the Intuitionistic Cooperative Games

In this subsection, similarly to the α−consensus value of cooperative games, the range of α, which makes the α−consensus value of intuitionistic cooperative games satisfy the individual rationality, is obtained.

For the intuitionistic fuzzy cooperative game $\tilde{v} \in \tilde{G}^N$, we have

$$\gamma_i(\tilde{v}) = \alpha Sh_i(\tilde{v}) + (1 - \alpha) CIS_i(\tilde{v})$$

$$= \alpha Sh_i(\tilde{v}) + CIS_i(\tilde{v}) - \alpha CIS_i(\tilde{v})$$

$$= \alpha(Sh_i(\tilde{v}) - CIS_i(\tilde{v})) + CIS_i(\tilde{v})$$

$$\geqslant \tilde{v}\{i\}.$$

By simple calculation, we get

$$\alpha \geqslant \frac{\tilde{v}\{i\} - CIS_i(\tilde{v})}{Sh_i(\tilde{v}) - CIS_i(\tilde{v})}. \tag{7}$$

It is easily seen that Eq. (7) involves the operations of subtraction and division of TIFNs, which are difficult. In order to avoid these operations of TIFNs, the λ weighted mean areas of TIFNs are used to transform the TIFNs into real numbers. Thus, according to Definition 5, Eq. (7) is transformed to

$$\alpha \geqslant \frac{S_\lambda(\tilde{v}\{i\}) - S_\lambda(CIS_i(\tilde{v}))}{S_\lambda(Sh_i(\tilde{v})) - S_\lambda(CIS_i(\tilde{v}))}.$$

Then, according to the range of α of α−consensus value of cooperative games, given in Subsection 3.3, we have the following conclusions.

i. If $S_\lambda(Sh_i(\tilde{v})) > S_\lambda(CIS_i(\tilde{v}))$, then we can get

$$\frac{S_\lambda(\tilde{v}\{i\}) - S_\lambda(CIS_i(\tilde{v}))}{S_\lambda(Sh_i(\tilde{v})) - S_\lambda(CIS_i(\tilde{v}))} < 0.$$

So, for arbitrary $\alpha \in [0, 1]$, the α−consensus value of the intuitionistic fuzzy cooperative game satisfies the individual rationality.

ii.   If $S_\lambda(Sh_i(\tilde{v})) < S_\lambda(CIS_i(\tilde{v}))$, then we have

$$0 \leqslant \alpha \leqslant \frac{S_\lambda(\tilde{v}\{i\}) - S_\lambda(CIS_i(\tilde{v}))}{S_\lambda(Sh_i(\tilde{v})) - S_\lambda(CIS_i(\tilde{v}))}.$$

And if $S_\lambda(Sh_i(\tilde{v})) < S_\lambda(\tilde{v}\{i\})$, we obtain

$$0 \leqslant \alpha \leqslant \frac{S_\lambda(\tilde{v}\{i\}) - S_\lambda(CIS_i(\tilde{v}))}{S_\lambda(Sh_i(\tilde{v})) - S_\lambda(CIS_i(\tilde{v}))} < 1.$$

Thus, for $\alpha \in \left[0, \frac{S_\lambda(\tilde{v}\{i\}) - S_\lambda(CIS_i(\tilde{v}))}{S_\lambda(Sh_i(\tilde{v})) - S_\lambda(CIS_i(\tilde{v}))}\right]$, the $\alpha-$consensus value of intuitionistic fuzzy cooperative game satisfies the individual rationality.

While $S_\lambda(Sh_i(\tilde{v})) > S_\lambda(\tilde{v}\{i\})$, for any $\alpha \in [0, 1]$, the $\alpha-$consensus value of the intuitionistic fuzzy cooperative game satisfies individual rationality.

5. NUMERICAL EXAMPLE

Suppose that there are three factories (i.e., players)1, 2 and 3, who have the ability to produce separately. Denoted the set of players by $N = \{1, 2, 3\}$. Given the coalition $S \subseteq N$, let characteristic function $\tilde{v}(S) = \langle (v_1^-(S), v(S), v_1^+(S)); v_2^-(S), v(S), v_2^+(S) \rangle$ be TIFN, all the characteristic function values of coalitions are obtained in Table 1.

According to Eqs. (4) and (5), we get Table 2.

According to Tables 1 and 2 and Definition 5, one can obtain $Sh_2(\tilde{v}) < \tilde{v}(2)$. Hence, $Sh_2(\tilde{v})$ does not satisfy the individual rationality.

According to condition (ii) of 4.3 and let $\lambda = \frac{1}{2}(\lambda \in [0, 1])$, we can get

$$0 \leqslant \alpha \leqslant \frac{S_\lambda(\tilde{v}\{i\}) - S_\lambda(CIS_i(\tilde{v}))}{S_\lambda(Sh_i(\tilde{v})) - S_\lambda(CIS_i(\tilde{v}))} = 0.558 < 1.$$

Thus, for $\alpha \in [0, 0.558]$, the $\alpha-$consensus value of the intuitionistic fuzzy cooperative game satisfies the individual rationality. Specially,

Table 1 │ The intuitionistic fuzzy characteristic functions.

Clear Coalition S	Intuitionistic Fuzzy Characteristic Functions
{1}	$\langle (8, 11, 6), (6, 11, 21) \rangle$
{2}	$\langle (11, 16, 21), (9, 16, 23) \rangle$
{3}	$\langle (21, 26, 31), (18, 26, 36) \rangle$
{1, 2}	$\langle (12, 18, 24), (10, 18, 30) \rangle$
{1, 3}	$\langle (24, 30, 36), (20, 30, 44) \rangle$
{2, 3}	$\langle (25, 32, 38), (22, 32, 46) \rangle$
{1, 2, 3}	$\langle (41, 54, 69), (36, 54, 81) \rangle$

Table 2 │ The intuitionistic fuzzy Shapley value and CIS value of the three factories.

	Shapley Value and CIS Value of the Three-person Intuitionistic Cooperative Game	
	Sh_i	CIS_i
1	$\langle (8.7, 12, 17), (7.2, 12, 21.2) \rangle$	$\langle (8.3, 11.3, 16.3), (6.3, 11.3, 21.3) \rangle$
2	$\langle (10.7, 15.5, 20.5), (9.7, 15.5, 23.2) \rangle$	$\langle (11.3, 16.3, 21.3), (9.3, 16.3, 23.3) \rangle$
3	$\langle (21.7, 26.5, 31.5), (19.2, 26.5, 36.7) \rangle$	$\langle (21.3, 26.3, 31.3), (18.3, 26.3, 36.3) \rangle$

let $\alpha = 0.4$, the consensus values of the three factories are obtained as follows, respectively.

$$\gamma_1(\tilde{v}) = \langle (8.46, 11.58, 16.58), (6.66, 11.58, 21.26) \rangle,$$
$$\gamma_2(\tilde{v}) = \langle (11.06, 15.98, 20.98), (9.46, 15.98, 23.26) \rangle,$$
$$\gamma_3(\tilde{v}) = \langle (21.46, 26.38, 31.38), (18.66, 26.38, 36.48) \rangle.$$

6. CONCLUSION

In this paper, the $\alpha-$consensus value of a cooperative game is improved and a formation mechanism of the $\alpha-$consensus value of the cooperative game is given. Furthermore, we study the ranges of α, which make the $\alpha-$consensus value of the cooperative game satisfy the individual rationality. In addition, the $\alpha-$consensus value of an intuitionistic fuzzy cooperative game is proposed, and its some properties are formulated and proved. By adjusting the parameter α, the $\alpha-$consensus value can make up for the disadvantage that Shapley value does not satisfy the individual rationality under the same cooperative game. This paper studies the $\alpha-$consensus value of the intuitionistic fuzzy cooperative game, other combination solutions of an intuitionistic fuzzy cooperative game can be researched in near future.

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