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Research Article

A Novel Probability Weighting Function Model with Empirical Studies

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ABSTRACT

Probability weighting is one of the key components of the modern risky decision-making theories, an effective probability weight function can more accurately describe the decision-makers' subjective response to the event probability. While the probability weighting functions (PWFs) with several different parametric forms and parameter-free elicitation methods have been proposed. This paper first introduces a Lagrange interpolation method (LIM) for building a parameter-free PWF model, then proposes a novel PWF model with the use of the LIM based on Prelec's PWF model. Furthermore, an experiment was designed and carried out. The results not only demonstrate that the novel PWF model could reflect the empirical regularities for maximizing the satisfaction degree of the curve fitting for the preference points obtained from experiment or questionnaire survey and better predict the preferences of decision-makers, but also are found to be consistent with the properties of PWF. This paper makes a significant methodological contribution to developing a numerical method, such as LIM, for constructing the probability weighting model. The finial error analysis suggests that the novel PWF model is a more effective approach.

1. INTRODUCTION

Situations that have to make decisions under an uncertain environment are more common than under a deterministic environment. Generally, people have a lot of choices under uncertainty and they may try to understand these choices. Therefore, it is useful to know the decision-maker's underlying preferences. To quantify this choice behaviors process, different theories had been proposed, i.e., an expected utility (EU) theory [1], a subjectively weighted utility (SWU) theory [2], a rank- and sign-dependent utility (RSDU) theory [3], a prospect theory (PT) [4], and a cumulative prospect theory (CPT) [5] have attracted an enormous amount of research attentions in both theories (Refs. [6–9]) and practical applications (Refs. [10–15]).

In a series of decision-making theories under risk and uncertainty, the probability of an event plays a pivotal role in different forms. In the EU theory, the utility of an uncertain prospect is the sum of the utilities of the outcomes, and each weighted by its probability. For example, a prospect $L = (x_1, p_1; \cdots; x_m, p_m)$ is assumed a contract that yields the possible monetary outcomes or possible wealth levels $x_i\,(i = 1, 2, \cdots, m)$ with probabilities $p_i\,(i = 1, 2, \cdots, m)$, where $p_1 + p_2 + \cdots p_m = 1$. Under well-known assumptions, the EU function takes the following form:

$$EU(L) = \sum_{i=1}^{n} p_i u(x_i) \tag{1}$$

where $u(x_i)$ is the utility of the outcome x_i. Owing to the fact that the relationship of preference and risky prospects are not linear in probability, Ref. [2] proposed the concept of the SWU theory, which used the weighting of decision-making instead of the linear probability, and its functional form takes

$$SWU(L) = \sum_{i=1}^{n} \pi(p_i)\, u(x_i) \tag{2}$$

where $\pi(p_i)$ is the subjective probability of p_i. At the same time, because several classes of choices problems' preferences (e.g., the Allais paradoxes [16]) systematically violate the axioms of the EU theory, Ref. [5] introduced the PT, in which the decision weight of an event is not its probabilities in the EU theory, it does not obey the probability axioms, and it should not be interpreted as measures of degree or belief. The PT is formulated as a reaction to the growing evidences that people do not follow the norms of economic EU maximization. Moreover, the PT as a new behavior under risk decision model does not only have impact on psychology fields, but also bring the psychology and some other subjects such as economics, philosophy and risk management closer together. The PT distinguishes two phases in a choice process: (1) A framing phase in which a decision-maker constructs a representation of the acts which reflects on the value function. (2) A valuation phase in which

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the decision-maker assesses the probability of each prospect and select the best one accordingly. The functional form of the PT can be defined as

$$PT(L) = \sum_{i=1}^{n} w\left(p_i\right) u\left(x_i\right) \tag{3}$$

where $w\left(p_i\right)(i = 1, 2, \cdots, m)$ is the value of probability $p_i(i = 1, 2, \cdots, m)$ in the PWF model. With the development of PWF, Ref. [17] considered that decision weight could not be applied to individual probability, but the overall probability instead, hence, he first proposed the Rank- dependent EU model. Ref. [3] also put forward a RSDU theory which combined rank and sign, and suggested that weighting is based on the rank order of outcomes and the outcomes related to sign. After a period of important development, Ref. [5] suggested that the PWF was not the probability's function of an event, but the cumulative probability distribution function of the decision-making results. So they absorbed RSDU and cumulative function theory, and improved the PT, and then put forward the CPT to resolve the constraint that early PT did not always satisfy the condition of stochastic dominance. A functional form of the CPT is defined as

$$CPT(L) = \begin{cases} \sum_{i=1}^{n}\left(v(x_i)\left(w\left(\sum_{r=i}^{n} p_r\right) - w\left(\sum_{s=i+1}^{n} p_s\right)\right)\right) \\ \quad if\ x_i > 0\ or\ x_i < 0\ for\ i = 1, 2, \cdots, n; \\ \sum_{i=1}^{k}\left(v(x_i)\left(w\left(\sum_{r=i}^{k} p_r\right) - w\left(\sum_{s=i}^{k-1} p_s\right)\right)\right) \\ + \sum_{i=k+1}^{n}\left(v(x_i)\left(w\left(\sum_{r=i}^{n} p_r\right) - w\left(\sum_{s=i+1}^{n} p_s\right)\right)\right) \\ \quad if\ x_i < 0\ for\ i = 1, 2, \cdots, k; x_i > 0\ for\ i = k+1, k+2, \cdots, n. \end{cases} \tag{4}$$

where, $w(p)$ is the cumulative PWF. Through the theory analysis of the above, it is not difficult to find that a decision-making theory comprises two key transformations, i.e., the value function and decision weights, known as PWF, which reflects not only the prospect's probability, but also the decision-maker's psychological preference for different outcomes. The focus of this paper is the novel PWF based on the different preferences of decision-makers by combining the Lagrange interpolation method (LIM) with Prelec's model.

Along with the development of decision-making theory under risk and uncertainty, a large number of scholars began to study the properties and applications of the PWF separately. The properties of PWF were significantly influenced by the source of the event's uncertainty and subjects' risk attitudes. Ref. [18] investigated the relationship between risk attitudes and decision weight. Ref. [19] described the curvature of the PWF, they believed that the PWF permits probabilities to be weighted nonlinearly, and investigated two sources of nonlinearity of decision weights, i.e., subadditively of probability judgements, and the overweighting of small probabilities and underweighting of medium and large probabilities. At the same time, Refs. [20–22] studied the shape of PWF. Ref. [20] presented a nonparametric estimation procedure for assessing the PWF and value function at the level of the individual subject, and discussed two features of the PWF. Ref. [23] demonstrated that gender differences in risk-taking behavior crucially depend on probabilities. Ref. [24] modeled the effect of people's risk choices by a more curved PWF, research suggests that people are less sensitive to variations in probability in affect-rich compared with affect-poor risky choices. Ref. [25] considered heterogeneity in probability distortion, through experiments, they concluded that 20% of the populations adhere to linear probability weighting, the choice of roughly 80% of the subject exhibit significant deviations from linear probability weighting of varying strength. The proposed PWF model is based on experimental data. Therefore, no matter how many the respondents differ, as long as the preference information is given, the novel PWF model can better reflect and predict respondents' underlying preferences.

Ref. [5] first offered a hypothesis to establish a psychological foundation for the PWF. Based on the weighting function of Ref. [5], Ref. [26] proposed the first axiomatically derived weighting function. Then, Ref. [27] put forward a simpler derivation of Prelec's function. Ref. [28] provided a further simplification derivation of Prelec's PWF. Ref. [29] showed that the Tversky-Kahneman PWF is not increasing for all parameter values and therefore can assign negative decision weighting to some outcomes. Ref. [30] outlined three stylized facts on nonlinear weighting that any alternative theory of risk must address. Ref. [31] utilized psychophysical theory for deriving the Prelec's PWF from psychophysical laws of perceived waiting time in probabilistic choices to study the PWF.

Previous research of PWF could be mainly divided into two parts: parametric PWF and parameter-free elicitation of the PWF. Refs. [5,19,26,32] proposed some descriptive models based on a decision theory, psychology of economic behavior, and experimental data fitting, so these descriptive models are called as parameter PWF models. Furthermore, Ref. [33] provided preference foundations for parametric weighting functions under the RDU. Ref. [34] presented a preference foundation for a two-parameter family of PWF. For the parameter-free elicitation method's research, Ref. [35] devised a simple and direct nonparametric method for measuring the change in relative probability weights resulting from a change in payoff ranks. Ref. [36] provided a parameter-free elicitation of the PWF, used aggregated and individual subject data, and obtained probability weights in a new domain. Ref. [37] proposed an optimally efficient elicitation method which took the inevitable distortion of preferences by random errors into account and minimizes the effect of such errors on the inferred PWFs. Ref. [38] reported the results of an experimental parameter-free elicitation and a decomposition of decision weights under uncertainty. Other studies are focused on making a methodological contribution to experimental development economics (i.e., Refs. [7,39,40]). The focus of this paper is to propose a novel parameter-free PWF model which utilizes a Lagrange interpolation approach and a prelec's PWF to build a parameter-free numerical model. The interpolated preference points are divided into two parts: (a) experimental preference points collected by processing the data of experiment or questionnaire survey and (b) inferred preference points collected by integrating the experimental preference points into the descriptive model (such as prelec's PWF model). The three key steps can be expounded as follows:

1. The novel PWF model starts with the collection of the experimental preference points. In this process, an experiment or a questionnaire survey is first designed and carried out to obtain several high-quality experimental data, and a group experimental preference points are collected by utilizing certainty equivalent method (it was defined in Definition 2 in Section 2) to deal with the obtained experimental data.
2. Prelec's PWF model is used to obtain several inferred preference points. In this process, the coefficients of Prelec's PWF model are first determined by curve fitting the experimental preference points. After then, it is easy to use the determined Prelec's PWF model to infer several preference points. Moreover, some mathematical methods are utilized to update these preference points into the inferred preference points.
3. A Lagrange interpolation approach is introduced to interpolate the experimental preference points and the inferred preference points to build a novel PWF model.

To display the application of the novel PWF model, an empirical study and an error analysis are then carried out in Sections 5 and 6.

The remainder of this paper is organized as follows: Section 2 outlines the preliminaries. Section 3 presents the existing PWF models and analyzes their characteristics. In Section 4, a novel PWF model is proposed to reflect decision-makers' preferences by combining the Lagrange interpolation approach and the existing PWF model. An empirical study is carried out through an experiment survey in Section 5. Section 6 is devoted to analyzing the errors between the proposed model and the existing ones. Finally, Section 7 gives the conclusions and proposes possible extensions.

2. PRELIMINARIES

Being the focus of this paper, the properties of PWF are summarized and redefined as follows:

Definition 1 (Ref. [4]). PWF $w(x)$ satisfies the next properties:

1. $w(x)$ is a weak increasing function for probability p in the interval $(0, 1)$.
2. Overweighting for small values of p, which means that a lower interval $(0, q]$ has more impact on decision-makers than an intermediate interval $[p, q + p]$, provided that $q + p$ is bounded away from 1.
3. Underweighting for big values of p, which says that a higher interval $[1 - q, 1)$ has more impact on decision-makers than an intermediate interval $[p, q + p]$, provided that p is bounded away from 0.
4. Loss aversion. People behave differently on gains and on losses. They are not uniformly risk-averse but distinctively more sensitive to losses than to gains.
5. A reference point. For different reference points, decision-makers have different subjective responses to the same objective probability p.

In Definition 1, PWF $w(p)$ are defined in the interval $(0, 1)$, it does not contain the boundary point 0 and 1. This is because a nonlinear PWF has a great significance at the end points of the property interval $[0, 1]$. Ref. [28] highlighted an important stylized fact on the nonlinear PWF that decision-makers ignore events of extremely low probability and treat extremely high-probability events as certain one. However, Ref. [4] concluded that probability weights are not well behaved near the end points. Consequently, the end points of the probability interval $[0, 1]$ are ignored as shown in Figure 1. For convenience, this paper is consistent with the latter, it does not account for the end points of the probability interval $[0, 1]$ to avoid the effect of boundary points.

The developments of decision-making theories from the EU to the CPT with gain–loss separability were to accommodate the need of behavioral decision-making. As a result, the value function and PWF are separated. In this process, a value function converts money into value, and a PWF converts a subjective probability into a decision weight. The focus of this paper is on the latter of the

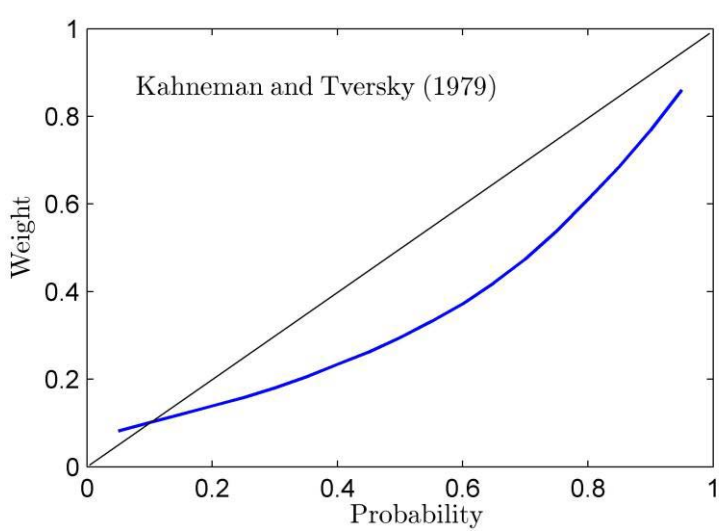

Figure 1 The probability weighting function (PWF) of Ref. [4].

two transformations. Through comprehensive reviews of literatures including Refs. [2,19,26,32,34,36,41], it can be found that research of PWF mainly focused on the parametric PWF and the parameter-free elicitation of the PWF. However, the analysis for the two ways both started with obtaining decision weights. To analyze decision weights quantitatively, a certainty equivalent method should be first introduced in order to convert experimental data to decision weights (probability weights).

Definition 2 (Ref. [5]). For each two-outcome prospect of the form $\left(x_i, p_i; 0, 1-p_i\right)$ $(i = 1, 2, \cdots, n)$, it is assumed that there is a certainty prospect $(c_i, 1)$ $(i = 1, 2, \cdots, n)$, and $\left(x_i, p_i; 0, 1-p_i\right) \sim (c_i, 1)$ (in where, $i = 1, 2, \cdots, n$, and the symbol $\sim$ represents the indifferent or equivalent preference relation). Let c_i/x_i $(i = 1, 2, \cdots, n)$ be the radio of the certainty equivalent of the prospect to the nonzero outcome c_i. The method is called as the certainty equivalent method:

$$w\left(p_i\right) = \left|\frac{c_i}{x_i}\right| \text{ and } p_i \to w\left(p_i\right) \ (i = 1, 2, \cdots, n) \tag{5}$$

For convenience, it assumes that n pairs preference information are collected by using the certainty equivalent method.

$$\left(p_i, w\left(p_i\right)\right) \ (i = 1, 2, \cdots, n) \tag{6}$$

The preference information is denoted as preference points, and the preference points defined as the specific reflection of decision-makers' preferences. In particular, those preference points can be collected by processing experimental data as shown in Section 5.

Definition 3 (Ref. [18]). The objective probability p_k with probability weight $w\left(p_k\right)$ constitute the dyadic arrays $\left(p_k, w\left(p_k\right)\right)$ $(k = 1, 2, \cdots, M)$, called preference points.

1. The objective probability p_k $(k = 1, 2, \cdots, M)$ is the probability of events.
2. The probability weight $w\left(p_k\right)$ $(k = 1, 2, \cdots, M)$ is the psychology probability of decision-makers.
3. The preference points $\left(p_k, w\left(p_k\right)\right)$ $(k = 1, 2, \cdots, M)$ reflect the decision-maker's psychological choices for different prospects.
4. The preference points are used to reflect the preference of decision-makers, obtain the fitting curve, called preference curve.
5. The preference points obtained by processing the experimental data are called the experimental preference points such as $\left(p_i, w\left(p_i\right)\right)$ $(i = 1, 2, \cdots, N)$.
6. The preference points which are obtained by integrating the existing PWF models $w\left(p\right)$ and the experimental preference points $\left(p_i, w\left(p_i\right)\right)$ $(i = 1, 2, \cdots, N)$ the into preference points $\left(p_k, w\left(p_k\right)\right)$ $(k = 1, 2, \cdots, M; N \leq M)$ are called inferred preference points such as $\left(p_j, w\left(p_j\right)\right)$ $\left(j = 1, 2, \cdots, L; L \leq M\right)$.

Through Definition 3, it can be seen that the preference points are made up of two parts: the experimental preference points and the inferred preference points. The experimental preference points are collected by processing the data of experiment or questionnaire survey as shown in Table 1. The inferred preference points are collected by utilizing the existing PWF models such as those in Table 2 and the experimental preference points to infer.

3. EXISTING PWF MODELS

3.1. Models Review

The development of the PWF can be traced back to the last century (Refs. [4,26,42]). Moreover, with the developments of decision theory, several forms of PWF models, such as a linear model, two one-parameter models and two two-parameter models, have been investigated. These three forms of PWF models recalled in Ref. [8] can be outlined in Table 2.

To analyze the existing PWF models in Table 2, the parameters changes of the PWF models were drawn as shown in Figure 2. By a visual inspection of the shapes of the probability weighting curves, it is not surprising that the PWF curves are both regressive and inverse S-shaped for the value of parameters.

For these PWF models, the linear model is no parameter, and its curve is a diagonal line; the one-parameter model has a parameter γ such as the models of Refs. [5,26]; the two-parameter model has two parameters γ and δ such as the models of Refs. [26,43]. The model's curve of Ref. [5] is shown in the upper left panel of Figure 2. $w(p)$ approximates the horizontal axis of p as $\gamma \to 0$; $w(p)$ approximates the linear model as $\gamma \to 1$. The upper right panel of Figure 2 shows the changes of curvature parameter and elevation parameter for the model of Ref. [43]. The two pictures of Ref. [26] display the changes of curvature parameter γ and elevation parameter δ in a respective way. The common loss–gain weighting functions of both the model of Ref. [43] and the model of Ref. [26] are intrinsically asymmetric, with a fixed point and inflection point at $p = 0.5$ (Ref. [43]), and with an affixed point and inflection point at $p = 1/e$ (Ref. [5]). As, $w(p)$ approximates the linear model as $\gamma \to 1$ and it approximates a step function as $\gamma \to 0$. So γ could visually be inferred by the slope of the function at the inflection point.

From Figure 2, it is not difficult to find that how the PWF accommodates three intuitions about the decision impact of small probabilities. Three intuitions are as follows:

1. It adopts a inverse S-shaped PWF that exhaustively divide the probability interval into a region where the PWF is concave for small probabilities and a region where PWF is convex for moderate and big probabilities.
2. When the weight of smaller probability tends to 0, the slope tends to infinity at 0, giving a qualitative character to the transition from impossibility to possibility. When the weight of ever bigger probability tends to 1, the slope tends to infinity at 1, giving a qualitative character to the transition from possibility to certainty. The image becomes relatively flatter at smaller probability and bigger probability.
3. The curvature and elevation of the PWF models are controlled by the models' parameters.

3.2. The Analysis of Advantages and Disadvantages of Existing PWF Models

The analysis in Section 3.1 shows some characteristics such as the inverse S-shaped PWF, the fourfold pattern of risk preferences, and the curvature and elevation of the existing PWF models are controlled by the model's parameters. Generally, more parameters,

Table 1 Experimental data for prospect (¥200, p; 0, 1 − p).

Question	Probability		Outcomes					
			c_{i1}	c_{i2}	c_{i3}	c_{i4}	c_{i5}	c_{i6}
1.1	0.01	Value	$0 \le c < 4$	$4 \le c < 8$	$8 \le c < 12$	$12 \le c < 16$	$16 \le c < 20$	$20 \le c < 24$
		q	2	4	7	21	46	10
1.2	0.05	Value	$8 \le c < 12$	$12 \le c < 16$	$16 \le c < 20$	$20 \le c < 24$	$24 \le c < 28$	$28 \le c < 32$
		q	0	1	4	5	19	61
1.3	0.10	Value	$18 \le c < 22$	$22 \le c < 26$	$26 \le c < 30$	$30 \le c < 34$	$34 \le c < 38$	$38 \le c < 42$
		q	2	8	3	5	29	43
1.4	0.25	Value	$38 \le c < 42$	$42 \le c < 46$	$46 \le c < 50$	$50 \le c < 54$	$54 \le c < 58$	$58 \le c < 62$
		q	6	3	2	2	38	39
1.5	0.50	Value	$56 \le c < 64$	$64 \le c < 72$	$72 \le c < 80$	$80 \le c < 88$	$88 \le c < 96$	$96 \le c < 104$
		q	7	8	8	35	19	13
1.6	0.75	Value	$95 \le c < 105$	$105 \le c < 115$	$115 \le c < 125$	$125 \le c < 135$	$135 \le c < 145$	$145 \le c < 155$
		q	3	80	4	3	0	0
1.7	0.90	Value	$125 \le c < 135$	$135 \le c < 145$	$145 \le c < 155$	$155 \le c < 165$	$165 \le c < 175$	$175 \le c < 185$
		q	7	11	66	6	0	0
1.8	0.95	Value	$135 \le c < 145$	$145 \le c < 155$	$155 \le c < 165$	$165 \le c < 175$	$175 \le c < 185$	$185 \le c < 195$
		q	10	31	37	8	4	0
1.9	0.99	Value	$165 \le c < 171$	$185 \le c < 195$	$177 \le c < 183$	$183 \le c < 189$	$189 \le c < 195$	$195 \le c < 200$
		q	3	12	62	8	5	0

Note: The decision data about 90 graduate students are displayed, (¥200, p; 0, 1 − p) is a two-outcome prospect. The first column and second column on the left are respectively the choice questions and the probabilities of outcomes. Value represents the probability corresponding to the range of certainty prospects, q is the number of decision-makers choosing corresponding to the range of certainty prospect (i.e., in question 1.1, there are 10 graduate students, they think that (¥200, 0.01; 0, 0.99) is equivalent to the certainty prospect in the range of between ¥20 and ¥24), the values of $20 \le c < 24$ in the last column of the upper right corner.

Table 2 The existing PWF models.

Model Classifications	Models	Parameters	References
Linear model	$w(p) = p$	Parameter-free	Ref. [1], EU theory
One-parameter model	$w(p) = \frac{p^{\gamma}}{\left(p^{\gamma}+(1-p)^{\gamma}\right)^{\frac{1}{\gamma}}}$	$0.28 < \gamma \le 1$	Ref. [5], PT and CPT
	$w(p) = \exp\left[-(-\ln p)^{\gamma}\right]$	$0 < \gamma \le 1$	Ref. [26], PT and CPT
Two-parameter model	$w(p) = \frac{\delta p^{\gamma}}{\delta p^{\gamma}+(1-p)^{\gamma}}$	$0 < \gamma < 1, \delta > 0$	Ref. [43], PT and CPT
	$w(p) = \exp\left[-\delta(-\ln p)^{\gamma}\right]$	$0 < \gamma < 1, \delta > 0$	Ref. [26], PT and CPT

PWF, probability weighting function; EU, expected utility; PT, prospect theory; CPT, cumulative prospect theory.

better to express PWF. However, it is difficult to solve the PWF model when the parameters become more. For this reason, most researchers focused on one-parameter models or two-parameter models. Because the number of the parameters is less, the existing PWF models can be determined using a small amount of experimental data. So it is convenience to obtain a PWF expression under a small amount of preference information about decision-makers. Despite the functional differences between forms of the PWF, they own the same advantage, i.e., using less preference information to obtain a good performance model.

However, the existing PWF models also have some disadvantages. For instance, a parametric estimation makes the models need more assumptions, the assumed functional form has also been determined when a parameter is fixed; if the true functional form is different from the assumed functional form, then it cannot flexibly reflect the preferences of decision-makers. Moreover, the existing PWF models are solved by curve fitting of the experimental preference points, so the solved models can't accurately reflect the choices of the decision-makers. To solve this problem, we first propose the LIM to interpolate the experimental preference points, and then, a novel PWF model is proposed based on the LIM and prelec's PWF model. Therefore, the novel PWF model combines the advantages of prelec's PWF model with the ones of the LIM. Moreover, the novel PWF model can dynamically be response to decision-makers' preferences by adding the preference information into the PWF model once the preference information has been collected.

4. THE PROPOSED NOVEL PWF MODEL

To build a novel PWF model, a group experimental data[1] response to the preferences of decision-makers are collected by a relative experiment by conducting the questionnaire survey, and the experimental data are processed to obtain the experimental preference points (the details are presented in Section 5). Here, we present the process of building a novel PWF model, which is derived in two

[1]Please refer to Table 3 in Tversky and Kahneman [5] for reference.

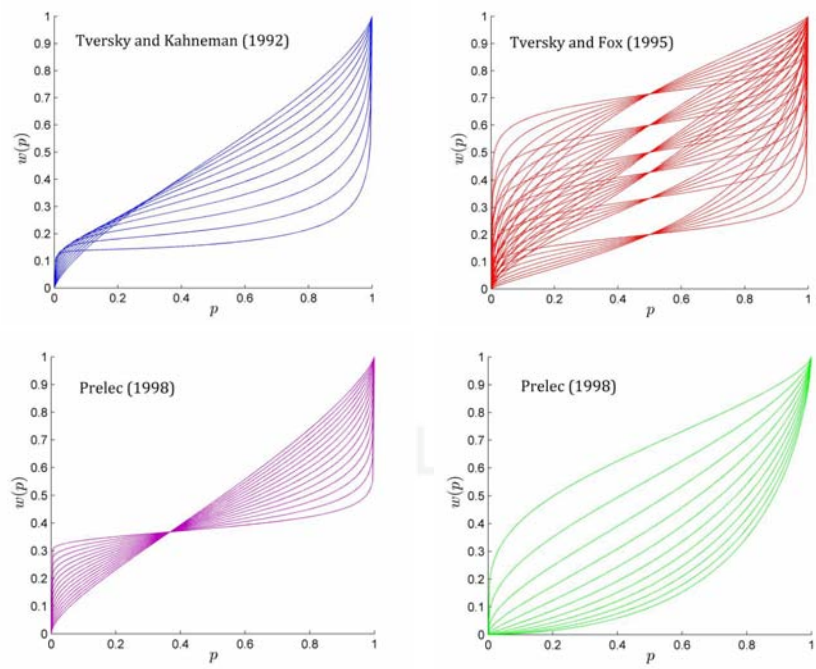

Figure 2 Three families of functions that have been proposed for the probability weighting function (PWF) in prospect theory (PT) and cumulative prospect theory (CPT). Each function is plotted for a range of its parameters: Ref. [5], plotted for different γ (the value of γ is from 0.3 to 0.8 by the steps of 0.05); Ref. [43] from 0.2 to 0.8 for its curvature parameter and 0.25 to 2.5 for its elevation parameter, each in increments of 0.1; Ref. [26] corresponds to two pictures of below left and below right, its curvature parameter changes from 0.1 to 0.8, each in increments of 0.05, its elevation parameter changes from 0.5 to 3, each in increments of 0.25.

steps. First, the LIM is introduced into interpolating the experimental preference points, and the advantages and disadvantages of this method are discussed in Subsection 4.1. Second, Subsection 4.2 presents a novel PWF model by combining the Lagrange interpolate approach with Prelec's model.

4.1. Lagrange Interpolation Method

A PWF used to reflect and predict decision-makers' underlying preferences is usually built by the curve fitting of preference points. However, the resulting PWF curve does not always go through the preference points, so it can not accurately reflect decision-makers' preferences. The LIM is a numerical method, which has the advantage of constructing a smooth curve which can possibly go through all the interpolation points in a fixed interval using a Lagrange interpolation polynomial [44].

For a certain data points, such as $\{(x_0, f(x_0)), (x_1, f(x_1)), \cdots, (x_m, f(x_m))\}$, a curve through these data points can be obtained as follows:

Lemma 1. Assume that there are $m+1$ nodes $x_i\,(i = 0, 1, \cdots, m)$ in the interval $[a, b]$, the m-order polynomial $l_j(x)$ has the property that

$$l_j(x_i) = \begin{cases} 1, & for\ \ i = j; \\ 0, & for\ \ i \neq j. \end{cases} \tag{7}$$

and a basis function of interpolation polynomial is that

$$\begin{aligned} l_j(x) &= \frac{(x - x_0)(x - x)_1 \cdots (x - x_{j-1})(x - x_{j+1}) \cdots (x - x_m)}{(x_j - x_0)(x_j - x_1) \cdots (x_j - x_{j-1})(x_j - x_{j+1}) \cdots (x_j - x_m)} \\ &= \prod_{\substack{i=0 \\ i \neq j}}^{m} \left(\frac{x - x_i}{x_j - x_i} \right) \end{aligned} \tag{8}$$

Then the Lagrange polynomial $L(x)$ for the original interpolation points can be given by the following equation:

$$L(x) = \sum_{j=0}^{m} f(x_j)\, l_j(x) = \sum_{j=0}^{m} \left(f(x_j) \prod_{\substack{i=0 \\ i \neq j}}^{m} \frac{(x - x_i)}{(x_j - x_i)} \right) \tag{9}$$

It is clear that this polynomial has a degree $\leq m$ and has the property that $L(x_j) = y_j\,(j = 0, 1, \ldots, m)$ as required. Note that the Lagrange polynomial, $L(x)$, is unique. If there were two such polynomials, $L(x)$ and $P(x)$, then $L(x) - P(x)$ would be a polynomial of degree $\leq m$ with $m+1$ zeros. If it would have to be identically zero. Thus, $L(x) \equiv P(x)$.

Assume that a group experimental preference points $(p_i, w(p_i))\,(i = 1, 2, \ldots, n)$ defined in Definition 3 have been

collected. The basis function of interpolation polynomial $l_i(p)$ $(i = 1, 2, \ldots, n)$ is constructed as

$$l_i(p) = \prod_{\substack{j=1 \\ j \neq i}}^{n} \left(\frac{p - p_j}{p_i - p_j} \right) \tag{10}$$

By using the basis function $l_i(p)$ $(i = 1, 2, \ldots, n)$, the Lagrange polynomial $L(p)$ is built as

$$L(p) = \sum_{i=1}^{n} w(p_i)\, l_i(p) = \sum_{i=1}^{n} \left(w(p_i) \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{(p - p_j)}{(p_i - p_j)} \right) \tag{11}$$

where the values of p_i and $w(p_i)$ $(i = 1, 2, \cdots, n)$ are obtained from the experimental preference points (Section 5).

Note that the numerical model in Equation (11) is a parameter-free model, and it is built by using the LIM to interpolate the experimental preference points that reflect decision-makers' preference. Its advantage is that the functional form can vary according to the information from the experimental preference points. Therefore, the proposed PWF model can dynamically reflect the change of decision-makers' preferences by changing the experimental preference points or its amount. On the basis of the collected experimental preference points $(p_i, w(p_i))$ $(i = 1, 2, \cdots, n)$, another m pairs preference information are collected. According to Definition 2, the new m experimental preference points are constructed as

$$(p_i, w(p_i)) \; (i = n+1, n+2, \cdots, n+m) \tag{12}$$

To update the PWF model, the m new experimental preference points are incorporated into the previous n experimental preference points. The basis function of interpolation polynomial $l_i(p)$ $(i = 1, 2, \cdots, n, n+1, \cdots, n+m)$ is updated as

$$l_i(p) = \prod_{\substack{j=1 \\ j \neq i}}^{n+m} \left(\frac{p - p_j}{p_i - p_j} \right) \tag{13}$$

Based on the basis function $l_i(p)$ $(i = 1, 2, \cdots, n+m)$, the numerical model in Equation (11) is updated as

$$L'(p) = \sum_{i=1}^{n+m} w(p_i)\, l_i(p) = \sum_{i=1}^{n+m} \left(w(p_i) \prod_{\substack{j=1 \\ j \neq i}}^{n+m} \frac{(p - p_j)}{(p_i - p_j)} \right) \tag{14}$$

Compared with Equation (11), Equation (14) is changed by adding the m new experimental preference points into Equation (11). In other words, the $L'(p)$ is obtained by utilizing the LIM to interpolate $n + m$ experimental preference points. So, it is easy to find that the PWF-based model can vary with the amount of known experimental preference points.

However, the numerical PWF models (11) and (14) also have some disadvantages due to the properties of Lagrange interpolation polynomial. For example, the obtained models would not reflect and predict decision-makers' preferences in an accurate way when the number of interpolation nodes is large or small. On the one hand, the more preference points collected from decision-makers, the more accurate response to preferences of decision-makers. Moreover, the PWF's properties defined in Definition 1 are overweight for small value of p and underestimate for lager value of p, so we need more preference points to express the preference curve when $p \to 0$ or $p \to 1$. However, the questionnaire survey cannot obtain large experimental data upon most occasions. On the other hand, suppose that we obtain a lot of experimental data by experiment or questionnaire survey, and then, the built PWF model is unstable due to the high-order oscillatory of Lagrange interpolation polynomial.

In order to intuitively display the advantages and disadvantages of a numerical PWF model, it is assumed that some experimental preference points such as $(0.01, 0.08)$, $(0.05, 0.14)$, $(0.10, 0.18)$, $(0.25, 0.28)$, $(0.50, 0.42)$, $(0.75, 0.56)$, $(0.90, 0.74)$, $(0.95, 0.78)$, $(0.99, 0.90)$ are collected by experiment or questionnaire survey. Using these experimental preference points, a numerical PWF such as Equations (11) and (14) model is built. Figure 3 displays the curve of the numerical PWF model. Based on Figure 3, although the proposed PWF model (such as Equations (11) and (14)) can make accurate response to the experimental preference points (i.e., the built PWF curve goes through all experimental preference points) and satisfies overweighting for small probabilities, underweighting for big probabilities, loss aversion and reference-dependent[2] in Definition 1, it will not be able to better predict the decision-makers' preferences than the existing PWF models in Table 2. For example, the built PWF curve isn't a strong increasing function for probability near the point $p = 0.5$.

A good PWF model not only makes accurate response to the experimental preference points possible, but also better predicts the decision-makers' preferences regardless of the amount of experimental data is great or small, as well as satisfying the properties of PWF in Definition 1. To this end, a novel PWF model is proposed to combine the advantages of the prelec's PWF models and the LIM introduced in Section 4.1. Specifically, the novel PWF model is built to utilize the LIM to interpolate the preference points collected by processing the experimental data, and by inferring the preference points from prelec's PWF models.

4.2. A Novel PWF Model

Two cases are considered in the process of building a novel PWF model. At first, it considers the case that the amount of collected experimental data is small. To present the processes of building the novel PWF model, a flow chart is depicted in Figure 4. The two main building blocks of this method are introducing into the prelec's PWF model and updating the preference points. The method begins

[2]For different reference points, the researchers can obtain different experimental preference points by experiments or questionnaires. Therefore, different numerical PWF model can be built by utilizing Lagrange interpolation approach to interpolate different experimental preference points.

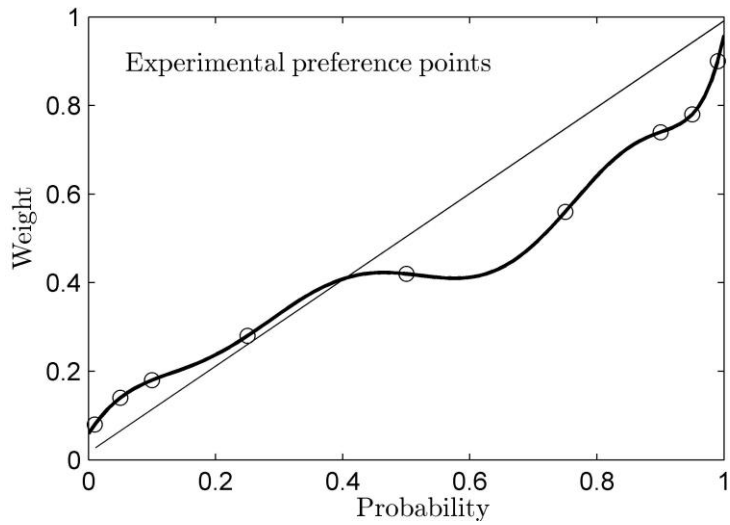

Figure 3 | The numerical probability weighting function (PWF) model is built based on experimental preference points.

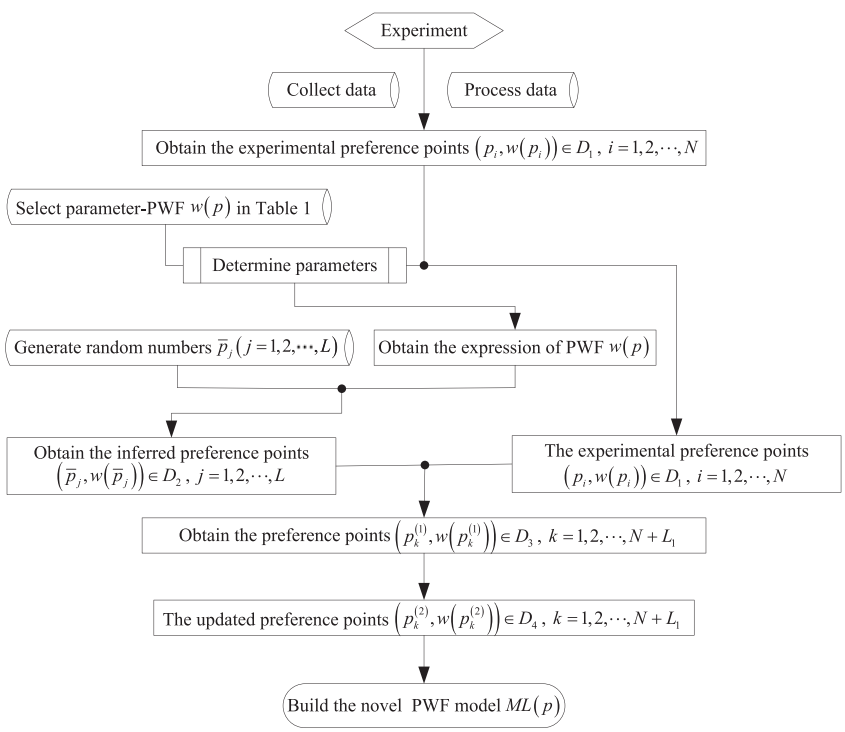

Figure 4 | The flow chart of building the modified probability weighting function (PWF) model.

with conducting experiment or questionnaire survey, ends with obtaining the updated preference points. Then, using the LIM to interpolate the updated preference points, a new PWF model is obtained. For convenience, the novel PWF model is termed as $ML(p)$. At last, it considers the case that the amount of collected experimental data is big. For this case, a piecewise method is introduced.

Case 1:

The detailed computation is outlined as follows:

Step 1. Conduct experiments. A set of experimental data is obtained by an experiment or a questionnaire survey (See Section 5.1 for detailed description).

Step 2. Process the experimental data. It first assumes that N groups experimental data are collected. To obtain the experimental preference points, the medial cash equivalent method is utilized to transform monetary outcomes into the psychology probability $w(p_i)$ of p_i. The experimental preference points are represented by the dyadic array $(p_i, w(p_i))$ $(i = 1, 2, \cdots, N)$, the

collection of dyadic arrays is denoted by the set D_1, i.e., $(p_i, w(p_i)) \in D_1$ $(i = 1, 2, \cdots, N)$ (see Section 3.2).

Step 3. Select one of the existing PWF models $w(p)$ (Table 2).

Step 4. Determine the parameters of the selected PWF model $w(p)$ using the dyadic arrays in D_1.

Step 5. Generate random numbers. To infer the preference information, the random numbers $\overline{p}_j$ $(j = 1, 2, \cdots, L)$ is generated on interval $[0, 1]$ with the aid of the MATLAB software. For convenience, the generated random numbers can be uniform distribution, also can be piecewise uniform distribution according to the density degree of $\overline{p}_j$ $(j = 1, 2, \cdots, L)$, but it must contain p_i $(i = 1, 2, \cdots, N)$ for convenience the next computation.

Step 6. Infer the preference points. Apply the generated random numbers $\overline{p}_j$ $(j = 1, 2, \cdots, L)$ obtained in step 5 into the selected PWF $w(p)$ obtained in step 2, a series of preference points $\left(\overline{p}_j, w\left(\overline{p}_j\right)\right)$ $(j = 1, 2, \cdots, L)$ can be determined. Specially, the collection of preference points is denoted by the set D_2, i.e., $\left(\overline{p}_j, w\left(\overline{p}_j\right)\right) \in D_2$ $(j = 1, 2, \cdots L)$. For convenience, $\left(\overline{p}_j, w\left(\overline{p}_j\right)\right)$ $(j = 1, 2, \cdots L)$ are called as inferred preference points.

Step 7. Add the inferred preference points into D_1 and update them. In order to be more specific to this step, the process of the adding and updating of preference points is divided into two sub-steps.

I. If $\overline{p}_j \neq p_i$ $(i = 1, 2, \cdots, N; j = 1, 2, \cdots, L)$, the inferred preference points $\left(\overline{p}_j, w\left(\overline{p}_j\right)\right)$ $(j = 1, 2, \cdots, L)$ are added into D_1. Suppose that the number of $\overline{p}_j$ that satisfies $\overline{p}_j \neq p_i$ $(i = 1, 2, \cdots, N; j = 1, 2, \cdots, L)$ is L_1 $(L_1 \leq L)$, so $N + L_1$ $(L_1 \leq L)$ preference points are collected. For convenience, we rearrange these preference points according to the size of p_i and $\overline{p}_j$, and express them by dyadic arrays $\left(p_k^{(1)}, w^{(1)}\left(p_k^{(1)}\right)\right)$ $(k = 1, 2, \cdots, N + L_1)$. The collection of dyadic arrays is denoted by the set D_3, i.e., $\left(p_k^{(1)}, w^{(1)}\left(p_k^{(1)}\right)\right) \in D_3$ $(k = 1, 2, \cdots, N + L_1)$.

II. If we assume that there are H $(H \leq L_1)$ inferred preference points that interpolated into the area between p_i and p_{i+1}, i.e., a new collection of preference points

$$\left(p_i, w\left(p_i\right)\right), \left(\overline{p}_f, w\left(\overline{p}_f\right)\right), \cdots, \left(\overline{p}_{f+H-1}, w\left(\overline{p}_{f+H-1}\right)\right), \left(p_{i+1}, w\left(p_{i+1}\right)\right), \tag{15}$$

where $i = 1, 2, \cdots, N-1; f = 1, 2, \cdots, H$, they are re-expressed as

$$\left(p_k^{(1)}, w^{(1)}\left(p_k^{(1)}\right)\right), \left(p_{k+1}^{(1)}, w^{(1)}\left(p_{k+1}^{(1)}\right)\right), \cdots, \left(p_{k+H_i}^{(1)}, w^{(1)}\left(p_{k+H_i}^{(1)}\right)\right), \left(p_{k+H_i+1}^{(1)}, w^{(1)}\left(p_{k+H_i+1}^{(1)}\right)\right) \tag{16}$$

in where,

$$\begin{aligned}\left(p_k^{(1)}, w^{(1)}\left(p_k^{(1)}\right)\right) &= \left(p_i, w\left(p_i\right)\right), \left(p_{k+1}^{(1)}, w^{(1)}\left(p_{k+1}^{(1)}\right)\right) \\ &= \left(\overline{p}_j, w\left(\overline{p}_j\right)\right), \cdots, \left(p_{k+H+1}^{(1)}, w^{(1)}\left(p_{k+H+1}^{(1)}\right)\right) \\ &= \left(p_{i+1}, w\left(p_{i+1}\right)\right).\end{aligned} \tag{17}$$

$\left(p_k^{(1)}, w^{(1)}\left(p_k^{(1)}\right)\right)$ and $\left(p_{k+H_i+1}^{(1)}, w^{(1)}\left(p_{k+H_i+1}^{(1)}\right)\right)$ are the experimental preference points in set D_1, the others are the inferred preference points in set D_2. To better reflect decision-makers' preferences according to the experimental preference points $\left(p_i, w\left(p_i\right)\right)$ $(i = 1, 2, \cdots, N)$, we modify the inferred preference points $\left(p_{k+1}^{(1)}, w^{(1)}\left(p_{k+1}^{(1)}\right)\right), \cdots, \left(p_{k+H}^{(1)}, w^{(1)}\left(p_{k+H}^{(1)}\right)\right)$ into $\left(p_{k+1}^{(2)}, w^{(2)}\left(p_{k+1}^{(2)}\right)\right), \cdots, \left(p_{k+H_i}^{(2)}, w^{(2)}\left(p_{k+H_i}^{(2)}\right)\right)$ by $\left(p_k^{(1)}, w^{(1)}\left(p_k^{(1)}\right)\right)$, the detail is given as follows:

$$\begin{aligned}
&w^{(2)}\left(p_k^{(2)}\right) = w^{(1)}\left(p_k^{(1)}\right), \quad k = 1, 2, \cdots, N + L_1 - H_i - 1 \\
&w^{(2)}\left(p_{k+1}^{(2)}\right) = w^{(2)}\left(p_k^{(2)}\right) + f\left(p_{k+1}^{(1)}\right) \\
&\cdots\cdots \\
&w^{(2)}\left(p_{i+h}^{(2)}\right) = w^{(2)}\left(p_{k+H-1}^{(2)}\right) + f\left(p_{k+H}^{(1)}\right) \\
&w^{(2)}\left(p_{k+H+1}^{(2)}\right) = w^{(2)}\left(p_{k+H}^{(2)}\right) + f\left(p_{k+H+1}^{(1)}\right) = w^{(1)}\left(p_{k+H+1}^{(1)}\right) \\
&p_k^{(2)} = p_k^{(1)}, p_{k+1}^{(2)} = p_{k+1}^{(1)}, \cdots, p_{k+H+1}^{(2)} = p_{k+H+1}^{(1)}
\end{aligned} \tag{18}$$

According to the inverse-S property of the PWF [19,20], the function f can be built as follows:

$$\Delta T_k = w^{(1)}\left(p_{k+H+1}^{(1)}\right) - w^{(1)}\left(p_k^{(1)}\right) \tag{19}$$

$$\Delta S_k = w\left(p_{k+H+1}^{(1)}\right) - w\left(p_k^{(1)}\right) \tag{20}$$

$$f\left(p_{k+r}^{(1)}\right) = \frac{\Delta T_k}{\Delta S_k} \cdot \left(w\left(p_{k+r}^{(1)}\right) - w\left(p_{k+r-1}^{(1)}\right)\right), r = 1, 2, \cdots, H_i + 1 \tag{21}$$

where $w\left(p_k^{(1)}\right)$ is the value of the probability $p_k^{(1)}$ corresponding to the existing PWF $w(p)$ selected in ***Step 3***. The updated sequence preference points are denoted by $\left(p_k^{(2)}, w^{(2)}\left(p_k^{(2)}\right)\right)$ $(k = 1, 2, \cdots, N + L_1)$, their set is denoted by D_4, i.e., $\left(p_k^{(2)}, w^{(2)}\left(p_k^{(2)}\right)\right) \in D_4$ $(k = 1, 2, \cdots, N + L_1)$.

Step 8. Build the LIM-based model. By using the preference points $\left(p_k^{(2)}, w\left(p_k^{(2)}\right)\right) \in D_4\,(k = 1, 2, \cdots, N + L_1)$, the basic functions of interpolation polynomial $l_k\left(p\right)\,(k = 1, 2, \cdots, N + L_1)$ are solved as

$$l_k'\left(p\right) = \prod_{\substack{j=1 \\ j \neq k}}^{N+L_1} \left(\frac{p - p_j^{(2)}}{p_k^{(2)} - p_j^{(2)}}\right) \tag{22}$$

Using the interpolation basic functions and the preference points $\left(p_k^{(2)}, w\left(p_k^{(2)}\right)\right) \in D_4\,(k = 1, 2, \cdots, N + L_1)$, the novel PWF model can be expressed as

$$ML(p) = \sum_{k=1}^{N+L_1} \left(w^{(2)}\left(p_k^{(2)}\right) \prod_{\substack{j=1 \\ j \neq k}}^{N+L_1} \left(\frac{p - p_j^{(2)}}{p_k^{(2)} - p_j^{(2)}}\right)\right) \tag{23}$$

According to the above 8 steps, the novel PWF model $ML\left(p\right)$ can be built in Equation (23). In Subsection 4.1, the advantages of using the LIM to interpolate the experimental preference points can be verified, i.e., the functional forms can vary according to information derived from the experimental preference points, and the expression of PWF can go through the experimental preference points completely. Intuitively, it is easy to find that the novel PWF model.

Case 2:

In this case, a large number of experimental preference points can be collected by using the certainty equivalent approach to process the experimental data. If the proposed approach is utilized to build a PWF model, the built PWF model may be unstable due to the high-order oscillatory of Lagrange interpolation polynomial. To solve this problem, a piecewise Lagrange interpolation approach is proposed. In this process, all steps are same to Steps 1–7 in addition to Step 8 (i.e., a piecewise Lagrange interpolation approach is used in the last step). For instance, it assumes that the updated preference points $\left(p_k^{(2)}, w\left(p_k^{(2)}\right)\right) \in D_4\left(k = 1, 2, \cdots, s, s+1, \cdots, g + L_1\right)$ are collected, in where, g is a big positive integer. Using a piecewise LIM to build a novel PWF model, it can divide the interval $(0, 1)$ into $(0, 0.5]$ and $[0.5, 1)$, and assume $p_1^{(2)}, p_2^{(2)} \cdots, p_s^{(2)} \in (0, 0.5]$ and $p_{s+1}^{(2)}, p_{s+2}^{(2)}, \cdots, p_{g+l}^{(2)} \in [0.5, 1)$. Next, the LIM is respectively used in two sub-intervals, a piecewise PWF model is built as follows:

$$ML(p) = \begin{cases} \displaystyle\sum_{k=1}^{s} \left(w^{(2)}\left(p_k^{(2)}\right) \prod_{\substack{j=1 \\ j \neq k}}^{s} \left(\frac{p - p_k^{(2)}}{p_k^{(2)} - p_j^{(2)}}\right)\right) & p \in (0, 0.5] \\ \displaystyle\sum_{k=s+1}^{g+L_1} \left(w^{(2)}\left(p_k^{(2)}\right) \prod_{\substack{j=s+1 \\ j \neq k}}^{g+L_1} \left(\frac{p - p_k^{(2)}}{p_k^{(2)} - p_j^{(2)}}\right)\right) & p \in [0.5, 1) \end{cases} \tag{24}$$

The result of dividing the interval into two parts is that the order of the built PWF is reduced by half. If the amount of collected experimental data is large enough, it can also be divided into three parts or more the interval $(0, 1)$ so that the built PWF model $ML\left(p\right)$ has a lower order. Apparently, this method is effective to avoid the phenomenon of the high-order oscillatory of Lagrange interpolation polynomial.

5. EMPIRICAL STUDIES: APPLICATION OF THE NOVEL PWF MODEL

Several empirical studies have demonstrated the important of PWF and have provided quantitative assessments of its effects (Refs. [5,20]). These studies are consistent with a PWF that satisfies the properties in Definition 1. The aim of the empirical studies in this paper is to examine the proposed novel PWF model's performance. Because of the probability weights are gained by asking a participant for the subjective weight given to a probability (i.e., PWF is the psychological reflects of decision-makers under risk and uncertainty), as they cannot be measured, and must be estimated indirectly through observed choices. Therefore, an empirical study is used to demonstrate the application of the proposed PWF model.

5.1. Experiment Design

An online experiment was designed to collect the preference information of decision-makers under risk and uncertainty. At first, participants were recruited with no special training in decision theory (ss well as the experiments in Ref. [5]). Secondly, the participants were asked to imagine that they were actually faced with the choices described in the questions, and the respondents were asked to indicate the decision they would have made in such a case, separately. For an uncertainty prospect, different decision-makers have different risk preferences, different risk preferences result in different certainty prospects. For this reason, the range of the chosen certainty prospects should be relatively big with a large amount of decision-makers. To solve this problem, the values of certainty prospects is set in a big range[3] to induce the respondents to make decision, and the change range of certainty prospects are set according to the properties of PWF in Definition 1. To this end, six options about the certainty prospect's range were given corresponding to an uncertainty prospect in experiment (in generally, the options are three). For convenience, every participant was asked to select one of the six options. The experimental operation interface for the prospect (¥200, 0.01; 0, 0.99) is presented in Figure 5. Due to the limit of space, the experimental operation interfaces for other prospects were not shown in here.

The experiment consisted of four treatments. Case 1: decision-maker has p chance to obtain ¥200, and $1 - p$ chance to obtain nothing, denoted by prospect $\left(¥200, p; 0, 1 - p\right)$. Case 2: decision-maker has p chance to obtain ¥2000, and $1 - p$ chance to obtain nothing, denoted by prospect $(¥2000, p; 0, 1 - p)$. Case 3: decision-maker has p chance to lose ¥200, and $1 - p$ chance to lose nothing, denoted by prospect $(-¥200, p; 0, 1 - p)$. Case 4: decision-maker

[3]Based on the reasons of psychology, when Giving an uncertainty prospect, participants are difficult to choose an equivalent certainty prospects, but, participants are easy to choose an equivalent certainty prospect in a range. So, we give the certainty prospects' range such as $0 \leq c < 4$.

has p chance to lose ¥2000, and $1-p$ chance to lose nothing, denoted by and prospect $(-¥2000, p; 0, 1-p)$. Each treatment consisted of 9 questions (i.e., 9 different p from 0.01 to 0.99 are chosen in every case in Figure 5). In total, the experiment had 36 questions, and every participant was asked to answer 36 questions. In order to ensure the validity of the experiment, all four treatments were conducted in sequence, and there is ten minutes rest between each two consecutive treatments in order to avoid mutual influence among the respondents. The 9 problems in each treatment reflect that the prospects combined with the different probabilities of an outcome are equivalent to some certainty prospects.

5.2. Experiment

The online experiment interviewed 90 graduate students from School of Transportation and Logistics in Southwest Jiaotong University (50 male students and 40 female students). Special efforts were made to obtain the high-quality experimental data by offering a token of appreciation to the respondents for their serious participation in the interview them. The 60-minute online experiment was carried out in a laboratory of the university. Furthermore, the participants were asked to dedicate to the problems of experiment on some computers without communicating with one another.

5.3. Experimental Result

Using statistical software, the distribution histogram of decision-makers' number for the equivalent prospects of prospect $(¥200, 0.01; 0, 0.99)$ is plotted as shown in Figure 6.

Conclusion: 2 respondents chose the option (A) (i.e., the value of $¥0 \leq c_{11} < ¥4$) from Figure 6, for convenience sake, $c_{11} = (0+4)/2 = 2$ is assumed, and the prospect $(¥200, 0.01; 0, 0.99)$ and $(¥2, 1)$ are indifferent for the 2 respondents. Similarly, the prospect $(¥200, 0.01; 0, 0.99)$ is collected and the certainty prospects $(¥6, 1)$, $(¥10, 1)$, $(¥14, 1)$, $(¥18,1)$ and $(¥22, 1)$ are

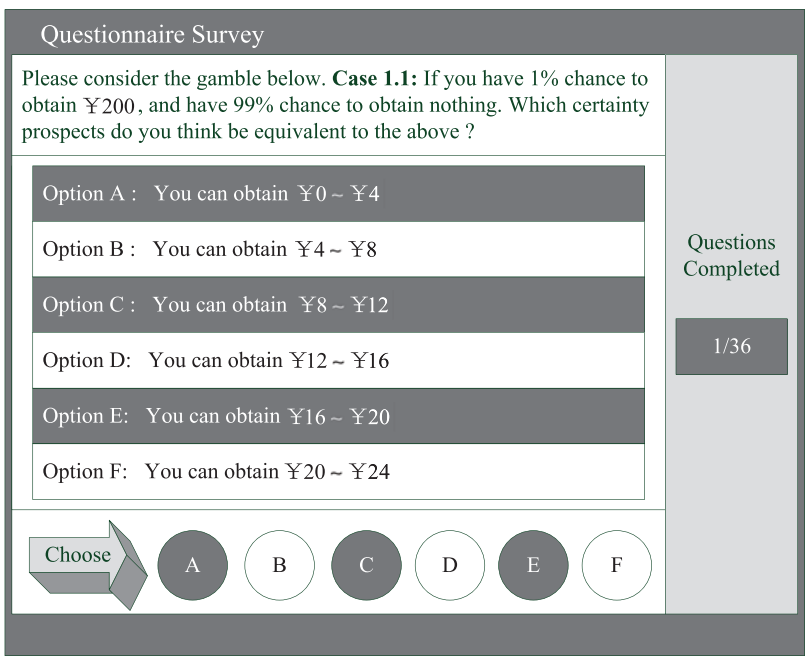

Figure 5 | Example of questionnaire survey for prospect $(¥200, 0.01; 0, 0.99)$.

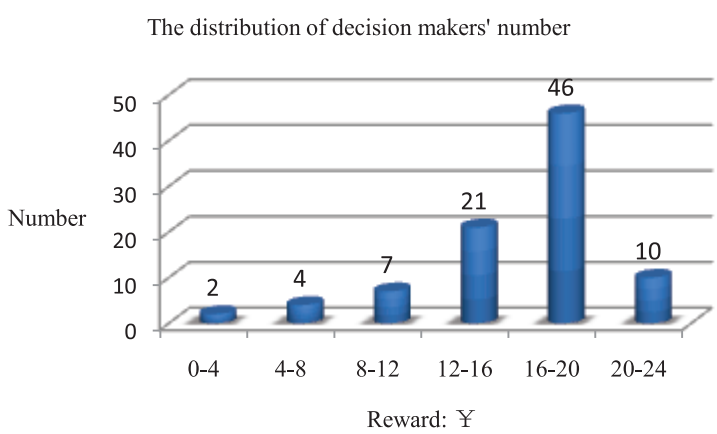

Figure 6 | The distribution histogram of decision-makers' number for the equivalent prospects of prospect $(¥200, p; 0, 1-p)$.

indifferent for 4 respondents, 7 respondents, 21 respondents, 46 respondents and 10 respondents respectively. Considering the probability weights of most people, decision-makers' average is expected as $\overline{E}(c_1)$ about certainty prospect as most people's certainty prospect c_1 being equivalent to the two-outcome uncertainty prospect (¥200, 0.01; 0, 0.99). For instance, most people's certainty prospect can be givens as

$$c_1 = \overline{E}(c_1) = \sum_{i=1}^{6} c_{1i}q_{1i}/90 = 2 \cdot \frac{2}{90} + 6 \cdot \frac{4}{90} + \cdots + 22 \cdot \frac{10}{90} = 16 \tag{25}$$

where, $c_{11} = 2$, $c_{12} = 6$, $c_{13} = 10$, $c_{14} = 14$, $c_{15} = 18$, $c_{16} = 22$, and q_{1i} $(i = 1, 2, \cdots, 6)$ is the number of decision-makers who think that the prospect (¥200, 0.01; 0, 0.99) is equivalent to the certainty prospect $(c_{1i}, 1)$ $(i = 1, 2, \cdots, 6)$. So, an equivalent relation was obtained such as (¥200, 0.01; 0, 0.99) $\sim$ (¥16, 1), where, $\sim$ represents preference relation for indifferent or equivalent.

To make experiment more intuitively, it provides a table representation for the case of monetary rewards level in $0 < x \leq 200$ (because of the limitation of space, the experimental data under this case are only presented). The experimental data are displayed in Table 1, it shows the number of decision-makers for different range of certainty prospects.

To obtain the certainty prospects from different probabilities' uncertainty prospects, a transformation was performed to convert a range of certainty prospects to a single certainty prospect using the method described in Equation (25). The computation formula can be given as follows:

$$\overline{E}(c_i) = \sum_{j=1}^{6} \frac{c_{ij}q_{ij}}{90}, i = 1, 2, \cdots, 36; \tag{26}$$

where c_{ij} is the certainty prospect of the i-th problem corresponding to the j-th option; q_{ij} is the decision-makers' number of the i-th problem corresponding to the j-th option. Using this method to process the experimental data in Table 1, the processed experimental data are shown in Table 3.

After experimental data are obtained, ***Step 1*** has been completed, ***Step 2*** then transforms these data into the experimental preference points. Hence, utilizing the certainty equivalent method in Equation (5) to process the experimental data in Table 3, the relationship of objective probability p and the corresponding probability weight $w(p)$ are given in Table 4, in which, the first row expresses the probability p, the second row expresses the probability p corresponding to probability weight $w(p)$ at the monetary rewards level of $0 < x \leq 200$. For instance, $w(p) = 0.08$ in the second row of the second column exhibits that the probability $p = 0.01$ is considered as probability weight $w(p) = 0.08$ for most people in risky decision-making. Moreover, for different levels' monetary rewards, decision-makers have different subjective responses to the same objective probability p, Table 4 also shows the relationship of p and $w(p)$ at the levels of four different monetary rewards.

5.4. Empirical Studies

In CPT, the monetary reward $x > 0$ represents gains and $x < 0$ represents losses. People behave differently on gains and on losses.

Table 3 | Median cash equivalents for all nonmixed prospects.

	Probability								
Outcomes	**0.01**	**0.05**	**0.10**	**0.25**	**0.50**	**0.75**	**0.90**	**0.95**	**0.99**
(0, 200)	16	28	36	56	84	112	148	156	180
(0, 2000)	120	240	320	500	760	1240	1560	1640	1880
(0, −200)	−6	−16	−24	−48	−84	−126	−160	−170	−192
(0, −2000)	−100	−200	−280	−420	−800	−1240	−1560	−1580	−1900

Note: The two outcomes of each prospect are given in the left-hand side of each row; the corresponding value of median cash equivalents are given in the right columns (i.e., (¥16, 1.0) is equivalent to (0, 0.99; ¥200, 0.01), the value of ¥16 in the upper left corner.).

Table 4 | The relationship of probability p and probability weight $w(p)$ at the different monetary rewards levels.

	Probability								
Outcomes	**0.01**	**0.05**	**0.10**	**0.25**	**0.50**	**0.75**	**0.90**	**0.95**	**0.99**
(0, 200)	0.08	0.14	0.18	0.28	0.42	0.56	0.74	0.78	0.90
(0, 2000)	0.06	0.12	0.16	0.25	0.38	0.62	0.78	0.82	0.94
(0, −200)	0.03	0.08	0.12	0.24	0.42	0.63	0.80	0.85	0.96
(0, −2000)	0.05	0.10	0.14	0.21	0.40	0.62	0.79	0.83	0.95

They are not uniformly risk-averse, but distinctively more sensitive to losses than to gains (Definition 1). In the light of the construction method of the novel PWF model, it is intuitive to find that the novel PWF model satisfies the experimental preference points. To intuitively show that the proposed PWF model also satisfies the properties of Definition 1. The applications of the novel PWF model are demonstrated using the experimental result in Section 5.2. The preference information is displayed in Table 4. Because the processing methods for the four cases are the same, for simplicity, only the case of $0 < x \leq 200$ is discussed here, it takes the following data:

$$p_1 = 0.01, p_2 = 0.05, \cdots, p_9 = 0.99$$

$$w(p_1) = 0.08, w(p_2) = 0.14, \cdots, w(p_9) = 0.90 \tag{27}$$

According to Definition 3, the experimental preference points can be collected.

$$(p_i, w(p_i)) \in D_1 \ (i = 1, 2, \cdots, 9) \tag{28}$$

Utilizing the Equation (8), the Lagrange interpolation basis functions $l_i(p)$ can be solved.

$$l_i(p) = \prod_{\substack{j=1 \\ j \neq i}}^{9} \left(\frac{p - p_j}{p_i - p_j} \right) \tag{29}$$

So, the numerical model is built by using the basis function $l_i(p)$ $(i = 1, 2, \cdots, 9)$

$$L(p) = \sum_{i=1}^{9} w(p_i)\, l_i(p) = \sum_{i=1}^{9} \left(w(p_i) \prod_{\substack{j=1 \\ j \neq i}}^{9} \frac{(p - p_j)}{(p_i - p_j)} \right) \tag{30}$$

A modified method is proposed to add the interpolation nodes in Equation (28) so that a novel PWF model can be built by using the updated preference points. To do that, it needs to collect the preference points which are obtained from two parts: one part is from the experimental data processed in Table 4 while the other part is from the inferred preference points. To collect the inferred preference information, ***Step 3*** should be carried out, i.e., a PWF $w(p)$ should be selected from the existing PWF models (see Table 2). The existing PWF models are compared based on the number of parameters they contained. The model of Ref. [5] has only a parameter γ, but the models of Refs. [26,43] both have two parameters, i.e.,, γ and δ. Having compared the model parameters, Ref. [20] found that only one parameter was required to describe aggregate choice data while two parameters were required to describe individual choice data, Ref. [45] deemed that the performances of one- and two-parameter forms depend on assumptions about the other component functions in CPT. At the same time, the main theoretical advantages of the one-parameter form over the two-parameter form are the stability of the model and the simplicity of the calculation. However, the two-parameter form can better express the preference of the decision-makers than one-parameters model.

For the above reason, the two-parameter form analysis model $w(p) = \exp\left[-\delta\left(-\ln p\right)^{\gamma}\right]$ is adopted (i.e., the model of Ref. [43]. Of course, the other analysis models, such as the model of Ref. [26]), can also be used. In ***Step 4***, in order to calculate the parameters γ and δ, the collected experimental preference points $\left(p_i, w\left(p_i\right)\right) \in D_1\ (i = 1, 2, \cdots, 9)$ in formula (28) are used. By the calculation, the PWF $w(p)$ is expressed in Equation (31).

$$w(p) = \exp\left[-1.09\left(-\ln p\right)^{0.53}\right] \tag{31}$$

To generate random number in ***Step 5***, since the number of probability $p_i\ (i = 1, 2, \cdots, 9)$ is relatively small, the probability $\overline{p}_j\ (j = 1, 2, \cdots, L)$ is chosen in interval $(0, 1)$. For convenience the next computation, the chosen probability $\overline{p}_j\ (j = 1, 2, \cdots, L)$ should contain probability $p_i\ (i = 1, 2, \cdots, 9)$, so the parameter $L = 29$ is set, the detailed forms are shown in Table 5.

In ***Step 6***, the distribution list of the inferred probabilities $\overline{p}_j\ (j = 1, 2, \cdots, 29)$ and its corresponding to the probability weights $w\left(\overline{p}_j\right)\ (j = 1, 2, \cdots, 29)$ are displayed in Table 5. Therefore, the preference information $\left(\overline{p}_j, w\left(\overline{p}_j\right)\right) \in D_2 (j = 1, 2, \cdots, 29)$ are obtained.

In ***Step 7***, to update the preference points, two small steps are carried out. First, if $\overline{p}_j \neq p_i\ \left(i = 1, 2, \cdots, 9; j = 1, 2, \cdots, 29\right)$, it is easy to find that $L_1 = 20$ (i.e., there are 20 inferred preference points which satisfy the condition $\overline{p}_j \neq p_i$). Then the inferred preference points $\left(\overline{p}_j, w\left(\overline{p}_j\right)\right) \in D_2\ \left(j = 1, 2, \cdots, 20\right)$ are added and expressed in thick types in Table 5 into D_1. The collected preference points are rearranged according to the size of probability p_i and $\overline{p}_j$, and re-expressed as $\left(p_k^{(1)}, w^{(1)}\left(p_k^{(1)}\right)\right) \in D_3 (k = 1, 2, \cdots, 29)$ as shown in Table 6.

Second, the values of the inferred preference points $\left(p_k^{(1)}, w^{(1)}\left(p_k^{(1)}\right)\right) \in D_2\ (k = 1, 2, \cdots, 20)$ are modified by using Equations (18–21). In order to intuitively explain the Equations (18–21), an example is used. If the preference points $\left(p_k^{(1)}, w^{(1)}\left(p_k^{(1)}\right)\right) = (0.010, 0.080)$ and $\left(p_{k+H+1}^{(1)}, w^{(1)}\left(p_{k+H+1}^{(1)}\right)\right) = (0.050, 0.140)$ are taken, and $p_k^{(1)} = 0.010$ and $p_{k+H+1}^{(1)} = 0.050$ are put into Equation (28), $w\left(p_k^{(1)}\right) = 0.086$ and $w\left(p_{k+H+1}^{(1)}\right) = 0.142$ can be obtained. From Table 6, it is easy to find that $k = 1$ and $H = 2$ in Equation (18). After that, using the Equations (19–21), the following results are obtained:

$$\Delta T_1 = w^{(1)}\left(p_4^{(1)}\right) - w^{(1)}\left(p_1^{(1)}\right) = 0.060 \tag{32}$$

$$\Delta S_1 = w\left(p_4^{(1)}\right) - w\left(p_1^{(1)}\right) = 0.056$$

$$f\left(p_2^{(1)}\right) = \frac{\Delta T_1}{\Delta S_1} \cdot \left(w\left(p_2^{(1)}\right) - w\left(p_1^{(1)}\right)\right) = 0.0214$$

$$f\left(p_3^{(1)}\right) = \frac{\Delta T_1}{\Delta S_1} \cdot \left(w\left(p_3^{(1)}\right) - w\left(p_2^{(1)}\right)\right) = 0.0279$$

$$f\left(p_4^{(1)}\right) = \frac{\Delta T_1}{\Delta S_1} \cdot \left(w\left(p_4^{(1)}\right) - w\left(p_3^{(1)}\right)\right) = 0.0107$$

Then, using the Equation (18), the following values are obtained:

$$w^{(2)}\left(p_1^{(2)}\right) = w^{(1)}\left(p_1^{(1)}\right) = 0.08,$$

$$w^{(2)}\left(p_2^{(2)}\right) = w^{(2)}\left(p_1^{(2)}\right) + f\left(w^{(1)}\left(p_2^{(1)}\right)\right)$$

Table 5 The distribution list of probability $\overline{p}_j$ and probability weight $w\left(\overline{p}_j\right)$ for gains.

	Probability and Probability Weight										
$\overline{p}_j$	0.010	0.020	0.040	0.050	0.060	0.080	0.100	0.150	0.200	0.250	0.300
$w\left(\overline{p}_j\right)$	0.086	0.106	0.132	0.142	0.152	0.169	0.183	0.216	0.246	0.274	0.300
$\overline{p}_j$	0.350	0.400	0.450	0.500	0.550	0.600	0.650	0.700	0.750	0.800	0.850
$w\left(\overline{p}_j\right)$	0.327	0.353	0.380	0.408	0.436	0.466	0.500	0.532	0.569	0.611	0.660
$\overline{p}_j$	0.900	0.920	0.940	0.950	0.960	0.980	0.990				
$w\left(\overline{p}_j\right)$	0.718	0.747	0.779	0.798	0.819	0.871	0.909				

$$= 0.08 + 0.0214 = 0.101 \tag{33}$$

$$w^{(2)}\left(p_3^{(2)}\right) = w^{(2)}\left(p_2^{(2)}\right) + f\left(w^{(1)}\left(p_3^{(1)}\right)\right)$$
$$= 0.101 + 0.0279 = 0.129$$
$$w^{(2)}\left(p_4^{(2)}\right) = w^{(2)}\left(p_3^{(2)}\right) + f\left(w^{(1)}\left(p_4^{(1)}\right)\right)$$
$$= 0.129 + 0.0107 = 0.140$$

Applying the above method, the part modified preference points are solved as follows:

$$\left(p_1^{(1)}, w^{(1)}\left(p_1^{(1)}\right)\right) = (0.010, 0.080) \tag{34}$$
$$\left(p_2^{(1)}, w^{(1)}\left(p_2^{(1)}\right)\right) = (0.020, 0.101)$$
$$\left(p_3^{(1)}, w^{(1)}\left(p_3^{(1)}\right)\right) = (0.040, 0.129)$$
$$\left(p_4^{(1)}, w^{(1)}\left(p_4^{(1)}\right)\right) = (0.050, 0.140)$$

The total preference points $\left(p_k^{(2)}, w^{(2)}\left(p_k^{(2)}\right)\right) \in D_4\,(k = 1, 2, \cdots, 29$ are displayed in Table 7, in where, the thick types express the inferred preference points $\left(p_k^{(2)}, w^{(2)}\left(p_k^{(2)}\right)\right) \in D_3\,(k = 1, 2, \cdots, 20)$ modified by utilizing Equations (16–20).

Utilizing the 29 preference points in Table 7 to build a novel PWF model, we first solve the Langrage interpolation basis functions $l'_k(p)\,(k = 1, 2, \cdots, 29)$.

$$l'_k(p) = \prod_{\substack{j=1 \\ j \neq k}}^{29} \left(\frac{p - p_j^{(2)}}{p_k^{(2)} - p_j^{(2)}}\right) \tag{35}$$

Utilizing Langrage interpolation basis functions $l'_i(p)\,(i = 1, 2, \cdots, 29)$ to build the novel PWF model, it is expressed in Equation (36)

$$ML(p) = \sum_{k=1}^{29} w^{(2)}(p_k)\, l'_k(p) \tag{36}$$
$$= \sum_{k=1}^{29} \left(w^{(2)}(p_k) \prod_{\substack{j=1 \\ j \neq k}}^{29} \frac{\left(p - p_j^{(2)}\right)}{\left(p_k^{(2)} - p_j^{(2)}\right)} \right)$$

In Equation (36), the PWF model $ML(p)$ is built by utilizing the LIM to interpolate the preference points $\left(p_k^{(2)}, w^{(2)}\left(p_k^{(2)}\right)\right) \in D_4\,(k = 1, 2, \cdots, 29)$. These preference points are collected from two parts, the experimental preference points $(p_i, w(p_i)) \in D_1\,(i = 1, 2, \cdots 9)$ and the inferred preference points $\left(\bar{p}_j, w\left(\bar{p}_j\right)\right) \in D_2\,(j = 1, 2, \cdots, 29)$. Compared with the PWF model $L(p)$ in Equation (21), the novel PWF model $ML(p)$ in Equation (36) increases 20 inferred preference points, so it can better reflect the decision-makers' preferences (i.e., the novel PWF model built by using 29 preference points $\left(p_k^{(2)}, w^{(2)}\left(p_k^{(2)}\right)\right) \in D_4\,(k = 1, 2, \cdots, 29)$ is better

Table 6 The distribution list of probability $p_k^{(1)}$ and modified probability weight $w^{(1)}\left(p_k^{(1)}\right)$.

	Probability and Probability Weight										
$p_k^{(1)}$	0.010	**0.020**	**0.040**	0.050	**0.060**	**0.080**	0.100	**0.150**	**0.200**	0.250	**0.300**
$w^{(1)}\left(p_k^{(1)}\right)$	0.080	**0.106**	**0.132**	0.140	**0.152**	**0.169**	0.180	**0.216**	**0.246**	0.280	**0.300**
$p_k^{(1)}$	**0.350**	**0.400**	**0.450**	0.500	**0.550**	**0.600**	**0.650**	**0.700**	0.750	**0.800**	**0.850**
$w^{(1)}\left(p_k^{(1)}\right)$	**0.327**	**0.353**	**0.380**	0.420	**0.436**	**0.466**	**0.500**	**0.532**	0.560	**0.611**	**0.660**
$p_k^{(1)}$	0.900	**0.920**	**0.940**	0.950	**0.960**	**0.980**	0.990				
$w^{(1)}\left(p_k^{(1)}\right)$	0.740	**0.747**	**0.779**	0.780	**0.819**	**0.871**	0.900				

Note: The bold values represent the unreplaced preference points in Table 5.

Table 7 The distribution list of probability $p_i^{(2)}$ and modified probability weight $w^{(2)}\left(p_i^{(2)}\right)$.

	Probability and Probability Weight										
$p_k^{(2)}$	0.010	**0.020**	**0.040**	0.050	**0.060**	**0.080**	0.100	**0.15**	**0.20**	0.250	**0.30**
$w^{(2)}\left(p_k^{(2)}\right)$	0.080	**0.101**	**0.129**	0.140	**0.149**	**0.165**	0.180	**0.217**	**0.249**	0.280	**0.308**
$p_k^{(2)}$	**0.350**	**0.400**	**0.450**	0.500	**0.550**	**0.600**	**0.650**	**0.700**	0.750	**0.800**	**0.850**
$w^{(2)}\left(p_k^{(2)}\right)$	**0.336**	**0.363**	**0.391**	0.420	**0.445**	**0.471**	**0.498**	**0.528**	0.560	**0.611**	**0.669**
$p_k^{(2)}$	0.900	**0.920**	**0.940**	0.950	**0.960**	**0.980**	0.990				
$w^{(2)}\left(p_k^{(2)}\right)$	0.740	**0.754**	**0.771**	0.780	**0.802**	**0.859**	0.900				

Note:The bold values represent the preference points in Table 6 updated using formula (16–20).

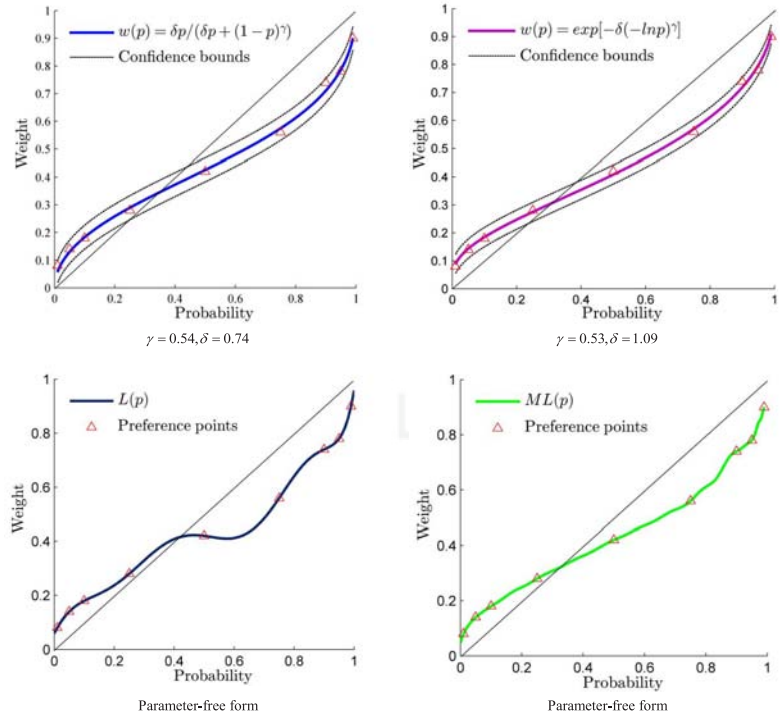

Figure 7 The objective probability p corresponds to subjective probability $w(p)$ with different expression forms of the preference curves. The dotted lines correspond to the 95% confidence interval across the preference points.

than the numerical PWF model built by using 9 preference points $(p_i, w(p_i)) \in D_1\ (i = 1, 2, \cdots 9)$).

In order to intuitively display the advantages of the proposed PWF model, the curves of $w(p)$ and the models $L(p)$ and $ML(p)$ are shown in Figure 7. It outlines the evolution of the estimates of each curve as a function of the experimental preference points $(p_i, w(p_i)) \in D_1\ (i = 1, 2, \cdots 9)$, in where, taking the value of prospects x is gain (i.e., $0 < x \leq 200$). For two-parameter models, the averages as well as the 95% confidence intervals of the estimates are plotted across the 9 preference points (the 95% confidence interval is bounded by the 2.5*th* and the 97.5*th* percentiles of the estimate across the preference points) by using the method of curve fitting. Compared with $w(p)$, the proposed PWF models such as $L(p)$ and $ML(p)$ can go through the experimental preference points reflecting the preferences of decision-makers. Moreover, the novel PWF model $ML(p)$ obtained by modifying $L(p)$ can combine the advantages of $L(p)$ and $w(p)$, so it can not only satisfy the properties of PWF in Definition 1, but also well reflect and predict the preferences of decision-makers.

In Definition 1, the properties of PWF are redefined, such as fourfold patterns of risk preferences (properties (2) and (3)), risk-aversion (property (4)), and reference point (property (5)). To explain how the novel PWF model satisfy these properties, the following four cases are compared. Firstly, when the monetary rewards are gains, for small probabilities, decision-makers are more risk-averse at larger levels' monetary rewards (i.e., $0 < x \leq 2000$) than at smaller levels' monetary rewards (i.e., $0 < x \leq 200$); for big probabilities, decision-makers are more risk-seeking at larger levels' monetary rewards (i.e., $0 < x \leq 2000$) than at smaller levels' monetary rewards (i.e., $0 < x \leq 200$), which are displayed on the upper left panel of Figure 8. Secondly, when the monetary rewards are losses, risk preference reverse, the upper right panel of Figure 4 shows the properties. Thirdly, when monetary rewards' levels are same such as $|x| \leq 200$, decision-makers are more risk-averse for small probabilities' losses than for small probabilities' gains and more risk-seeking for big probabilities' losses than for big probabilities' gains (the PWF departs more from the diagonal line for gains than for losses in same monetary rewards' levels), which are displayed on the lower left panel of Figure 8. Lastly, the risk attitudes of decision-makers under the case of $|x| \leq 2000$ are similar to under the case of $|x| \leq 200$, the lower right panel of Figure 8 reveals the characteristics. Figure 8 shows that, both for low levels' monetary rewards and for high levels' monetary rewards or for gains and for losses, decision-makers overweight small probabilities, and underweight moderate and big probabilities (the four charts in Figure 8 both display that the PWF curves are above the diagonal for small probabilities and under the diagonal for moderate and big probabilities). Figure 8 also shows that, for small probabilities' gains, the risk-seeking of decision-makers in high levels' monetary rewards is more than in low levels' monetary rewards; for big probabilities' gains, the risk-aversion of decision-makers in high levels' monetary rewards is less than in low levels' monetary rewards; for losses, the preferences reversal phenomenon happens.

The advantages of the novel PWF model is that its functional form can vary according to information derived from the experimental preference points. To verify this end, we add some new experimental preference points by mean of increasing the number of question subjects, some new experimental data can be obtained and processed. Specifically, the two-outcome prospects with monetary outcomes and numerical probabilities are set in experiment survey. For simplicity, we only consider

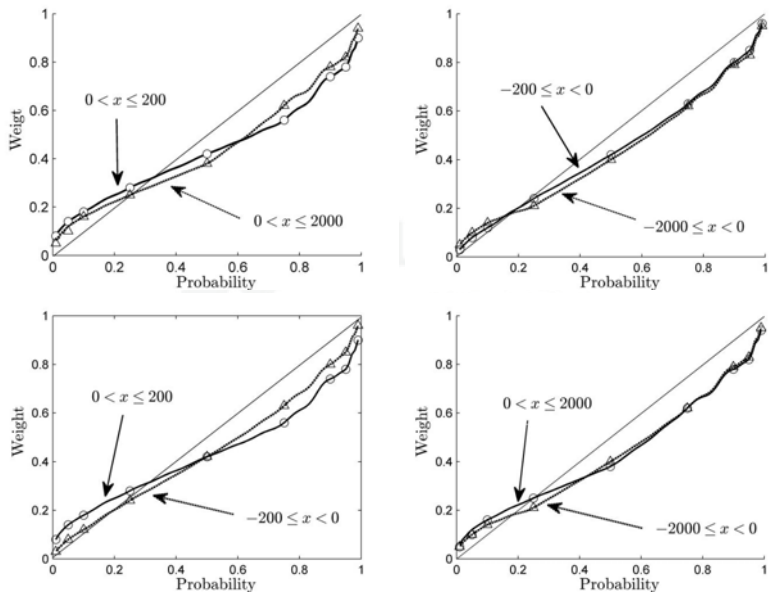

Figure 8 | The probability weighting function (PWF) curves with different levels' monetary rewards for gains or losses. x represents the levels of monetary rewards.

Table 8 | Different levels' monetary rewards corresponding to probability p and probability weight $w(p)$.

	Probability p and Probability Weight $w(p)$										
Outcomes	**0.01**	**0.05**	**0.10**	**0.25**	**0.40**	**0.50**	**0.75**	**0.85**	**0.90**	**0.95**	**0.99**
$(0, 200)$	0.08	0.14	0.18	0.28	**0.33**	0.42	0.56	**0.70**	0.74	0.78	0.90
$(0, -200)$	0.03	0.08	0.12	0.24	**0.35**	0.42	0.63	**0.72**	0.80	0.85	0.96

Note:The bold values represent the preference points added relative to Table 4.

the cases of monetary rewards $|x| \leq 200$. Taking the positive prospects (¥200, 0.4; ¥0, 0.6) and (¥200, 0.85; ¥0, 0.15), the negative prospects (−¥200, 0.4; ¥0, 0.6), and (−¥200, 0.85; ¥0, 0.15), they are respectively corresponding to the certainly-prospects (¥66, 1), (¥140, 1), (−¥70, 1), and (−¥144, 1) by conducting the experiment survey and data processing, i.e., four groups equivalent relation can be obtained, i.e.,

$$\begin{aligned}&(¥200, 0.4; 0, 0.6) \sim (¥66, 1)\\&(¥200, 0.85; 0, 0.15) \sim (¥140, 1)\\&(-¥200, 0.4; 0, 0.6) \sim (-¥70, 1)\\&(-¥200, 0.85; 0, 0.15) \sim (-¥144, 1)\end{aligned} \tag{37}$$

Hence, two new experimental preference points for every level's monetary rewards can be collected respectively. Utilizing the certainty equivalent method to process the above experimental data, the new results can be shown in Table 8 with the thick types representing the new increasing experimental preference information.

To build the novel PWF model, the modified method described in Subsection 4.2 is used. Then, in order to express intuitively the advantages, Figure 9 gives the curves of the modified PWF model and the existing PWF model, they are both built by using the new increasing preference points in Table 8. By comparing the modified PWF model $ML(p)$ and the existing PWF model $w(p)$, it is easy to notice that $w(p)$ is determined when the parameters γ and

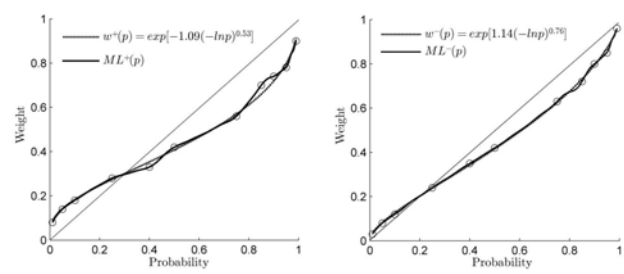

Figure 9 | The change curves of the probability weighting function (PWF) by objective probability p_i. $w^-(p)$ and $w^-(p)$ expressing the existing PWF model for positive prospects and negative prospects respectively. $ML^+(p)$ and $ML^-(p)$ represent the modified PWF model for positive prospects and negative prospects, respectively. The circles represent the experimental preference points $(p_i, w(p_i))$.

δ are fixed. However, $ML(p)$ can vary as the experimental preference points change to dynamic reflect decision-makers' preferences. Figure 9 also shows whether positive prospects or negative prospects can both dynamic reflect decision-makers' preferences according to the change of the experimental preference points.

6. ERROR ANALYSIS

In existing PWF models, some functions were fitted using the experimental preference points obtained from the experiment or questionnaire survey, their performances are assessed by whether they can provide a reasonably good approximation to both the aggregate and the individual data for probabilities in the range between 0.01 and 0.99 (Ref. [5]). Other functions were proposed by satisfying some axioms and some properties (such as, diagonal concavity, sub-proportionality and compound invariance), their performances were verified by the theoretical analysis (Ref. [26]).

In this paper, a novel parameter-free PWF model is built by proposing a numerical approach, and performances analysis is carried out based on errors between the predicted value from the PWF models and the experimental preference points from experiment or questionnaire survey.

Accounting for the effectiveness of the proposed PWF model, some comparisons for the response errors of the existing analysis model [5], the numerical model (such as Equation (21)) and the proposed PWF model (such as Equation (36)) are made. Of course, similar comparisons can be done with some other analysis model (Refs. [5,29,45]), and the results are expected to be the same.

In order to estimate the response errors of different models, the best square estimates are used. The computational formulas are in the context as follows:

$$e_w = \sum_{i=1}^{11} \left| w\left(p_i\right) - w^{(0)}\left(p_i\right) \right|^2 \tag{38}$$

$$e_L = \sum_{i=1}^{11} \left| w\left(p_i\right) - L\left(p_i\right) \right|^2 \tag{39}$$

$$e_{ML} = \sum_{i=1}^{11} \left| w\left(p_i\right) - ML\left(p_i\right) \right|^2 \tag{40}$$

where $\left(p_i, w\left(p_i\right)\right)$ $(i = 1, 2, \cdots, 11)$ are the experimental preference points (the preference points obtained from Table 8), $w^{(0)}\left(p_i\right)$, $L\left(p_i\right)$ and $ML\left(p_i\right)$ are respectively the values of probability p_i corresponding to the PWFs $w^{(0)}\left(p\right)$, $L\left(p\right)$ and $ML\left(p\right)$. Specially, $w^{(0)}\left(p\right)$ expresses the existing PWF model $w\left(p\right)$ such as Prelec [26] whose parameters are solved by curve fitting $q\left(q = 2, 3, \cdots 10\right)$ experimental preference points $\left(p_j, w\left(p_j\right)\right)$ $\left(j = 1, 2, \cdots, 11\right)$. The analysis compares the errors between the experimental preference points $\left(p_i, w\left(p_i\right)\right)$ $(i = 1, 2, \cdots, 11)$ and the inferred preference points (i) $\left(p_i, w^{(0)}\left(p_i\right)\right)$ $(i = 1, 2, \cdots, 11)$, (ii) $\left(p_i, L\left(p_i\right)\right)$ $(i = 1, 2, \cdots, 11)$, and (iii) $\left(p_i, ML\left(p_i\right)\right)$ $(i = 1, 2, \cdots, 11)$. These inferred preference points are respectively inferred by the different PWFs $w^{(0)}\left(p\right)$, $L\left(p\right)$ and $ML\left(p\right)$ by curve fitting with known experimental preference points $\left(p_j, w\left(p_j\right)\right)$ $\left(j = 1, 2, \cdots, 11\right)$. For example, to explain the case of $q = 2$, it assumes that only two experimental preference points are known. They can be used to obtain the PWFs $w^{(0)}\left(p\right)$, $L\left(p\right)$ and $ML\left(p\right)$, respectively. Then, three groups of inferred preference (i) $\left(p_i, w^{(0)}\left(p_i\right)\right)$ $(i = 1, 2, \cdots, 11)$, (ii) $\left(p_i, L\left(p_i\right)\right)$ $(i = 1, 2, \cdots, 11)$, and (iii) $\left(p_i, ML\left(p_i\right)\right)$ $(i = 1, 2, \cdots, 11)$ are collected. Using Equations (38–40), the response errors e_w, e_L and e_{ML} in the case of the known experimental preference points' number $q = 2$ can be solved.

In order to present the response error intuitively, the logarithm of the response error $\log \mathrm{e}$ is taken as the vertical coordinate and the known experimental preference points' amount q as horizontal coordinate. Figures 6 and 7 offer a graphical representation of the response errors' change tendency with the known experimental preference points' amount from $q = 2$ to $q = 10$ for gains $(0 < x \leq 200)$ and for losses $(-200 \leq x < 0)$. The blue line shows the trend of the errors of the existing PWF model (i.e., Ref. [26]) with the known experimental preference points' number q; while the purple line shows the trend of the errors of the numerical PWF model with q; moreover, the red line displays that of the novel PWF model with q.

Not surprisingly, the response errors with each model tend to be smaller when the number q of the experimental preference points is larger. Indeed, more valid information brings more the high-quality experimental preference points and the lower the level of response errors. Moreover, the response error of Prelec's PWF $w\left(p\right)$ has a small change with the increase of the experimental preference points' number q in Figures 10 and 11. However, when q is increasing, the response error of the numerical PWF model $L\left(p\right)$ and the novel PWF model $ML\left(p\right)$ have a large decrease. Owing to the properties of Lagrange interpolation polynomial, the model $L\left(p\right)$ is not stable and it has a larger response error as the small number of the experimental preference points. So, the model $L\left(p\right)$ is modified for building the novel PWF model $ML\left(p\right)$. In Figures 10 and 11, the red line represents the response error of the novel PWF model $ML\left(p\right)$ with known experimental preference points' number q. It possesses the advantages of both $w\left(p\right)$ and $L\left(p\right)$, so it can better reflect and predict the decision-makers' preferences than others.

7. CONCLUSION AND EXTENSIONS

PWFs relate objective probabilities to their psychological weight attached to an outcome, and they play a pivotal role in modeling choices under risk with PT and CPT. In contrast with a parametric form and a parametric-free form of the PWFs, in this paper, we combined the parametric form with parametric-free form by using numerical fitting the experimental preference points and proposed a novel PWF model. The model can not only reflect empirical regularities, but also make the maximum possible satisfaction recent theoretical work. Hence, it can play a very important role in the development of the risky decision-making theory and is better able to reflect a decision's preference for events.

The more preferences information of the decision-makers is collected, and these are high-quality from experiment or questionnaire survey, the more accurate the PWF model is in reflecting and predicting the preferences of decision-makers. Establishing the PWF from a larger population is crucially important yet time-consuming as concluded by Refs. [5,46]. However, improvements in hardware and more efficient programming techniques should be able to solve the above problem. By introducing a numerical method to deal with economy questions, the novel PWF model can be built. At the same time, the error analysis of the novel model has demonstrated that it could not only accurately reflect known preferences of decision-maker, but also better predict unknown preference of decision-makers in a certainty environment (because the preferences of decision-makers are different as the different environment). At last, because of the properties of Lagrange interpolating polynomial (volatility and concussive), some interpolation nodes (i.e., the inferred preference points) are added among the existing interpolation nodes (i.e., the experimental preference points). The modification give rise to more inferred preference points, and help obtain a better performance when the PWF model is built using LIM.

The novel PWF model proposed in this paper is built using LIM when compared to other analysis models shown in Table 2 bring a

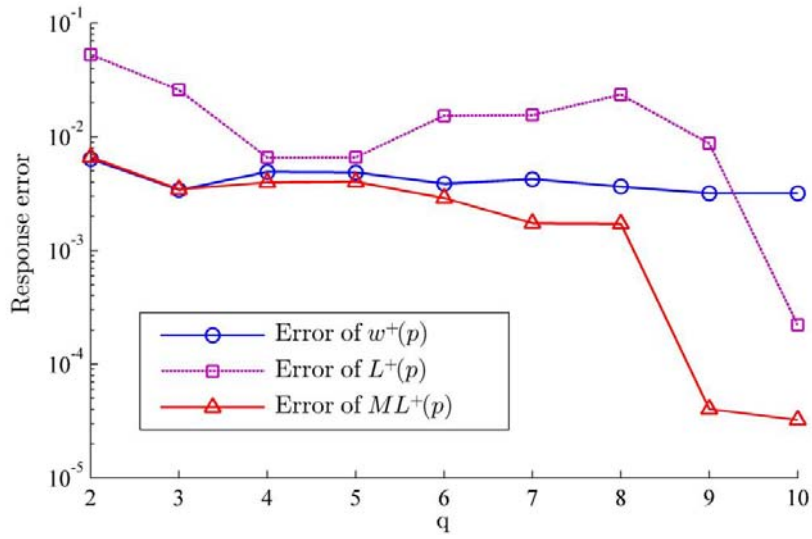

Figure 10 The response errors of the probability weighting function (PWF) for gains in semi-log scale. When increasing the number of the prospect (q), the error of analysis model $w^+(p)$ is decrease more slowly, than those of the numerical model $L^+(p)$ and the proposed novel PWF model $ML^+(p)$.

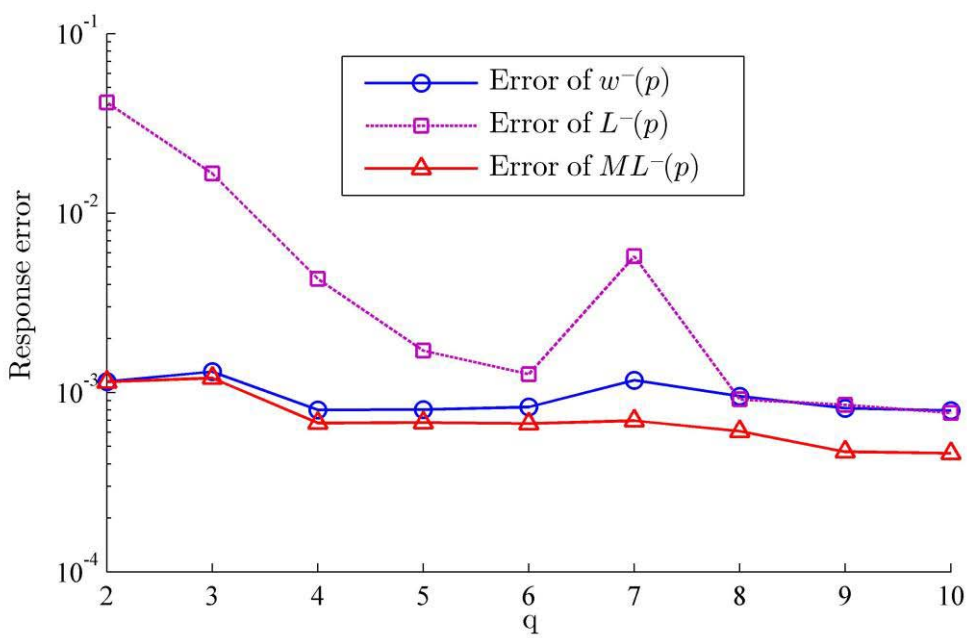

Figure 11 The response errors of the probability weighting function (PWF) for losses in semi-log scale. When increasing the number of the prospect (q), the trend of the errors of analysis model $w^-(p)$, the numerical model $L^-(p)$ and the proposed novel PWF model $ML^-(p)$ are similar to $w^+(p)$, $L^+(p)$ and $ML^+(p)$ respectively.

number of benefits. First, the proposed PWF model can go through the experimental preference points so that it can more accurately reflect the decision-making process. Second, the novel PWF model is flexible as the experimental preference points and their amount change. At last, the proposed PWF model combines prelec's PWF model and numerical model and performs better in predicting the preferences of decision-makers as illustrated in Subsection 5.3.

Detailed explanations have been given on construction and application of the novel PWF model beginning with the collection of experimental preference points, because they are an important component of building PWF model. In other words, if some preference information of decision-makers could not be collected, or the collected preference information are not high quality, the novel PWF model could not be established. However, as long as some preference information of decision-makers is available (even if only a few), the method in Section 4.2 can be applied, and it could build a better performance's PWF model than existing PWF model in Table 2.

This study also suggests several areas for future research. First, as a good performance's PWF model built in PT and CPT is crucially, it would be interesting to apply this novel PWF model in different scenarios, such as option prices (Ref. [47]), lottery decisions (Ref. [10]), inventory problem (Ref. [13]) in risky decision-making. Second, this paper mainly discusses the decision-making in general. More research is required to explore the application of this novel PWF models in the decision-making in individual cases and their applications in weight assignment in the context of multiple criteria decision-making (Refs. [48,49]). Finally, some other numerical methods can be used in researching PWF model.

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