A Modification of the Gompertz Distribution Based on the Class of Extended-Weibull Distributions

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ABSTRACT

This paper introduces a new four-parameter extension of the generalized Gompertz distributions. This distribution involves some well-known distributions such as extension of generalized exponential, generalized exponential, and generalized Gompertz distributions. In addition, it can have a decreasing, increasing, upside-down bathtub, and bathtub-shaped hazard rate function depending on its parameters. Some mathematical properties of this new distribution, such as moments, quantiles, hazard rate function, and reversible hazard rate function are obtained. In addition, the density function and the moments of the ordered statistics of this new distribution is provided. The parameters of model are estimated using the maximum likelihood method. Also, a real data set was used to illustrate the validity of the proposed distribution.

1. INTRODUCTION

The class of extended-Weibull (EW) distributions is defined by [1] and has the following cumulative distribution function (cdf):

$$G(x) = 1 - e^{-\tau \Phi(x;\eta)}, \quad \tau > 0, \ x \geq 0, \tag{1}$$

where $\Phi(x;\eta)$ is a nonnegative, continuous, increasing, and differentiable function of x. The probability density function (pdf) of EW model is

$$g(x) = \tau \phi(x;\eta) e^{-\tau \Phi(x;\eta)}, \quad x \geq 0, \tag{2}$$

where $\phi(x)$ is the first derivative of $\Phi(x;\eta)$. The class of EW distribution contains various well-known distributions. We summarized several of these models in Table 1. For more details see [1] and [2].

Kundu and Gupta [19] proposed an extension of generalized exponential (GE) distribution [20]. It is a flexible model such that it is positively skewed, and has increasing, decreasing, unimodal, and bathtub-shaped hazard rate function (hrf). It is included exponential, GE, Pareto, and generalized Pareto [3] distributions. Cordeiro *et al.* [21] introduced a five-parameter called the McDonald extended exponential distribution [19] as a generalization of extended generalized exponential (EGE) distribution. Kazemi *et al.* [22] introduced an extension of the generalized linear failure rate (GLFR) distribution [23]. It is included the EGE, GLFR, generalized Rayleigh [24,25], Rayleigh, and linear failure rate distributions. By compounding the EW distribution and method of [19] and [22], we can define an extension of EW (EEW) distribution.

For $\alpha > 0, \tau > 0$ and $-\infty < \beta < \infty$, consider

$$F(x) = \begin{cases} \left(1 - (1 - \beta\tau\Phi(x;\eta))^{1/\beta}\right)^{\alpha} & \text{if } \beta \neq 0 \\ \left(1 - e^{-\tau\Phi(x;\eta)}\right)^{\alpha} & \text{if } \beta = 0. \end{cases} \tag{3}$$

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Table 1 | Special cases of extended-Weibull (EW) distribution and corresponding $\Phi(x; \eta)$ function.

Distribution	Support	$\Phi(x; \eta)$	τ	η	Reference
Exponential	$x \geq 0$	x	τ	$\varnothing$	[3]
Pareto	$x \geq c$	$\log(x/k)$	τ	c	[3]
Gompertz	$x \geq 0$	$c^{-1}\left(\exp(cx) - 1\right)$	τ	c	[4]
Weibull	$x \geq 0$	x^c	τ	c	[5]
Weibull Kies	$0 < \mu < x < \sigma$	$(x - \mu)^a / (\sigma - x)^b$	τ	(μ, σ, a, b)	[6]
Linear failure rate	$x \geq 0$	$ax + bx^2/2$	1	(a, b)	[7]
Exponential power	$x \geq 0$	$\exp((cx)^a - 1)$	1	(a, c)	[8]
Rayleigh	$x \geq 0$	x^2	τ	$\varnothing$	[9]
Phani	$0 < \mu < x < \sigma$	$((x - \mu)/(\sigma - x))^b$	τ	(μ, σ, b)	[10]
Additive Weibull	$x \geq 0$	$(x/a)^c + (x/b)^d$	1	(a, b, c, d)	[11]
Chen	$x \geq 0$	$\exp(x^b - 1)$	τ	b	[12]
Pham	$x \geq 0$	$(a^x)^b - 1$	1	(a, b)	[13]
Weibull extension	$x \geq 0$	$a\left(\exp(cx)^b - 1\right)$	τ	(a, b, c)	[14]
Modified Weibull	$x \geq 0$	$x^b \exp(ax)$	τ	(a, b)	[15]
Traditional Weibull	$x \geq 0$	$x^b \exp(ax^c - 1)$	τ	(a, b, c)	[1]
Generalized Weibull power	$x \geq 0$	$[1 + (x/a)^b]^c - 1$	1	(a, b, c)	[16]
Flexible Weibull extension	$x \geq 0$	$\exp(ax - b/x)$	1	(a, b)	[17]
Almalki additive Weibull	$x \geq 0$	$ax^d + bx^\gamma e^{cx}$	1	(a, b, c, d, γ)	[18]

Depending on whether the parameter β be negative or positive, the support of EEW distribution varies in $(0, \infty)$ or $\left(0, \Phi^{-1}\left(\frac{1}{\beta\tau}\right)\right)$, where $\Phi^{-1}(.)$ is the inverse function of $\Phi(.; \eta)$. The EEW is a flexible family and extends many exponentiated distributions such as GE [20], exponentiated Weibull [26], generalized Rayleigh distribution [24,27], generalized modified Weibull [28], GLFR [23], generalized Gompertz (GG) [29] distributions.

As a special case of the class of EEW distribution, in this paper, we consider the GG distribution and investigate the properties of this new four-parameter distribution which is called extended generalized Gompertz (EGG) distribution and contains EGE distribution. The paper is organized as follows. In Section 2, the model EGG was introduced and described. Some statistical properties such as moments, quantiles, and ordered statistics are provided in Section 3. The parameters are estimated using he maximum likelihood method in Section 4. An application of the EGG is illustrated using a real data set in Section 5.

2. PROPOSED DISTRIBUTION

By considering $\Phi(x; \eta) = c^{-1}\left(e^{cx} - 1\right)$, $c > 0$ in (3), we obtain the EGG distribution by the support $S_X = (0, \infty)$ if $\beta \leq 0$ and $S_X = \left(0, \frac{1}{c}\log\left(\frac{c}{\tau\beta} + 1\right)\right)$ if $\beta > 0$, and the following cdf:

$$F_X(x) = \begin{cases} \left[1 - (1 - \dfrac{\beta\tau}{c}\left(e^{cx} - 1\right))^{1/\beta}\right]^\alpha & \text{if } \beta \neq 0 \\ \left(1 - e^{-\frac{\tau}{c}(e^{cx}-1)}\right)^\alpha & \text{if } \beta = 0. \end{cases} \tag{4}$$

The pdf of this new distribution is

$$f_X(x) = \begin{cases} \alpha\tau e^{cx}(1 - \beta z)^{\frac{1}{\beta} - 1}\left(1 - (1 - \beta z)^{1/\beta}\right)^{\alpha - 1} & \text{if } \beta \neq 0 \\ \alpha\tau e^{cx}e^{-z}\left(1 - e^{-z}\right)^{\alpha - 1} & \text{if } \beta = 0, \end{cases} \tag{5}$$

and the hrf has the following form:

$$h_1(x) = \begin{cases} \dfrac{\alpha \tau e^{cx} (1 - \beta z)^{1/\beta - 1} \left(1 - (1 - \beta z)^{1/\beta} \right)^{\alpha - 1}}{1 - \left(1 - (1 - \beta z)^{1/\beta} \right)^{\alpha}} & \text{if } \beta \neq 0 \\[4mm] \dfrac{\alpha \tau e^{cx} e^{-z} \left(1 - e^{-z} \right)^{\alpha - 1}}{1 - (1 - e^{-z})^{\alpha}} & \text{if } \beta = 0, \end{cases} \tag{6}$$

where $z = \frac{\tau}{c} \left(e^{cx} - 1 \right)$. We denote this new distribution by EGG(α, β, τ, c). The new model reduced to GG model which is introduced by [29] when $\beta = 0$. The GG includes the GE (If c tends to zero), exponential (If c tends to zero, and $\alpha = 1$), Gompertz (If $\alpha = 1$), distributions. If c tends to zero, then EGG distribution reduces to the EGE distribution introduced by [19]. The EGE distribution includes GE, exponential, generalized Pareto [3], and Pareto distributions. Also, If X has a EGG distribution, then $Y = \frac{\tau}{c} \left(e^{cX} - 1 \right)$ has a EGE distribution. Figure 1 obtains the shapes of pdf and hrf of EGG distribution for some values of the parameters when $\beta \neq 0$.

The limiting behaviors of pdf and hrf of the EGG distribution are as follows:

$$\lim_{x \to 0^+} f_X(x) = \begin{cases} 0 & \text{if } \alpha > 1 \\ \alpha \tau & \text{if } \alpha = 1 \\ \infty & \text{if } \alpha < 1 \end{cases} \text{ and } \lim_{x \to v^-} f_X(x) = \begin{cases} \infty & \text{if } \beta > 1 \\ 0 & \text{if } \beta < 1, \end{cases}$$
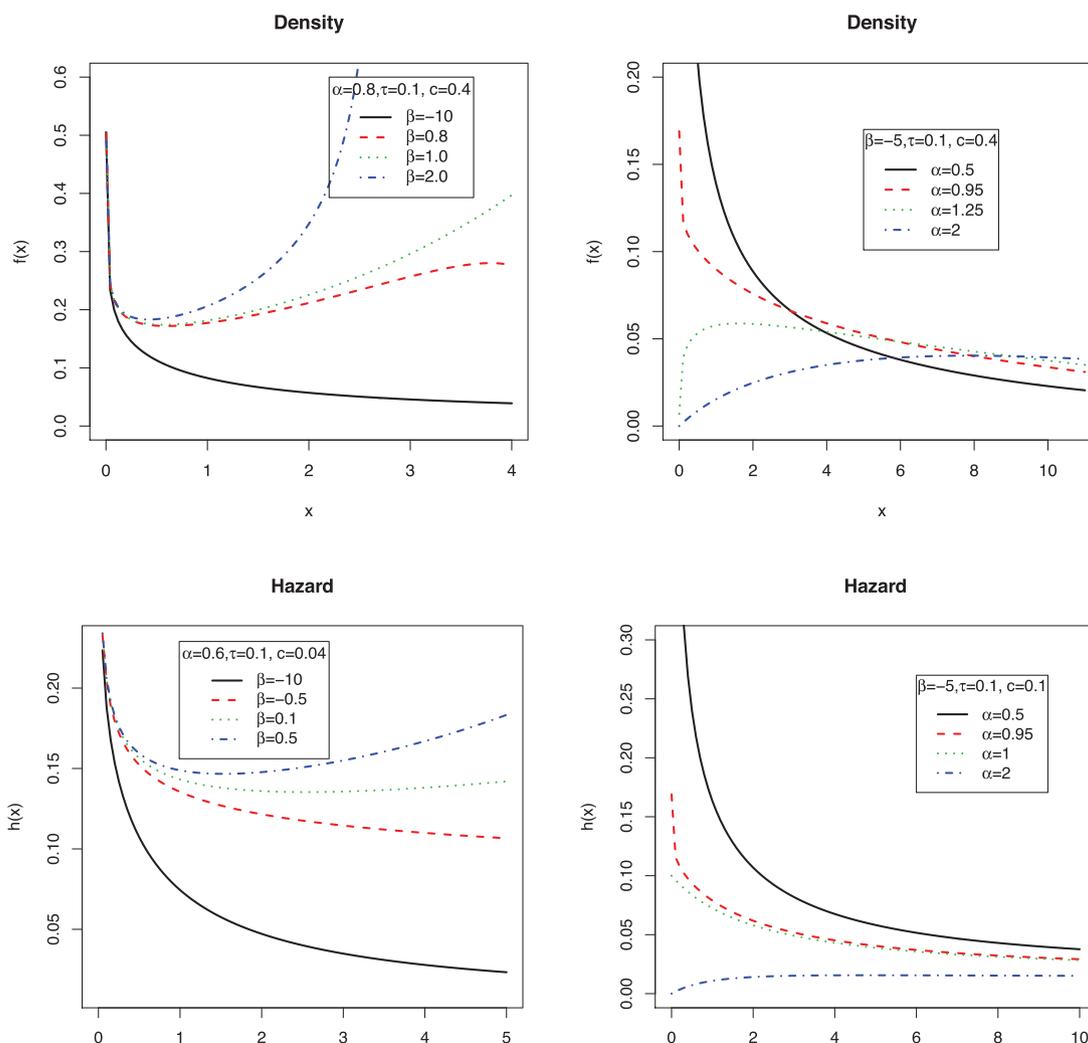
Figure 1 | probability density function (pdf) and hazard rate function (hrf) of extended generalized Gompertz (EGG) distribution.

$$\lim_{x \to 0^+} h_1(x) = \begin{cases} 0 & \text{if} \quad \alpha > 1 \\ \alpha\tau & \text{if} \quad \alpha = 1 \\ \infty & \text{if} \quad \alpha < 1 \end{cases} \text{ and } \lim_{x \to v^-} h_1(x) = \begin{cases} \infty & \text{if} \quad \beta > 0 \\ 0 & \text{if} \quad \beta < 0, \end{cases}$$

where $v = \frac{1}{c} \log\left(\frac{c}{\tau\beta} + 1\right)$ for $\beta > 0$, and $v = \infty$ for $\beta < 0$.

3. PROPERTIES

In this section, some measure such as the quantile function, non-central moment and entropy measure for EGG distribution are obtained and discussed.

3.1. Quantiles

The quantile function of EGG distribution is

$$Q(u) = \begin{cases} \frac{1}{c} \log\left(\frac{c}{\tau\beta}\left[1 - \left(1 - \sqrt[\alpha]{u}\right)^\beta\right] + 1\right) & \text{if} \quad \beta \neq 0 \\ \frac{1}{c} \log\left(1 - \frac{c}{\tau}\left[\log\left(1 - \sqrt[\alpha]{u}\right)\right]\right) & \text{if} \quad \beta = 0, \end{cases}$$

3.2. Moments and Characterization

Here, first, we obtain a theorem to compute the noncentral moment, $\mu^{(r)}$, of EGG distribution when $\beta > 0$. Also, we show that all moments of X exist when $\beta < 0$.

Theorem 3.1. *For $\beta > 0$, the r-th non-central moment of $EGG(\alpha, \beta, \tau, c)$ is*

$$\mu^{(r)} = E(X^r) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{m} \sum_{i=0}^{r} \frac{\alpha\tau^{m+1}(-1)^{i+n+2m-k} r! \psi^{r-i}}{((k+1)c)^{i+1}(r-i)!} \tau_{n,m,k} \beta^m c^{-m} e^{(k+1)cv}, \tag{7}$$

where $\tau_{n,m,k} = \binom{\alpha-1}{n}\binom{\frac{n+1}{\beta}-1}{m}\binom{m}{k}$ and $v = \frac{1}{c}\log\left(\frac{c}{\tau\beta} + 1\right)$.

Proof. The proof is done by using binomial series expansion and following formula resulted from [30], Section 2.321, as

$$\int_0^\psi x^r e^{(k+1)cx} dx = e^{(k+1)c\psi} r! \sum_{i=0}^{r} \frac{(-1)^i}{((k+1)c)^{i+1}(r-i)!} \psi^{r-i}.$$

$\square$

Theorem 3.2. *All moments of $EGG(\alpha, \beta, \tau, c)$ exist when $\beta < 0$.*

Proof. See the Appendix. $\square$

Using Theorem 3.2, the moments of the EEG distribution exist when $\beta < 0$. Therefore, Table 2 obtains them for some selected values of model parameters.

Remark 3.1. Consider $Y = \frac{\tau}{c}\left(e^{cX} - 1\right)$, where X has EGG distribution with parameters α, β, τ, c. Therefore, (see [19])

$$E\left[e^{cX}\right] = \frac{c}{\beta\tau}\left(1 - \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\right) + 1, \quad \beta \neq 0.$$

3.3. Entropy

In this part, for measuring the uncertainty amount of the $EGG(\alpha, \beta, \tau, c)$ distribution, we use the Shannon's entropy defined by [31] as

$$H_{Sh}(f) = -E_f[\log f_X(X)] = -\int_0^\infty f_X(X)\log(f_X(x))dx.$$

Table 2 Computation of $\mu^{(r)}$ for $\tau = 2$ and $c = 3$.

α	r	β					
		-1	-1.5	-2	-4	-4.5	-5
2	1	0.595	0.750	0.922	1.703	1.913	2.127
	2	0.512	0.875	1.384	5.037	6.376	7.891
	3	0.582	1.406	2.922	21.186	30.189	41.537
	4	0.829	2.906	7.998	115.83	185.72	283.98
	5	1.429	7.383	27.012	781.781	1410.264	2396.170
0.5	1	0.253	0.307	0.366	0.641	0.717	0.794
	2	0.165	0.272	0.421	1.493	1.887	2.335
	3	0.165	0.389	0.799	5.732	8.166	11.237
	4	0.221	0.763	2.087	30.085	48.238	73.767
	5	0.369	1.891	6.896	199.151	359.255	610.443

If $\alpha = 1$, then the cdf of the EGG$(1, \beta, \tau, c)$ distribution can be rewritten as

$$F^*(x) = \begin{cases} 1 - (1 - \beta z)^{\frac{1}{\beta}} & \text{if } \beta \neq 0 \\ 1 - e^{-z} & \text{if } \beta = 0 \end{cases}$$

respectively, where $z = \frac{\tau}{c}(e^{cx} - 1)$ and consequently, the cdf in (4) can be written as $(F^*(x))^\alpha$. Let $W \sim \text{beta}(\alpha, 1)$. Then, following the result of [32], the Shannon entropy for EGG(α, β, τ, c) distribution is

$$H(X) = -\ln(\alpha) + \frac{\alpha - 1}{\alpha} - E_W[\ln f_1(F_1^{-1}(W))]. \tag{8}$$

3.4. Ordered Statistics

In this section, the cdf and noncentral moments of ordered statistics from the EGG distribution are provided. Let $x_{(1)}, \ldots, x_{(n)}$ be the ordered statistics of a random sample. Then, the pdf of the ℓ-th ordered statistic $X_{(\ell)}$, is

$$f_{X_{(\ell)}}(x) = n \binom{n-1}{\ell-1} f_X(x) \sum_{j=0}^{n-\ell} (-1)^j \binom{n-\ell}{j} [F_X(x)]^{\ell+j-1} = \sum_{j=0}^{n-\ell} \frac{n(-1)^j \binom{n-\ell}{j} \binom{n-1}{\ell-1}}{(j+\ell)} f_X^*(x), \tag{9}$$

where f_X^* is the pdf of EGG with parameters $\alpha(j + \ell), \beta, \tau, c$. In the following, the r-th non-central moment of $X_{(\ell)}$ is given.

Theorem 3.3. *For $\beta > 0$,*

$$\mu_{\ell:n}^{(r)} = \sum_{j=0}^{n-\ell} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{m} \alpha \left(j + \ell\right) \tau^{m+1} \beta^m (k+1)^{-r-1} c^{-r-c} (-1)^{n+2m-k} \tau_{n,m,k}^* \Delta \left(r; (k+1) c\psi\right),$$

where $\tau_{n,m,k}^* = \binom{\alpha j + \alpha \ell - 1}{n} \binom{\frac{n+1}{\beta} - 1}{m} \binom{m}{k}$ and $\Delta(r; t) = \int_0^t x^i e^x dx$.

Proof. We can proof the theorem Using (7) and (9). □

4. ESTIMATION

In this section, we discuss the maximum likelihood method to estimate the parameters of the EGG model based on a random sample of size n. When $\beta \neq 0$, the log-likelihood function is

$$\ell(\theta) = n \log(\alpha) + n \log(\tau) + c \sum_{i=1}^{n} x_i + \left(\frac{1}{\beta} - 1\right) \sum_{i=1}^{n} \log(1 - \beta z_i) + (\alpha - 1) \sum_{i=1}^{n} \log\left(1 - (1 - \beta z_i)^{\frac{1}{\beta}}\right),$$

where $z_i = \frac{\tau}{c}(e^{cx_i} - 1)$. By taking the derivative of log-likelihood function, we obtain the score vector $U(\theta) = \left(U_\alpha(\theta), U_\beta(\theta), U_\tau(\theta), U_c(\theta)\right)^T$ where

$$U_\alpha(\theta) = \frac{\partial \ell(\theta)}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log\left(1 - (1 - \beta z_i)^{\frac{1}{\beta}}\right),$$

$$U_\beta(\theta) = \frac{\partial \ell(\theta)}{\partial \beta} = -\frac{1}{\beta^2} \sum_{i=1}^{n} \log(1 - \beta z_i) - \left(\frac{1}{\beta} - 1\right) \sum_{i=1}^{n} \frac{z_i}{1 - \beta z_i}$$

$$+ \frac{(\alpha - 1)}{\beta^2} \sum_{i=1}^{n} \frac{(1 - \beta z_i)^{\frac{1}{\beta} - 1}}{1 - (1 - \beta z_i)^{\frac{1}{\beta}}} [\beta z_i + (1 - \beta z_i) \log(1 - \beta z_i)],$$

$$U_\tau(\theta) = \frac{\partial \ell(\theta)}{\partial \tau} = \frac{n}{\tau} - \frac{\beta}{c}\left(\frac{1}{\beta} - 1\right) \sum_{i=1}^{n} \frac{(e^{cx_i} - 1)}{1 - \beta z_i} + \frac{(\alpha - 1)}{c} \sum_{i=1}^{n} \frac{(e^{cx_i} - 1)(1 - \beta z_i)^{\frac{1}{\beta} - 1}}{1 - (1 - \beta z_i)^{\frac{1}{\beta}}},$$

$$U_c(\theta) = \frac{\partial \ell(\theta)}{\partial c} = \sum_{i=1}^{n} x_i - \frac{\beta \tau}{c^2}\left(\frac{1}{\beta} - 1\right) \sum_{i=1}^{n} \frac{e^{cx_i} + cx_i e^{cx_i} - 1}{(1 - \beta z_i)}$$

$$+ \frac{(\alpha - 1)\tau}{c^2} \sum_{i=1}^{n} \frac{(e^{cx_i} + cx_i e^{cx_i} - 1)(1 - \beta z_i)^{\frac{1}{\beta} - 1}}{\left(1 - (1 - \beta z_i)^{\frac{1}{\beta}}\right)}.$$

Unfortunately, the close form solution for the maximum likelihood estimation (MLE) of parameters does not exist, but one can provide them. As we see, when β be negative then the support of the model is $(0, \infty)$. So, by checking that the regularity conditions hold, one can say that the asymptotic distribution of vector $\hat{\theta}$ is multivariate normal. But the story is different when β be positive, because in this case the support of the distribution depends on the unknown parameters and so we can say that the asymptotic normality distribution does not satisfy. Here, We first find the MLE of the threshold parameter using [33]. Then, similar to the method of [19], asymptotic distribution of the MLE's are obtained.

As we know $v = \frac{1}{c}\log(\frac{c}{\tau\beta} + 1)$ is the thresholding parameter. It is easily verified that the MLE of v is $\tilde{v} = x_{(n)}$. Following the method of [19], the log-likelihood function is

$$\ell(\alpha, \tau, c, \tilde{v}) = (n - 1)\log(\alpha) + \sum_{i=1}^{n-1} \log(w(x_{(i)})) + (W(\tilde{v}) - 1) \sum_{i=1}^{n-1} \log(1 - \tilde{q}_{(i)})$$

$$+ (\alpha - 1) \sum_{i=1}^{n-1} \log\left(1 - (1 - \tilde{q}_{(i)})^{W(\tilde{v})}\right),$$

where $q = W(x)/W(v)$, $w(x)$ is the derivative of $W(x)$ with respect to x and $x_{(i)}$ and $q_{(i)}$ are the i-th observed ordered statistics from random samples x_i and q_i, respectively. At first, we provide that $\tilde{\alpha} = -(n - 1)/\tilde{u}$, where $\tilde{u} = \sum_{i=1}^{n-1} \log\left(1 - (1 - \tilde{q}_{(i)})^{W(\tilde{v})}\right)$ and in the next stage, the MLE of other parameters can be obtain by maximizing $\ell(\tilde{\alpha}, \tau, c, \tilde{v})$ with respect to τ, and c.

To determine the asymptotic distribution of the MLEs of (α, τ, c) based on the log-likelihood function $\ell(\alpha, \tau, c, \tilde{v})$, we present the following theorem.

Theorem 4.1. a. $n^{1/W(v)}(\tilde{v} - v)$ converges to $-\frac{W(v)}{\tau e^{cv}} V^{1/W(v)}$ in distribution, where V is distributed as an exponential distribution with mean $\frac{1}{\alpha}$.

b. Given $X_{(n)}$, the asymptotic distribution of the modified MLE for θ is multivariate normal distribution.

c. The asymptotic distribution of $(\tilde{\alpha}, \tilde{\tau}, \tilde{c})$ is (i) multivariate normal if $W(v) < \frac{1}{2}$, (ii) multivariate Weibull if $W(v) > \frac{1}{2}$, and (iii) a mixture of normal and Weibull if $W(v) = \frac{1}{2}$.

Proof. See [22] and [19]. □

5. MODELING A REAL DATA SET

The following data set has been provided by [34] and also analyzed by [23]. It represents the lifetimes of 50 devices.

0.1 0.2 1 1 1 1 1 2 3 6 7 11 12 18 18 18 18 1821 32 36 40 45 46 47 50 55
60 63 63 67 67 67 67 72 7579 82 82 83 84 84 84 85 85 85 85 85 86 86

To find the best model for above data, we compare EGG, EGE, GG, and GE distributions as competing models. For each model, we obtain the MLEs of parameters. Then, we calculate some statistics that are useful in detecting the fitting effect of above proposed distributions. These statistics as well as their p-values are famous in all fitting distribution problems. In Table 3, we provide these statistics. From the p-value of Kolmogorov-Smirnov (K-S) statistic, we find that all proposed distributions can be fitted to this data set. Also in Table 3, we provide some statistics such as Akaike information criterion (AIC), Corrected Akaike's Information Criterion (AICC), and Bayesian information criterion (BIC) to find the best fit between all proposed distributions. The lowest values of AIC, AICC, and BIC are related to EGG model. Also, all p-values of likelihood ratio test (LRT) statistic are less than 0.0001 which results in favor of EEG distribution. Totally, we can claim that the EGG model is the best model to fit among others. The plots of fitted pdfs with the histogram and plots of cdfs with the empirical cdf of the data set are presented in Figure 2.

Table 3 Fit criteria based on EGG, EGE, GG, and GE distributions.

	Distribution			
Statistic	**EGG**	**EGE**	**GG**	**GE**
$\hat{\alpha}$	0.3098	0.5368	0.2624	0.7798
$\hat{\beta}$	5.0000	1.8199	—	—
$\hat{\tau}$	0.0010	0.0064	0.0001	0.0187
$\hat{c}$	0.0173	—	0.0828	—
$-\log(L)$	173.69	189.1973	222.2441	239.9951
K-S	0.1371	0.1558	0.1146	0.2042
p-value (K-S)	0.3041	0.1763	0.5273	0.0309
AIC	355.3747	384.3945	454.2548	483.9903
AICC	356.2636	384.9163	454.7765	484.2456
BIC	363.0228	390.1306	459.9908	487.8143
LRT	—	31.0198	100.8801	132.6156
p-value (LRT)	—	0.0000	0.0000	0.0000

EGG, extended generalized Gompertz; EGE, extended generalized exponential; GG, generalized Gompertz; GE, generalized exponential; LRT, likelihood ratio test; K-S, Kolmogorov-Smirnov.
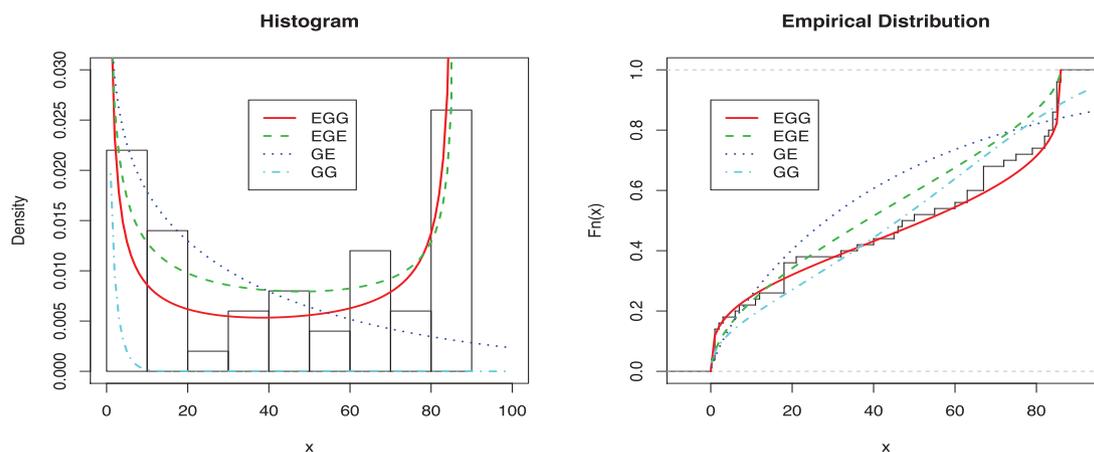
Figure 2 Fitting extended generalized Gompertz (EGG), extended generalized exponential (EGE), generalized Gompertz (GG), and generalized exponential (GE) distributions to the histogram (left) and the empirical cumulative distribution function (cdf) of the data (right).

CONFLICTS OF INTEREST

AUTHORS' CONTRIBUTIONS

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APPENDIX

A. Proof of Theorem 3.2

i.   Let $\alpha = 1$. Then

$$E\left(X^k\right) = \int_0^\infty \overline{F}\left(x^{1/k}\right) dx = \int_0^\infty \frac{1}{\left(1 + \frac{\tau}{cs}\left(e^{cx^{1/k}} - 1\right)\right)^s} dx,$$

where $\overline{F} = 1 - F$ and $s = -\frac{1}{\beta}$. Since $\frac{\left(cs - \tau + \tau e^{cx^{1/k}}\right)}{cs} \sim \frac{\tau}{cs}e^{cx^{1/k}}$, and integral $\int_0^\infty \frac{\tau}{cs}e^{-csx^{1/k}}\,dx$ converges for all positive values of τ, s and c.

ii. Since

$$\sum_{i=1}^\alpha (-1)^i \binom{\alpha}{i}\left(1 + \frac{z}{s}\right)^{-is} < \sum_{i=0}^\alpha \binom{\alpha}{i}\left(1 + \frac{z}{s}\right)^{-is} < 2^\alpha \left(1 + \frac{z}{s}\right)^{-s},$$

then using part (i), we can conclude that $E\left[X^k\right]$ exists for $\alpha \in \mathbb{Z}$.

iii. Since $E\left[X^k\right]$ exists for all $\alpha \in \mathbb{Z}$ and $\overline{F}$ is an increasing function of α then using (i) and (ii), we can conclude that $E\left[X^k\right]$ exists for all $\alpha > 0$.