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Research Article

Ideals Generated by Traces or by Supertraces in the Symplectic Reflection Algebra $H_{1,\nu}(I_2(2m+1))$ II

I.A. Batalin, S.E. Konstein*, I.V. Tyutin

I.E. Tamm Department of Theoretical Physics, P.N. Lebedev Physical Institute, RAS 119991, Leninsky Prosp., 53, Moscow, Russia

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ABSTRACT

The algebra $\mathcal{H} := H_{1,\nu}(I_2(2m+1))$ of observables of the Calogero model based on the root system $I_2(2m+1)$ has an m-dimensional space of traces and an $(m+1)$-dimensional space of supertraces. In the preceding paper we found all values of the parameter ν for which either the space of traces contains a degenerate nonzero trace tr_ν or the space of supertraces contains a degenerate nonzero supertrace str_ν and, as a consequence, the algebra $\mathcal{H}$ has two-sided ideals: one consisting of all vectors in the kernel of the form $B_{tr_\nu}(x, y) = tr_\nu(xy)$ or another consisting of all vectors in the kernel of the form $B_{str_\nu}(x, y) = str_\nu(xy)$. We noticed that if $\nu = \frac{z}{2m+1}$, where $z \in \mathbb{Z} \setminus (2m+1)\mathbb{Z}$, then there exist both a degenerate trace and a degenerate supertrace on $\mathcal{H}$. Here we prove that the ideals determined by these degenerate forms coincide.

1. INTRODUCTION

This paper is a continuation of [5]; we advise the reader to recall [5].

1.1. Definitions

Let $\mathcal{A}$ be an associative $\mathbb{Z}_2$-graded algebra with unit; let ε denote its parity. All expressions of linear algebra are given for homogenous elements only and are supposed to be extended to inhomogeneous elements via linearity.

A linear complex-valued function tr on $\mathcal{A}$ is called a *trace* if $tr(fg - gf) = 0$ for all $f, g \in \mathcal{A}$. A linear complex-valued function str on $\mathcal{A}$ is called a *supertrace* if $str(fg - (-1)^{\varepsilon(f)\varepsilon(g)}gf) = 0$ for all $f, g \in \mathcal{A}$. These two definitions can be unified as follows.

Let $\varkappa = \pm 1$. A linear complex-valued function $sp^\varkappa$ on $\mathcal{A}$ is called *$\varkappa$-trace* if $sp^\varkappa(fg - \varkappa^{\varepsilon(f)\varepsilon(g)}gf) = 0$ for all $f, g \in \mathcal{A}$.

Each nonzero $\varkappa$-trace $sp^\varkappa$ defines the nonzero symmetric[1] bilinear form $B_{sp^\varkappa}(f, g) := sp^\varkappa(fg)$.

If $B_{sp^\varkappa}$ is degenerate, then the set of the vectors of its kernel is a proper ideal in $\mathcal{A}$. We say that the $\varkappa$-trace $sp^\varkappa$ is *degenerate* if the bilinear form $B_{sp^\varkappa}$ is degenerate.

1.2. The Goal and Structure of the Paper

The simplicity (or, alternatively, existence of ideals) of Symplectic Reflection Algebras or, briefly, SRA (for definition, see [3]) was investigated in a number of papers, see, e.g., [2,9]. In particular, it is shown that all SRA $H_{1,\nu}(G)$ with $\nu = 0$ are simple (see [2,10]).

It follows from [4] and [7] that an associative algebra of observables of the Calogero model with harmonic term in the potential and with coupling constant ν based on the root system $I_2(2m+1)$ (this algebra is SRA denoted $H_{1,\nu}(I_2(2m+1))$) has an m-dimensional space of traces and an $(m+1)$-dimensional space of supertraces.

We say that the parameter ν is *singular*, if the algebra $H_{1,\nu}(I_2(n))$ has a degenerate trace or a degenerate supertrace.

*Corresponding author. Email: konstein@lpi.ru

[1]Initially, we used the term "(super)symmetric bilinear form" currently used by many, e.g., in the paper [1], even in its title. However, in a recent preprint [8], it is explained that the supersymmetry $B(v, w) = (-1)^{p(v)p(w)} B(w, v)$ is related with the isomorphism $V \otimes W \simeq W \otimes V$; of superspaces and has nothing to do with the (anti)symmetry of the bilinear form B on $V = W$.

In [5], we found all singular values of ν for the algebras $\mathcal{H} := H_{1,\nu}(I_2(n))$ in the case of n odd ($n = 2m + 1$) and found the corresponding degenerate traces and supertraces; the result is formulated in Theorem 10.1.

We noticed that if $\nu = \frac{z}{2m+1}$, where $z \in \mathbb{Z} \setminus (2m+1)\mathbb{Z}$, then there exist both a degenerate trace and a degenerate supertrace on H. Denote this degenerate trace by tr_z and the degenerate supertrace by str_z

Theorem 10.1 proved in [5] implies that if $z \in \mathbb{Z} \setminus n\mathbb{Z}$, then

(i) the trace given by the formula (10.1) in [5] is degenerate and generates the ideal $\mathcal{I}_{tr_z}$ consisting of all the vectors in the kernel of the degenerate form $B_{tr_z}(x, y) = tr_z(xy)$,

(ii) the supertrace (10.2) is degenerate and generates the ideal $\mathcal{I}_{str_z}$ consisting of all the vectors in the kernel of the degenerate form $B_{str_z}(x, y) = str_z(xy)$.

The goal of this paper is Theorem 13.1, which proves

Conjecture 1.1.

([5, Conjecture 9.1]) $\mathcal{I}_{tr_z} = \mathcal{I}_{str_z}$.

In Sections 2–10 we recall the necessary definitions and preliminary facts.

2. THE GROUP $I_2(2m + 1)$

Hereafter in this paper, $n = 2m + 1$.

Definition 2.1.

The group $I_2(n)$ is a finite subgroup of the orthogonal group $O(2, \mathbb{R})$ generated by the root system $I_2(n)$.

The group $I_2(n)$ is the symmetry group of a flat regular n-gon; $I_2(n)$ consists of n reflections R_k and n rotations S_k, where $k = 0, 1, \ldots, 2m$. We consider the indices k as integers modulo n.

These elements (R_k and S_k for all k) satisfy the relations

$$R_k R_l = S_{k-l}, \quad S_k S_l = S_{k+l}, \quad R_k S_l = R_{k-l}, \quad S_k R_l = R_{k+l}.$$

The element S_0 is the unit in the group $I_2(n)$. Obviously, since n is odd, all the reflections R_k are in the same conjugacy class.

The rotations S_k and S_l constitute a conjugacy class if $k + l = n$.

Let

$$\lambda := \exp\left(\frac{2\pi i}{n}\right).$$

Let

$$G := \mathbb{C}[I_2(n)] \tag{2.1}$$

be the group algebra of the group $I_2(n)$. In G, it is convenient to introduce the following basis

$$L_p := \frac{1}{n}\sum_{k=0}^{n-1} \lambda^{kp} R_k, \quad Q_p := \frac{1}{n}\sum_{k=0}^{n-1} \lambda^{-kp} S_k.$$

3. SYMPLECTIC REFLECTION ALGEBRA $H_{1,\nu}(I_2(2m + 1))$

Definition 3.1.

The symplectic reflection algebra $\mathcal{H} := H_{1,\nu}(I_2(2m+1))$ is the associative algebra of polynomials in the noncommuting elements a^α and b^α, where $\alpha = 0, 1$, with coefficients in G [see Eq. (2.1)], satisfying the relations

$$\begin{aligned} L_p a^\alpha &= -b^\alpha L_{p+1}, & L_p b^\alpha &= -a^\alpha L_{p-1}, \\ Q_p a^\alpha &= a^\alpha Q_{p+1}, & Q_p b^\alpha &= b^\alpha Q_{p-1}, \\ L_k L_l &= \delta_{k+l} Q_l, & L_k Q_l &= \delta_{k-l} L_l, \\ Q_k L_l &= \delta_{k+l} L_l, & Q_k Q_l &= \delta_{k-l} Q_l, \end{aligned} \tag{3.1}$$

$$\left[a^{\alpha}, b^{\beta}\right] = \varepsilon^{\alpha\beta}\left(1+\mu L_0\right),$$
$$\left[a^{\alpha}, a^{\beta}\right] = \varepsilon^{\alpha\beta}\mu L_1,$$
$$\left[b^{\alpha}, b^{\beta}\right] = \varepsilon^{\alpha\beta}\mu L_{-1},$$

where $\delta k := \delta_{k0}$, and $\varepsilon^{\alpha\beta}$ is the skew-symmetric tensor normalized so that $\varepsilon^{01} = 1$, and

$$\mu := n\nu.$$

Defining the parity in $\mathcal{H}$ by setting

$$\varepsilon(a^{\alpha}) = \varepsilon(b^{\alpha}) = 1, \quad \varepsilon(R_k) = \varepsilon(S_k) = 0,$$

we turn this algebra into a superalgebra.

The algebra $H_{1,\nu}(I_2(2m+1))$ depends on one complex parameter ν.

4. SUBALGEBRA OF SINGLETS

Consider the elements[2] $T^{\alpha\beta} := \frac{1}{2}(\{a^{\alpha}, b^{\beta}\} + \{b^{\alpha}, a^{\beta}\})$ of the algebra $\mathcal{H}$, and the inner derivations of $\mathcal{H}$ they generate:

$$D^{\alpha\beta}: \ f \mapsto \left[f, T^{\alpha\beta}\right] \qquad \text{for any } f \in \mathcal{H}.$$

It is easy to verify that the linear span of these derivations is a Lie algebra isomorphic to sl_2.

Definition 4.1.

A *singlet* is any element $f \in \mathcal{H}$ such that $[f, T^{\alpha\beta}] = 0$ for all α, β. The subalgebra $H^0 \subset \mathcal{H}$ consisting of all singlets of the algebra $\mathcal{H}$ is called the *subalgebra of singlets*.

One can consider the algebra $\mathcal{H}$ as an sl_2-module and decompose it into the direct sum of irreducible submodules.

Observe, that any $\varkappa$-trace is identically zero on all irreducible sl_2-submodules of $\mathcal{H}$, except for singlets.

Let the skew-symmetric tensor $\varepsilon_{\alpha\beta}$ be normalized so that $\varepsilon_{01} = 1$ and so $\sum_{\alpha}\varepsilon_{\alpha\beta}\varepsilon^{\alpha\gamma} = \delta^{\gamma}_{\beta}$. We set

$$\mathfrak{s} := \frac{1}{4i}\sum_{\alpha,\beta=0,1}\left(\{a^{\alpha}, b^{\beta}\} - \{b^{\alpha}, a^{\beta}\}\right)\varepsilon_{\alpha\beta}.$$

Proposition 4.1.

([5, Proposition 4.2]) The subalgebra of singlets H_0 is the algebra of polynomials in the element $\mathfrak{s}$ with coefficients in the group algebra $\mathbb{C}[I_2(2m+1)]$.

The commutation relations of the singlet $\mathfrak{s}$ with generators of the algebra $\mathcal{H}$ have the form:

$$[\mathfrak{s}, Q_p] = [\mathfrak{s}, S_k] = [T^{\alpha\beta}, \mathfrak{s}] = 0,$$
$$\mathfrak{s}L_p = -L_p\mathfrak{s}, \ \mathfrak{s}R_k = -R_k\mathfrak{s},$$
$$(\mathfrak{s} - i\mu L_0)a^{\alpha} = a^{\alpha}(\mathfrak{s} + i + i\mu L_0).$$

5. THE FORM OF IDEALS IN $\mathcal{H}$ AND IN H^0

Theorem 5.1.

([5, Theorem 4.3]) Let $\mathcal{I}$ be a proper ideal in the algebra $\mathcal{H}$, and $\mathcal{I}_0 := \mathcal{I} \cap H^0$. Then, there exist nonzero polynomials $\phi^0_k \in \mathbb{C}[\mathfrak{s}]$, where $k = 0, \ldots, n-1$, such that $\mathcal{I}_0$ is the span over $\mathbb{C}[\mathfrak{s}]$ of the elements

$$\phi^0_k(\mathfrak{s})Q_k, \quad \phi^0_{n-k}L_k, \quad \text{where } k = 0, \ldots, n-1 \text{ and } \phi^0_n := \phi^0_0.$$

Proposition 5.1.

([5, Proposition 4.4]) If $\mathcal{I} \subset \mathcal{H}$ is a proper ideal, then $\mathcal{I}_0 = \mathcal{I} \cap H^0$ is a proper ideal in H^0.

Definition 5.1.

For each $p = 0, \ldots, 2m$, we define the ideals $\mathcal{J}_p$ and $\mathcal{J}^p$ in the algebra $\mathbb{C}[\mathfrak{s}]$ by setting

$$\mathcal{J}_p := \{f \in \mathbb{C}[\mathfrak{s}] \mid f(\mathfrak{s})Q_p \in \mathcal{I}\}, \quad \mathcal{J}^p := \{f \in \mathbb{C}[\mathfrak{s}] \mid f(\mathfrak{s})L_p \in \mathcal{I}\}.$$

[2]Here the brackets $\{\cdot,\cdot\}$ denote anticommutator.

Proposition 5.2.

([5, Proposition 4.7]) We have $\mathcal{J}_p = \mathcal{J}^{-p}$.

Proposition 5.3.

([5, Proposition 4.8]). We have $\mathcal{J}_p \neq 0$ *for any* $p = 0, \ldots, 2m$.

Since $\mathbb{C}[\mathfrak{s}]$ is a principal ideal ring, we have the following statement:

Corollary 5.1.

For any $p = 0, \ldots, 2m$, *there exists a nonzero polynomial* $\phi_p^0 \in \mathbb{C}[\mathfrak{s}]$ *such that* $\mathcal{J}_p = \phi_p^0 \mathbb{C}[\mathfrak{s}]$.

Theorem 5.1 evidently follows from Corollary 5.1.

6. GENERATING FUNCTIONS OF $\varkappa$-TRACES

For each $\varkappa$-trace $sp^{\varkappa}$ on $\mathcal{H}$, one can define the following set of generating functions which allow one to calculate the $\varkappa$-trace of arbitrary element in H^0 via finding the values of the derivatives of these functions with respect to parameter t at zero:

$$\begin{aligned} F_p^{sp^{\varkappa}}(t) &:= sp^{\varkappa}(\exp(t(\mathfrak{s} - i\mu L_0))Q_p), \\ \Psi_p^{sp^{\varkappa}}(t) &:= sp^{\varkappa}(\exp(t\mathfrak{s})L_p), \qquad \text{where } p = 0, \ldots, 2m. \end{aligned} \tag{6.1}$$

Since $L_0 Q_p = 0$ for any $p \neq 0$, it follows from the definition (6.1) that

$$\begin{aligned} F_p^{sp^{\varkappa}}(t) &= sp^{\varkappa}(\exp(t\mathfrak{s})Q_p) \qquad \text{if } p \neq 0, \\ F_0^{sp^{\varkappa}}(t) &= sp^{\varkappa}(\exp(t(\mathfrak{s} - i\mu L_0))Q_0). \end{aligned}$$

It is easy to find $\Psi_p^{sp^{\varkappa}}$ for $p \neq 0$. Since $\mathfrak{s}L_q = -L_q\mathfrak{s}$ for any $q = 0, \ldots, 2m$, we have

$$\Psi_q^{sp^{\varkappa}}(t) = sp^{\varkappa}(\exp(t\mathfrak{s})L_q) = sp^{\varkappa}(L_q). \tag{6.2}$$

Next, since $sp^{\varkappa}(R_k)$ does not depend on k, we have $sp^{\varkappa}(L_p) = 0$ for any $p \neq 0$ and

$$\Psi_p^{sp^{\varkappa}}(t) \equiv 0 \qquad \text{for any } p \neq 0. \tag{6.3}$$

The value of $sp^{\varkappa}(L_0)$ will be calculated later, in Section 9.

We consider also the functions

$$\Phi_p^{sp^{\varkappa}}(t) := sp^{\varkappa}(\exp(t(\mathfrak{s} + i\mu L_0))Q_p).$$

It is easily verified, by expanding the exponential in a series, that these functions are related with the functions $F_p^{sp^{\varkappa}}$ by the formula

$$\Phi_p^{sp^{\varkappa}}(t) = F_p^{sp^{\varkappa}}(t) + 2i\Delta_p^{sp^{\varkappa}}(t), \qquad \text{where } \Delta_p(t)^{sp^{\varkappa}} = \delta_p \sin(\mu t) sp^{\varkappa}(L_0).$$

The form of generating functions is related with (non)degeneracy of the form $B_{sp^{\varkappa}}$ as described in Proposition 7.1 below.

7. DEGENERACY CONDITIONS FOR THE $\varkappa$-TRACE

Proposition 7.1.

([5, Proposition 6.1]). The $\varkappa$-trace on the algebra $\mathcal{H}$ is degenerate if and only if the generating functions $F_p^{sp^{\varkappa}}$ *defined by formula (6.1) have the following form*

$$F_p^{sp^{\varkappa}}(t) = \sum_{j=1}^{j_p} \exp(t\omega_{j,p})\varphi_{j,p}(t), \tag{7.1}$$

where $\omega_{j,p} \in \mathbb{C}$ *and* $\varphi_{j,p} \in \mathbb{C}[t]$ *might depend on* $\varkappa$.

8. EQUATIONS FOR THE GENERATING FUNCTIONS $F_p^{sp^{\varkappa}}$

In [5, Eq. (7.1)], the following system of differential equations for the generating functions is obtained:

$$\frac{d}{dt}F_p^{sp^{\varkappa}} - \varkappa e^{it}\frac{d}{dt}F_{p+1}^{sp^{\varkappa}} = iF_p^{sp^{\varkappa}} + \varkappa ie^{it}F_{p+1}^{sp^{\varkappa}} + 2\varkappa i\frac{d}{dt}\left(e^{it}\Delta_{p+1}^{sp^{\varkappa}}\right). \tag{8.1}$$

The initial conditions for this system are:

$$F_p^{sp^{\varkappa}}(0) = sp^{\varkappa}(Q_p).$$

To solve the system (8.1), we consider its Fourier transform. Let

$$\begin{aligned} &\lambda := e^{2\pi i/(2m+1)}, \\ &G_k^{sp^{\varkappa}} := \sum_{p=0}^{2m}\lambda^{kp}F_p^{sp^{\varkappa}}, && \text{where } k = 0, \ldots, 2m, \\ &\widetilde{\Delta}_k^{sp^{\varkappa}} := \sum_{p=0}^{2m}\lambda^{kp}\Delta_{p+1}^{sp^{\varkappa}} = \lambda^{-k}\left(\sin(\mu t)sp^{\varkappa}(L_0)\right), && \text{where } k = 0, \ldots, 2m. \end{aligned} \tag{8.2}$$

For the functions $G_k^{sp^{\varkappa}}$, we then obtain the equations

$$\frac{d}{dt}G_k^{sp^{\varkappa}} = i\frac{\lambda^k + \varkappa e^{it}}{\lambda^k - \varkappa e^{it}}G_k^{sp^{\varkappa}} + \frac{2i\varkappa\lambda^k}{\lambda^k - \varkappa e^{it}}\frac{d}{dt}\left(e^{it}\widetilde{\Delta}_k^{sp^{\varkappa}}\right) \tag{8.3}$$

with the initial conditions

$$G_k^{sp^{\varkappa}}(0) = sp^{\varkappa}(S_k). \tag{8.4}$$

We choose the following form of the solution of the system (8.3):

$$G_k^{sp^{\varkappa}}(t) = \frac{\varkappa e^{it}}{(\varkappa e^{it} - \lambda^k)^2}\lambda^k g_k^{sp^{\varkappa}}(t), \tag{8.5}$$

where

$$g_k^{sp^{\varkappa}}(t) = \left(\frac{2}{\mu}(\cos(t\mu) - 1) + 2i\lambda^{-k}(\lambda^k - \varkappa e^{it})\sin(t\mu)\right)sp^{\varkappa}(L_0) + \varkappa\lambda^{-k}(\varkappa - \lambda^k)^2 sp^{\varkappa}(S_k). \tag{8.6}$$

Evidently, this solution satisfies the initial condition (8.4) for each $\varkappa$ and k, except for the case where $\varkappa = +1$ and $k = 0$.

If $\varkappa = +1$ and $k = 0$, then the expression (8.5) for G_0^{tr} has a removable singularity at $t = 0$. In this case, instead of the condition $G_0^{tr}(0) = tr(S_0)$ we consider the condition $\lim_{t\to 0}G_0^{tr}(t) = tr(S_0)$.

When $\varkappa = +1$ the solution (8.5) and (8.6) gives

$$G_0^{tr}(t) = \frac{e^{it}}{(e^{it} - 1)^2}\left(\frac{2}{\mu}(\cos(t\mu) - 1) + 2i(1 - e^{it})\sin(t\mu)\right)tr(L_0),$$

and one can easily see that

$$\lim_{t\to 0}G_0^{tr}(t) = -\mu tr(L_0).$$

It is shown in Subsection 9.1 that if $\varkappa = +1$, then

$$tr(S_0) = -\mu tr(L_0)$$

for any trace tr on $\mathcal{H}$.

So, $G_0^{tr}(t)$ satisfies the initial conditions (8.4) also.

In the case where $\varkappa = -1$, the $\varkappa$-trace is a supertrace (see [4]). In this case, the $m + 1$ values $str(S_k) = str(S_{2m+1-k})$ for $k = 0, \ldots, m$ completely define the supertrace on $\mathcal{H}$ (see [7]).

In the case where $\varkappa = +1$, the $\varkappa$-trace is a trace (see [4]). In this case, the m values $tr(S_k) = tr(S_{2m+1-k})$ for $k = 1, \ldots, m$ completely define the trace on $\mathcal{H}$ (see [7]). The value $tr(S_0)$ linearly depends on parameters $tr(S_k)$, where $k = 1, \ldots, m$, and this value is found in Subsection 9.1 (see Eqs. (9.4 and 9.5)).

9. VALUES OF THE $\varkappa$-TRACE ON $\mathbb{C}[I_2(2m + 1)]$

From [5] we have

$$sp^{\varkappa}(R_k) = -\frac{2\mu}{2m+1}\left(\frac{1+\varkappa}{2}X^{tr} + \frac{1-\varkappa}{2}Y^{str}\right), \tag{9.1}$$

where

$$X^{tr} := \sum_{r=1}^{2m} \sin^2\left(\frac{\pi r}{2m+1}\right) tr(S_r), \tag{9.2}$$

$$Y^{str} := \sum_{r=0}^{2m} \cos^2\left(\frac{\pi r}{2m+1}\right) str(S_r). \tag{9.3}$$

Below we consider these values for the traces and supertraces separately.

9.1. Values of the Traces ($\varkappa = +1$) on $\mathbb{C}[I_2(2m + 1)]$

The group $I_2(2m + 1)$ has m conjugacy classes without the eigenvalue +1 in the spectrum: $\{S_p, S_{2m+1-p}\}$, where $p = 1, \ldots, m$.

By Theorem 2.3 in [4], the values of the trace on these conjugacy classes

$$s_k := tr(S_k), \qquad \text{where } s_{2m+1-k} = s_k,\ k = 1, \ldots, m,$$

are arbitrary and completely define the trace on the algebra $\mathcal{H}$. Therefore, the dimension of the space of traces is equal to m.

Further, the group $I_2(2m + 1)$ has one conjugacy class with one eigenvalue +1 in its spectrum: $\{R_1, \ldots, R_{2m+1}\}$. The value of $tr(R_k)$ is expressed via s_k by formula (9.1).

Besides, the group $I_2(2m + 1)$ has one conjugacy class with two eigenvalues +1 in its spectrum: $\{S_0\}$.

The traces on conjugacy classes with two eigenvalues +1 in the spectrum is calculated in [5] using Ground Level Conditions (for their definition, see [4]):

$$tr(S_0) = 2\nu^2(2m+1)X^{tr}. \tag{9.4}$$

We also note that

$$tr(L_0) = -\frac{2\mu}{2m+1}X^{tr}, \quad tr(L_p) = 0 \text{ for } p \neq 0, \qquad tr(S_0) = -\mu tr(L_0). \tag{9.5}$$

9.2. Values of the Supertraces ($\varkappa = -1$) on $\mathbb{C}[I_2(2m + 1)]$

The group $I_2(2m + 1)$ has $m + 1$ conjugacy classes without the eigenvalue −1 in the spectrum:

$$\{S_0\}, \{S_p, S_{2m+1-p}\}, \text{ where } p = 1, \ldots, m.$$

By [4, Theorem 2.3], the values of the supertrace on these conjugacy classes

$$u_k := str(S_k) = str(S_{2m+1-k}), \text{ where } k = 0, \ldots, m,$$

are arbitrary parameters that completely define the supertrace *str* on the algebra $\mathcal{H}$, and therefore the dimension of the space of supertraces is equal to $m + 1$.

Besides, the group $I_2(2m+1)$ has one conjugacy class with one eigenvalue −1 in the spectrum: $\{R_1, ..., R_{2m+1}\}$.

The supertraces of the conjugacy class with eigenvalue −1 in its spectrum are given by Eq. (9.1): $str(R_k) = -2\nu Y^{str}$, where $k = 0, 1, \ldots, 2m$, and where Y^{str} is defined by Eq (9.3).

10. SINGULAR VALUES OF THE PARAMETER μ

The solution Eq. (8.5) and (8.6) determines the generation functions of traces and supertraces on H^0 for any trace and any supertrace on $\mathcal{H}$. Generally speaking, $G_k^{sp^\varkappa}$ is a meromorphic function on t, but if μ and $sp^\varkappa$ are such that the form $B_{sp^\varkappa}$ is degenerate, then $G_k^{sp^\varkappa}$ is an integer function on t for each k. The complete list of such pairs of μ and $sp^\varkappa$ is given in Theorem 10.1. For these values of μ and $sp^\varkappa$, the functions $G_k^{sp^\varkappa}$ are Laurent polynomials in $\exp(it)$.

Theorem 10.1.

([5, Theorem 9.1]). Let $m \in \mathbb{Z}$, where $m \geq 1$, and $n = 2m+1$. Then

(1) *The associative algebra $H_{1,\nu}(I_2(n))$ has a one-parameter set of nonzero traces tr_z such that the symmetric invariant bilinear form $B_{tr_z}(x, y) = tr_z(xy)$ is degenerate if and only if $\nu = \frac{z}{n}$, where $z \in \mathbb{Z} \setminus n\mathbb{Z}$. These traces are completely defined by their values at S_k for $k = 1, \ldots, m$:*

$$tr_z(S_k) = \frac{\tau}{n\sin^2\frac{\pi k}{n}}\left(1-\cos\frac{2\pi kz}{n}\right), \quad \text{where } \tau \in \mathbb{C},\ \tau \neq 0. \tag{10.1}$$

Here τ is an arbitrary parameter specifying the trace in one-dimensional space of traces.

(2) *The associative superalgebra $H_{1,\nu}(I_2(n))$ has a one-parameter set of nonzero supertraces str_z such that the symmetric invariant bilinear form $B_{str_z}(x, y) = str_z(xy)$ is degenerate if $\nu = \frac{z}{n}$, where $z \in \mathbb{Z} \setminus n\mathbb{Z}$. These supertraces are completely defined by their values at S_k for $k = 0, \ldots, m$:*

$$str_z(S_k) = \frac{\tau}{n\cos^2\frac{\pi k}{n}}\left(1-(-1)^z\cos\frac{2\pi kz}{n}\right), \quad \text{where } \tau \in \mathbb{C},\ \tau \neq 0. \tag{10.2}$$

Here τ is an arbitrary parameter specifying the supertrace in one-dimensional space of supertraces.

(3) *The associative superalgebra $H_{1,\nu}(I_2(n))$ has a one-parameter set of nonzero supertraces $str_{1/2}$ such that the symmetric invariant bilinear form $B_{str_{1/2}}(x, y) = str_{1/2}(xy)$ is degenerate if $\nu = z + \frac{1}{2}$, where $z \in \mathbb{Z}$. These supertraces are completely defined by their values at S_k for $k = 0, \ldots, m$:*

$$str_{1/2}(S_k) = \frac{\tau}{n\cos^2\frac{\pi k}{n}}, \quad \text{where } \tau \in \mathbb{C},\ \tau \neq 0.$$

Here τ is an arbitrary parameter specifying the supertrace in one-dimensional space of supertraces.

(4) *For all other values of ν, all nonzero traces and supertraces are nondegenerate.*

11. GENERATING FUNCTIONS $F_p^{sp^\varkappa}$ FOR THE DEGENERATE $\varkappa$-TRACE

Let $\mu \in \mathbb{Z} \setminus n\mathbb{Z}$. Substitute the solutions (10.1) for the case $\varkappa = +1$ and (10.2) for the case $\varkappa = -1$ to Eqs. (8.5) and (8.6). We obtain the formula for both values of $\varkappa$

$$g_k^{sp^\varkappa} = -\frac{4\tau}{n}\left[\cos(t\mu) + i\mu\lambda^{-k}(\lambda^k - \varkappa e^{it})\sin(t\mu) - \varkappa^\mu \cos\frac{2\pi k\mu}{n}\right]. \tag{11.1}$$

Introducing the new variable y instead of t

$$y := \varkappa e^{it} \tag{11.2}$$

we can rewrite Eq. (11.1) in the form

$$g_k^{sp^{\varkappa}} = -\frac{2\tau}{n}\varkappa^{\mu}\left[(y^{\mu}+y^{-\mu})+\mu\lambda^{-k}(\lambda^k-y)(y^{\mu}-y^{-\mu})-2\cos\frac{2\pi k\mu}{n}\right]$$

and Eq. (8.5) in the form

$$G_k^{sp^{\varkappa}} = \frac{\lambda^k y}{(y-\lambda^k)^2}g_k^{sp^{\varkappa}}. \tag{11.3}$$

Now we see that $G_k^{sp^{\varkappa}}$ are the Laurent polynomials in y with the highest degree $\leq|\mu|$ and the lowest degree $\geq 1-|\mu|$.

Note, that the expressions (11.3) are even functions of the parameter μ, so we can assume that μ is a positive integer.

Let $\mu > 0$ in what follows.

Thus, $G_k^{sp^{\varkappa}}$ can be expressed in the form

$$G_k^{sp^{\varkappa}} = \varkappa^{\mu}\sum_{\ell=\mu}^{1-\mu}\beta_{\ell}^k y^{\ell}, \tag{11.4}$$

where the β_{ℓ}^k are constants not depending on $\varkappa$ and not all of them equal to zero.

Equation (11.4) implies that

$$\beta_{\mu}^k = \frac{2\tau\mu}{n}. \tag{11.5}$$

Further, Eq. (8.2) implies

$$F_p^{sp^{\varkappa}} = \frac{1}{n}\sum_{k=0}^{2m}\lambda^{-kp}G_k^{sp^{\varkappa}}$$

and the generating functions $F_p^{sp^{\varkappa}}$ have the form

$$F_p^{sp^{\varkappa}} = \varkappa^{\mu}\sum_{\ell=\mu}^{-\mu}\alpha_{\ell}^p y^{\ell}, \tag{11.6}$$

where the α_{ℓ}^k are constants not depending on $\varkappa$. Observe that $F_p^{sp^{\varkappa}}$ can be equal to zero for some $p \neq 0$ (e.g., if $\mu = 1$, then $F_p^{sp^{\varkappa}} = 0$ for each $p \neq 0$), but $F_0^{sp^{\varkappa}} \neq 0$ since Eq. (11.5) implies $\alpha_{\mu}^0 = \frac{2\tau\mu}{n} \neq 0$. Equation (11.5) implies also that $\alpha_{-\mu}^0 = 0$.

12. THE GENERATING FUNCTION $\mathcal{F}^{sp^{\varkappa}} = sp^{\varkappa}(\exp(t\mathfrak{s})Q_0)$ FOR THE DEGENERATE $\varkappa$-TRACE

Let $\mu \in \mathbb{Z}\setminus n\mathbb{Z}$ and $\varkappa$-trace be defined by Eq. (10.1) in the case $\varkappa = +1$ and by Eq. (10.2) in the case $\varkappa = -1$.

In this section we introduce the function

$$\mathcal{F}^{sp^{\varkappa}} := sp^{\varkappa}\left(\exp(t\mathfrak{s})Q_0\right)$$

and express it via $F_0^{sp^{\varkappa}}$.

Proposition 12.1.

$\mathcal{F}^{sp^{\varkappa}}$ is an even function of t:

$$\mathcal{F}^{sp^{\varkappa}} = sp^{\varkappa}(\cosh(t\mathfrak{s})Q_0. \tag{12.1}$$

Indeed, $\mathcal{F}^{sp^{\varkappa}} = sp^{\varkappa}(\cosh(t\mathfrak{s})Q_0 + \sinh(t\mathfrak{s})Q_0)$ and $sp^{\varkappa}(\sinh(t\mathfrak{s})Q_0) = 0$ since

$$sp^{\varkappa}(\sinh(t\mathfrak{s})Q_0) = sp^{\varkappa}((\sinh(t\mathfrak{s})L_0)L_0) = sp^{\varkappa}(L_0(\sinh(t\mathfrak{s})L_0)) = sp^{\varkappa}((L_0\sinh(t\mathfrak{s}))L_0) = sp^{\varkappa}((-(\sinh(t\mathfrak{s})L_0))L_0) = sp^{\varkappa}(-\sinh(t\mathfrak{s})Q_0)$$

Now, decompose $F_0^{sp^{\varkappa}}$:

$$\begin{aligned}
F_0^{sp^{\varkappa}} &= sp^{\varkappa}\left(e^{t(\mathfrak{s}-i\mu L_0)}Q_0\right) = F_{\text{even}} + F_{\text{odd}} \qquad \text{where} \\
F_{\text{even}} &= sp^{\varkappa}\left(\sum_{s=0}^{\infty}\frac{1}{(2s)!}(t(\mathfrak{s}-i\mu L_0))^{2s}Q_0\right) = sp^{\varkappa}\left(\sum_{s=0}^{\infty}\frac{1}{(2s)!}t^{2s}(\mathfrak{s}^2-\mu^2)^s Q_0\right),
\end{aligned} \tag{12.2}$$

$$\begin{aligned}
F_{odd} &= sp^{\varkappa}\left(\sum_{s=0}^{\infty}\frac{1}{(2s+1)!}(t(\mathfrak{s}-i\mu L_0))^{2s+1}Q_0\right) \\
&= sp^{\varkappa}\left(\sum_{s=0}^{\infty}\frac{1}{(2s+1)!}t^{2s+1}(\mathfrak{s}^2-\mu^2)^s(\mathfrak{s}-i\mu L_0)Q_0\right) \\
&= sp^{\varkappa}\left(\sum_{s=0}^{\infty}\frac{1}{(2s+1)!}t^{2s+1}(\mathfrak{s}^2-\mu^2)^s(-i\mu L_0)Q_0\right) \\
&= \sum_{s=0}^{\infty}\frac{1}{(2s+1)!}t^{2s+1}(-\mu^2)^s(-i\mu)sp^{\varkappa}L \;= \\
&= \sinh(-i\mu)sp^{\varkappa}L_0 = -\frac{\varkappa^{\mu}}{2}(y^{\mu}-y^{-\mu})sp^{\varkappa}L_0.
\end{aligned} \tag{12.3}$$

Equation (11.6) implies that

$$F_{\text{odd}} = \frac{\varkappa^{\mu}}{2}\left(\sum_{\ell=\mu}^{-\mu}\alpha_{\ell}^{0}y^{\ell} - \sum_{\ell=\mu}^{-\mu}\alpha_{-\ell}^{0}y^{\ell}\right). \tag{12.4}$$

Comparing Eq. (12.4) with Eq. (12.3) implies

$$\begin{aligned}
&\alpha_{\ell}^{0} = \alpha_{-\ell}^{0}, \text{ if } \ell \neq \mu, \ell \neq -\mu, \\
&\alpha_{\mu}^{0} - \alpha_{-\mu}^{0} = -sp^{\varkappa}L_0,
\end{aligned} \tag{12.5}$$

and

$$F_{\text{even}} = \frac{\varkappa^{\mu}}{2}\alpha_{\mu}^{0}(y^{\mu}+y^{-\mu}) + \frac{\varkappa^{\mu}}{2}\sum_{\ell=0}^{\mu-1}\alpha_{\ell}^{0}(y^{\ell}+y^{-\ell}) = \alpha_{\mu}^{0}\cosh(it\mu) + \varkappa^{\mu}\sum_{\ell=0}^{\mu-1}\varkappa^{\ell}\alpha_{\ell}^{0}\cosh(it\ell). \tag{12.6}$$

Proposition 12.2.

$$\mathcal{F}^{sp^{\varkappa}}(t) = \alpha_{\mu}^{0} + \varkappa^{\mu}\sum_{\ell=0}^{\mu-1}\varkappa^{\ell}\alpha_{\ell}^{0}\cosh\left(t\sqrt{\mu^2-\ell^2}\right).$$

Proof. Taking Proposition 12.1 into account let us decompose Eq. (12.1) into the Taylor series:

$$\mathcal{F}^{sp^{\varkappa}}(t) = \sum_{s=0}^{\infty}a_{2s}\frac{t^{2s}}{(2s)!},$$

where $a_{2s} := sp^{\varkappa}(\mathfrak{s}^{2s}Q_0)$ for $s = 0, 1, 2, \ldots$.

Equation (12.2) implies

$$a_{2s} = \left(\frac{d^2}{dt^2}+\mu^2\right)^s F_{\text{even}}\big|_{t=0},$$

and Eq. (12.6) implies

$$a_{2s} = \begin{cases} a_{\mu}^{0} + \varkappa^{\mu}\sum_{\ell=0}^{\mu-1}\varkappa^{\ell}\alpha_{\ell}^{0} & \text{if } s=0 \\ \varkappa^{\mu}\sum_{\ell=0}^{\mu-1}\varkappa^{\ell}\alpha_{\ell}^{0}(-\ell^2+\mu^2)^s & \text{if } s\neq 0. \end{cases}$$

So

$$\mathcal{F}^{sp^{\varkappa}}(t) = \sum_{s=0}^{\infty}a_{2s}\frac{t^{2s}}{(2s)!} = \alpha_{\mu}^{0} + \varkappa^{\mu}\sum_{\ell=0}^{\mu-1}\varkappa^{\ell}\alpha_{\ell}^{0}\cosh\left(t\sqrt{\mu^2-\ell^2}\right).$$

13. IDEALS GENERATED BY DEGENERATE $\varkappa$-TRACES

Let $\mu = z \in \mathbb{Z} \setminus n\mathbb{Z}$ and the $\varkappa$-trace be defined by Eq. (10.1) for the case $\varkappa = +1$ and by Eq. (10.2) for the case $\varkappa = -1$.

These degenerate $\varkappa$-traces are denoted in Theorem 10.1 by tr_z and str_z.

Denote the ideals generated by these traces $sp^{\varkappa}$ by $\mathcal{I}^{\varkappa}$; in H^0, consider the ideals $\mathcal{I}_0^{\varkappa} := \mathcal{I}^{\varkappa} \cap H^0$.

Now we can prove Conjecture 1.1 ([5, Conjecture 9.1]):

Theorem 13.1.

$\mathcal{I}^{+1} = \mathcal{I}^{-1}$.

To prove Theorem 13.1 we use Theorem 4.2 from [6] which in our case implies

Theorem 13.2.

([6, Theorem 4.2]) $\mathcal{I}^{+1} = \mathcal{I}^{-1}$ *if and only if* $\mathcal{I}^{+1} \cap H^0 = \mathcal{I}^{-1} \cap H^0$.

So, Theorem 13.1 follows from

Theorem 13.3.

$$\mathcal{I}^{+1} \cap H^0 = \mathcal{I}^{-1} \cap H^0 .$$

Proof. For degenerate $sp^{\varkappa}$, we established the following facts:

$$F_p^{sp^{\varkappa}}(t) = sp^{\varkappa}\left(e^{t\mathfrak{s}}Q_p\right) = \varkappa^{\mu} \sum_{\ell=\mu}^{-\mu} \alpha_\ell^p \varkappa^{\ell} e^{it\ell} \qquad \text{for } p = 1, 2, \ldots, n-1,$$

$$\mathcal{F}^{sp^{\varkappa}}(t) = sp^{\varkappa}\left(e^{t\mathfrak{s}}Q_0\right) = \alpha_\mu^0 + \varkappa^{\mu} \sum_{\ell=0}^{\mu-1} \varkappa^{\ell} \alpha_\ell^0 \cosh\left(t\sqrt{\mu^2 - \ell^2}\right) \qquad \text{where } \alpha_\mu^0 \neq 0,$$

and where the α-s do not depend on $\varkappa$.

For any $p = 1, \ldots, n$, it is easy to find the lowest degree polynomial differential operators with constant coefficients $D_p^{\varkappa}(d/dt)$ such that $D_p^{\varkappa}(d/dt)F_p^{sp^{\varkappa}}(t) = 0$:

$$D_p^{\varkappa}\left(\frac{d}{dt}\right) = \begin{cases} \prod\limits_{\ell=-\mu:\, \alpha_\ell^p \neq 0}^{\mu} \left(\frac{d}{dt} - i\ell\right) & \text{if } \mathcal{F}_p^{sp^{\varkappa}} \neq 0, \\ 1 & \text{if } \mathcal{F}_p^{sp^{\varkappa}} = 0, \end{cases}$$

and $D_0^{\varkappa}(d/dt)$ such that $D_0^{\varkappa}(d/dt)\mathcal{F}^{sp^{\varkappa}}(t) = 0$:

$$D_0^{\varkappa}\left(\frac{d}{dt}\right) = \frac{d}{dt} \prod_{\ell=0:\, \alpha_\ell^0 \neq 0}^{\mu-1} \left(\frac{d^2}{dt^2} - \mu^2 + \ell^2\right).$$

Further, it is a simple exercise to prove that

$$D_p^{\varkappa}(\mathfrak{s})Q_p,\ D_p^{\varkappa}(\mathfrak{s})L_{-p} \in \mathcal{I}_0^{\varkappa} \text{ for any } p = 0, \ldots n-1,$$

namely,

$$B_{sp^{\varkappa}}(D_p^{\varkappa}(\mathfrak{s})Q_p, f) = B_{sp^{\varkappa}}(D_p^{\varkappa}(\mathfrak{s})L_{-p}, f) = 0 \text{ for any } f \in H^0 \text{ and } p = 0, \ldots, n-1.$$

Consider, for example, $B_{sp^{\varkappa}}(D_0^{\varkappa}(\mathfrak{s})Q_0, f)$ for $f = g(\mathfrak{s})Q_p$ and $f = g(\mathfrak{s})L_p$:

$$sp^{\varkappa}(D_0^{\varkappa}(\mathfrak{s})Q_0 g(\mathfrak{s})Q_p) = sp^{\varkappa}(D_0^{\varkappa}(\mathfrak{s})Q_0 g(\mathfrak{s})L_p) = 0 \text{ for } p \neq 0,$$

since $Q_0 Q_p = Q_0 L_p = 0$ for $p \neq 0$,

$$sp^{\varkappa}(D_0^{\varkappa}(\mathfrak{s})Q_0 g(\mathfrak{s})Q_0) = sp^{\varkappa}(D_0^{\varkappa}(\mathfrak{s})g(\mathfrak{s})Q_0) = D_0^{\varkappa}\left(\frac{d}{dt}\right)g\left(\frac{d}{dt}\right)sp^{\varkappa}\left(e^{t\mathfrak{s}}Q_0\right)|_{t=0} = g\left(\frac{d}{dt}\right)D_0^{\varkappa}\left(\frac{d}{dt}\right)\mathcal{F}^{sp^{\varkappa}}(t)|_{t=0} = 0,$$

$$sp^{\varkappa}(D_0^{\varkappa}(\mathfrak{s})Q_0 g(\mathfrak{s})L_0) = sp^{\varkappa}(D_0^{\varkappa}(\mathfrak{s})g(\mathfrak{s})L_0) = D_0^{\varkappa}\left(\frac{d}{dt}\right)g\left(\frac{d}{dt}\right)sp^{\varkappa}\left(e^{t\mathfrak{s}}L_0\right)|_{t=0} = g\left(\frac{d}{dt}\right)D_0^{\varkappa}\left(\frac{d}{dt}\right)sp^{\varkappa}(L_0) = 0$$

due to Eq. (6.2) and since the operator $D_0^{\varkappa}\left(\frac{d}{dt}\right)$ contains the factor $\frac{d}{dt}$.

Further, it is easy to see that for each of the ideals $\mathcal{I}_0^{\varkappa}$, where $\varkappa = \pm 1$, the polynomials $\phi_p^0 \in \mathbb{C}[\mathfrak{s}]$ defined in Corollary 5.1 satisfy the relations $\phi_p^0(\mathfrak{s}) = D_p^{\varkappa}(\mathfrak{s})$ for $p = 0, \ldots, n-1$.

So, Theorem 5.1 implies that the $\mathbb{C}[\mathfrak{s}]$-span of the $D_p^{\varkappa}(\mathfrak{s})Q_p$ and $D_p^{\varkappa}(\mathfrak{s})L_{-p}$ for $p = 0, \ldots, n-1$ is $\mathcal{I}_0^{\varkappa}$.

Since $D_p^{+1} = D_p^{-1}$, we have $\mathcal{I}_0^{+1} = \mathcal{I}_0^{-1}$, and as result, $\mathcal{I}^{+1} = \mathcal{I}^{-1}$.

CONFLICTS OF INTEREST

The authors declare they have no conflicts of interest.

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REFERENCES

[1] S. Bouarroudj, A. Krutov, D. Leites, I. Shchepochkina, Non-degenerate invariant (super)symmetric bilinear forms on simple Lie (super)algebras, Algebras Repr. Theory 21 (2018), 897–941.
[2] K.A. Brown, I. Gordon, Poisson orders, symplectic reflection algebras and representation theory, J. Reine Angew. Math. 559 (2003), 193–216.
[3] P. Etingof, V. Ginzburg, Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism, Inv. Math. 147 (2002), 243–348.
[4] S.E. Konstein, I.V. Tyutin, Traces on the superalgebra of observables of rational Calogero model based on the root system, J. Nonlinear Math. Phys. 20 (2013), 271–294.
[5] S.E. Konstein, I.V. Tyutin, Ideals generated by traces or by supertraces in the symplectic reflection algebra $H_{1,\nu}(I_2(2m+1))$, J. Nonlinear Math. Phys. 24 (2017), 405–425.
[6] S.E. Konstein, I.V. Tyutin, Connection between the ideals generated by traces and by supertraces in the superalgebras of observables of Calogero models, J. Nonlinear Math. Phys. 27 (2020), 7–11.
[7] S.E. Konstein, R.B. Stekolshchik, Klein operator and the number of traces and supertraces on the superalgebra of observables of rational Calogero model based on the root system, J. Nonlinear Math. Phys. 20 (2013), 295–308.
[8] A. Krutov, A. Lebedev, D. Leites, I. Shchepochkina, Non-degenerate invariant symmetric bilinear forms on simple Lie superalgebras in characteristic 2, Oberwolfach preprint OWP 2020-02, Available from: http://publications.mfo.de/handle/mfo/3697.
[9] I. Losev, Completions of symplectic reflection algebras, Selecta Math. 18 (2012), 179–251.
[10] D.S. Passman, Infinite crossed products, Pure and Applied Mathematics, vol. 135, Academic Press, San Diego, 1989.