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Research Article

On the Coupled Dispersionless-type Equations and the Short Pulse-type Equations

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ABSTRACT

In this paper, we study the correspondence between the Coupled Dispersionless (CD)-type equations and the Short Pulse (SP)-type equations. From the real and complex modified CD equations, we construct the real and complex Modified Short Pulse (mSP) equations geometrically and algebraically. From the geometric point of view, we establish the link of the motions of space curves to the real and complex modified CD equations, then to the real and complex mSP equations via hodograph transformations. The integrability of these equations are confirmed by constructing their Lax pairs geometrically. By using hodograph transformation, we construct the two-component SP equation from the CD-type equations, the multi-component real and complex SP and mSP equations from the multi-component CD equations. The multi-soliton solutions in the determinant form for the mSP and two-component SP equations are provided.

1. INTRODUCTION

The Short Pulse (SP) equation was proposed by Schäfer and Wayne [16]

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}, \tag{1.1}$$

to describe the propagation of ultra-short optical pulses in nonlinear media [2]. Both the SP equation and the Nonlinear Schrödinger (NLS) equation are important in studying the dynamics of optical solitons in nonlinear optics. The efficiency and shortcoming of the SP compared with the NLS equation are explained in Feng [4]. The SP equation has received considerable attention in studies of the soliton theory and its integrable properties were investigated from various mathematical points of view, such as geometric meaning, soliton solutions and dynamics. The SP equation is integrable with Wadati-Konno-Ichikawa spectral problem [22], and it is also related to the Ablowitz–Kaup–Newell–Suger (AKNS) spectral problem under suitable hodograph transformation. It was shown that the SP equation can be transformed into the sine-Gordon (sG) equation through the appropriate hodograph transformation [15]. Recently, the links among the SP, AB system [5,13,19] and the first negative order AKNS(−1) system were clarified in Chen et al. [1] and nonlocal SP equation was also proposed therein.

Multi-component integrable SP equation was given when considering the effects of polarization or anisotropy. Matsuno proposed the two-component SP system [11]

$$u_{xt} = u + \frac{1}{2}(uvu_x)_x, \quad v_{xt} = v + \frac{1}{2}(uvv_x)_x, \tag{1.2}$$

and more general n-component system

$$u_{i,xt} = u_i + \frac{1}{2}(Fu_{i,x})_x, \quad i = 1, 2, \cdots, n, \tag{1.3}$$

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with

$$F = \frac{1}{2}\sum_{1\le j,k\le n} c_{jk}u_j u_k. \tag{1.4}$$

Here, c_{jk} are arbitrary constants with the symmetric relation $c_{jk} = c_{kj}$. It is easy to check that when $n = 2$ and $u_1 = u$, $u_2 = v$ with $c_{11} = c_{22} = 0$, $c_{12} = 1$, the multi-component system (1.3) yields the two-component SP system (1.2).

Starting from the close relation between the SP and sG equation and using the Bäcklund transformation, Feng proposed an alternative integrable coupled SP system [3], which reads

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx} + \frac{1}{2}v^2 u_{xx}, \quad v_{xt} = v + \frac{1}{6}(v^3)_{xx} + \frac{1}{2}u^2 v_{xx}. \tag{1.5}$$

When $u = 0$ or $v = 0$, the above coupled system is reduced to the SP equation (1.1). Note that the Matsuno's two-component system (1.2) degenerates to the SP equation (1.1) when we identify v with u.

Recently, Sakovich considered the integrability of the following nonlinear PDE [14]

$$u_{xt} = u + au^2 u_{xx} + buu_x^2, \tag{1.6}$$

where a and b are arbitrary constants. When $a/b = 1/2$, Eq. (1.6) corresponds to the SP equation (1.1), whereas the case $a/b = 1$ yields, after rescaling the variable u, a so-called modified SP (mSP) equation

$$u_{xt} = u + uu_x^2 + u^2 u_{xx}. \tag{1.7}$$

The mSP equation (1.7) can be derived directly from Feng's coupled SP equation (1.5) by putting $v = u$ and hence its integrability is assured.

Matsuno proposed an integrable multi-component generalization of the mSP equation [12],

$$u_{i,xt} = u_i + (Fu_{i,x})_x - \frac{1}{2}\Big(\sum_{1\le j,k\le n} c_{jk}u_{j,x}u_{k,x}\Big)u_i = 0, \quad i = 1, 2, \ldots, n, \tag{1.8}$$

where F is given by (1.4). For the special case $n = 2$ with $u_1 = u$, $u_2 = v$ and $c_{12} = 1$, $c_{11} = c_{22} = 0$, the n-component mSP (1.8) reduces to

$$u_{xt} = u + v(uu_x)_x, \quad v_{xt} = v + u(vv_x)_x. \tag{1.9}$$

The further degeneration $u = v$ yields the mSP equation (1.7). Especially, by imposing complex conjugate condition $v^* = u(\equiv q)$, the two-component mSP system (1.9) reduces to the so-called modified complex short pulse equation

$$q_{xt} = q + q^*(qq_x)_x. \tag{1.10}$$

Similar to the complex short pulse equation proposed by Feng [4], that admits both the focusing and defocusing type, the modified complex short pulse equation (1.10) has also the defocusing type [18]

$$q_{xt} = q - q^*(qq_x)_x. \tag{1.11}$$

It was shown in Shen et al. [17] that the SP equation is closely related to the Coupled Dispersionless (CD) equations,

$$\rho_s + 2uu_y = 0, \tag{1.12}$$

$$u_{ys} = 2\rho u, \tag{1.13}$$

proposed by Konno and Oono [9]. The CD equations are integrable with Lax pair

$$\psi_y = U\psi, \qquad \psi_s = V\psi, \tag{1.14}$$

$$U = -i\lambda\begin{pmatrix}\rho & u_y\\ u_y & -\rho\end{pmatrix}, \quad V = \begin{pmatrix}\frac{i}{2\lambda} & -u\\ u & -\frac{i}{2\lambda}\end{pmatrix}. \tag{1.15}$$

One can check that the compatibility condition $U_s - V_y + [U, V] = 0$ gives the CD equations. Soon after the proposition of the CD equations, it was pointed out that the CD equations were closely related to the sG equation [6]. The complex version of the CD equations were presented and solved by the inverse scattering method [8].

Shortly, Kakuhata and Konno [7] proposed a more general CD equations (K–K general CD for short),

$$\rho_s + (uv)_y = 0, \tag{1.16}$$

$$u_{ys} = (2\rho - 1)u, \tag{1.17}$$

$$v_{ys} = (2\rho - 1)v, \tag{1.18}$$

with Lax pair

$$\Psi_y = U\Psi, \qquad \Psi_s = V\Psi, \tag{1.19}$$

$$U = -i\lambda \begin{pmatrix} \rho - \frac{1}{2} & u_y \\ v_y & -\rho + \frac{1}{2} \end{pmatrix}, \quad V = \begin{pmatrix} \frac{i}{2\lambda} & -u \\ v & -\frac{i}{2\lambda} \end{pmatrix}. \tag{1.20}$$

By using the idea to derive Feng's coupled SP system, an integrable Generalized CD (GCD) equations were proposed [10],

$$u_{ys} - 4\rho uv + (\rho v)^{-1} r r_y u_y = 0, \tag{1.21}$$

$$r_{ys} - 4\rho rv + (\rho v)^{-1} u u_y r_y = 0, \tag{1.22}$$

$$\rho_s + v^{-1} u u_y = 0, \tag{1.23}$$

$$v_s + \rho^{-1} r r_y = 0. \tag{1.24}$$

When $v = 1/2$, $r = 0$, the GCD equations degenerate into the CD equations (1.12) and (1.13).

Upon the reduction $r = u$ and $v = \rho$, we obtain from the above GCD equations

$$\rho\rho_s + u u_y = 0, \tag{1.25}$$

$$u_{ys} - 4u\rho^2 + \rho^{-2} u u_y^2 = 0, \tag{1.26}$$

that could be called the modified CD (mCD) equations.

In this paper, we prove that there exists general correspondence between the SP- and CD-type equations. The rest of the paper is organized as follows. In Section 2, we show that the coupled SP equations proposed by Feng can be derived from the GCD equations under suitable hodograph transformation. In Section 3, we study the connection between the mCD and mSP equations. The link of the motions of space curves to the mCD and mSP equations is derived in Section 4. The relation between the mSP and sG equation is given in Section 5. Section 6 is devoted to the connection between the K–K general CD and two-component SP. We construct the multi-component real and complex SP/mSP from the multi-component CD in Section 7. Determinant expressions for multi-soliton solutions of the mSP and two-component mSP are given in Section 8. Finally, the concluding remarks and discussion are presented.

2. FROM THE GCD TO THE COUPLED SP EQUATIONS BY FENG

From Eqs. (1.23) and (1.24), we have the conservation law

$$(\rho v)_s + \left(\frac{r^2 + u^2}{2}\right)_y = 0, \tag{2.1}$$

that hints the hodograph transformation

$$dx = 2\rho v dy - (r^2 + u^2)ds, \qquad dt = 2ds, \tag{2.2}$$

or equivalently,

$$\partial_s = -(r^2 + u^2)\partial_x + 2\partial_t, \qquad \partial_y = 2\rho v \partial_x. \tag{2.3}$$

By using the above transformation and Eqs. (1.23) and (1.24), we have

$$(r^2+u^2)(qv)_x = 2(\rho v)_t + \rho v(r^2+u^2)_x. \tag{2.4}$$

With Eq. (2.4) and the hodograph transformation (2.3), Eq. (1.22) is transformed into

$$u_{xt} = u + \frac{1}{2}(r^2+u^2)u_{xx} + uu_x^2. \tag{2.5}$$

Similarly, from (1.21), we can derive

$$r_{xt} = r + \frac{1}{2}(r^2+u^2)r_{xx} + rr_x^2. \tag{2.6}$$

Thus we get Feng's coupled SP equation (1.5) from the GCD equation (1.21)–(1.24).

3. FROM THE MODIFIED CD EQUATIONS TO THE MODIFIED SP EQUATION

The first equation of the modified CD system represents a conservation law

$$(\rho^2)_s + (u^2)_y = 0. \tag{3.1}$$

We introduce the hodograph transformation $(y, s) \to (x, t)$ by

$$dx = 2\rho^2 dy - 2u^2 ds, \qquad dt = 2ds. \tag{3.2}$$

It is obvious that

$$\frac{\partial x}{\partial y} = 2\rho^2, \qquad \frac{\partial x}{\partial s} = -2u^2, \tag{3.3}$$

or equivalently,

$$\partial_s = -2u^2\partial_x + 2\partial_t, \qquad \partial_y = 2\rho^2\partial_x. \tag{3.4}$$

Substitution of (3.4) into Eq. (1.26) yields

$$2\rho^2\partial_x(-2u^2\partial_x + 2\partial_t)u - 4u\rho^2 + \rho^{-2}u(2\rho^2\partial_x u)^2 = 0, \tag{3.5}$$

or in the simplified form

$$u_{xt} - u - uu_x^2 - u^2u_{xx} = 0, \tag{3.6}$$

which is exactly the modified SP equation (1.7).

4. THE LINK OF THE MOTIONS OF SPACE CURVES TO THE mCD AND mSP

Let $\gamma(y, s) : [0, l] \times [0, S] \to R^3$ be a family of smooth space curve parameterized by the arc length $y \in [0, l]$ at each time s. The unit tangent vector $\mathbf{t}(y, s)$, principal normal vector $\mathbf{n}(y, s)$ and binormal vector $\mathbf{b}(y, s)$ are defined by

$$\mathbf{t} = \gamma', \qquad \mathbf{n} = \frac{\gamma''}{|\gamma''|}, \qquad \mathbf{b} = \mathbf{t}\times\mathbf{n},$$

respectively. Here $'$ denotes the differentiation with respect to y. The equation for the orthogonal triad $\mathbf{n}, \mathbf{t}, \mathbf{b}$ along the curve takes the form

$$\begin{bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix}_y = \begin{bmatrix}0 & k_g & k_n\\ -k_g & 0 & \tau\\ k_n & \tau & 0\end{bmatrix}\begin{bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix}. \tag{4.1}$$

While the temporal evolution can be expressed as

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}_s = \begin{bmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ \beta & \gamma & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}. \tag{4.2}$$

The compatibility condition of (4.1) and (4.2) implies a more general class of integrable system

$$k_{g,s} = \alpha_y - k_n \gamma + \tau \beta, \tag{4.3}$$

$$k_{n,s} = \beta_y - k_g \gamma + \tau \alpha, \tag{4.4}$$

$$\tau_s = \gamma_y + k_g \beta - k_n \alpha. \tag{4.5}$$

Specially, if we choose

$$\alpha = 4u, \quad \beta = 0, \quad \gamma = \lambda^{-1}, \tag{4.6}$$

$$k_g = 0, \quad k_n = 4\lambda u_y, \quad \tau = 2\lambda\left(2\rho^2 - \frac{1}{2}u_y^2 \rho^{-2}\right), \tag{4.7}$$

Equation (4.3) holds automatically and Eqs. (4.4) and (4.5) become

$$u_{ys} - 4u\rho^2 + \rho^{-2} u u_y^2 = 0, \tag{4.8}$$

$$\rho \rho_s + u u_y = 0. \tag{4.9}$$

Thus the link between the motion of space curves and the modified CD system is established. It is well known that the real Lorentz Lie algebra so(2, 1) in the plane is isomorphic to sl(2, R) and the correspondence reflects as $[L_i, L_j] \leftrightarrow [e_i, e_j]$, where

$$L_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{4.10}$$

is the basis of so(2, 1) and

$$e_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.11}$$

is the basis of sl(2, R). Based on this fact, we can construct the Lax pair for the modified CD equations geometrically as follows

$$\Psi_y = U\Psi, \quad \Psi_s = V\Psi, \tag{4.12}$$

where

$$\begin{aligned} U &= k_n e_1 - k_g e_2 + \tau e_3 \\ &= \lambda \begin{pmatrix} 2\rho^2 - \dfrac{u_y^2}{2\rho^2} & 2u_y \\ 2u_y & \dfrac{u_y^2}{2\rho^2} - 2\rho^2 \end{pmatrix}, \end{aligned} \tag{4.13}$$

$$\begin{aligned} V &= \beta e_1 - \alpha e_2 + \gamma e_3 \\ &= \begin{pmatrix} \dfrac{1}{2\lambda} & -2u \\ 2u & -\dfrac{1}{2\lambda} \end{pmatrix}. \end{aligned} \tag{4.14}$$

By using the hodograph transformation between mCD and mSP equation, we can construct the Lax pair for the mSP equation (1.7) as

$$\Psi_x = P\Psi, \qquad \Psi_t = Q\Psi, \tag{4.15}$$

with

$$P=\frac{1}{2\rho^2}U=\lambda\begin{pmatrix}1-u_x^2 & 2u_x\\ 2u_x & u_x^2-1\end{pmatrix}, \tag{4.16}$$

$$Q=\frac{1}{2}\left(V+\frac{u^2}{\rho^2}U\right)=\begin{pmatrix}\frac{1}{4\lambda}+\lambda u^2(1-u_x^2) & 2\lambda u^2u_x-u\\ u+2\lambda u^2u_x & -\frac{1}{4\lambda}-\lambda u^2(1-u_x^2)\end{pmatrix}. \tag{4.17}$$

One can check that the compatibility condition $P_t - Q_x + [P, Q] = 0$ gives the mSP equation (1.7).

5. THE RELATION BETWEEN THE MODIFIED CD AND sG EQUATION

It is known that we can solve the SP equation by use of solutions of the sG equation. Here we show that it is also true for the mSP case. We first introduce the variable transformation

$$v=\frac{1}{2}\cos\left(\frac{\phi}{2}\right),\qquad u=\frac{1}{4}\phi_s, \tag{5.1}$$

and suppose the function $\phi(y, s)$ satisfies sG equation

$$\phi_{ys}=\sin\phi. \tag{5.2}$$

By making the use of the trigonometric identity, we have

$$v^2=\frac{1}{4}\cos^2\left(\frac{\phi}{2}\right)=\frac{1+\cos\phi}{8}.$$

We can verify the mCD equation by direct calculation,

$$(v^2)_s+(u^2)_y=-\frac{1}{8}\phi_s\sin\phi+\frac{1}{8}\phi_s\phi_{ys}=0, \tag{5.3}$$

$$u_{ys}-4uv^2+uu_y^2v^{-2}=\frac{1}{4}\left[\phi_s\cos\phi-\phi_s\cos^2\frac{\phi}{2}+\frac{1}{4}\phi_s\sin^2\phi\cos^{-2}\frac{\phi}{2}\right]=0. \tag{5.4}$$

Thus, solutions of the sG equation give solutions of the mCD equations through the transformation (5.1). It was shown in Hirota and Tsujimoto [6] that CD equations

$$q_s+2rr_y=0,\qquad r_{ys}-2qr=0, \tag{5.5}$$

were solved by the sG equation $\phi_{ys} = \sin\phi$ with

$$q=\frac{1}{2}\cos\phi,\qquad r=\frac{1}{2}\phi_s. \tag{5.6}$$

Transformations (5.1) and (5.6) hint the substitution

$$r=2u,\quad q=4v^2-\frac{1}{2}, \tag{5.7}$$

that leads to the system

$$(v^2)_s+(u^2)_y=0, \tag{5.8}$$

$$u_{ys}-8uv^2+u=0. \tag{5.9}$$

With the substitution $v^2 = \rho$, we find that above equations are reduction of AKNS(−1) equations [25].

6. K–K GENERAL CD EQUATIONS AND TWO-COMPONENT MODIFIED SP EQUATION

6.1. From K–K General CD to Two-component mSP

The K–K general CD equations read

$$\rho_s + (uv)_y = 0, \tag{6.1}$$

$$u_{ys} = (2\rho - 1)u, \tag{6.2}$$

$$v_{ys} = (2\rho - 1)v. \tag{6.3}$$

Equation (6.1) represents a conservation law and can be used to define a reciprocal transformation $(y, s) \to (x, t)$ as

$$dx = \rho dy - uvds, \qquad dt = ds, \tag{6.4}$$

or equivalently,

$$\partial_y = \rho\partial_x, \qquad \partial_s = -uv\partial_x + \partial_t. \tag{6.5}$$

Upon using (6.2) and (6.3), we have

$$\rho_s = -(u_y v + u v_y) = -\frac{1}{2\rho - 1}(u_y v_{ys} + v_y u_{ys}), \tag{6.6}$$

which implies a conserved quantity

$$\rho - \rho^2 - u_y v_y \tag{6.7}$$

for the K–K general CD equations. Under the boundary conditions $\rho \to 1$, $u \to 0$, $v \to 0$ as $y \to \infty$, the above conserved quantity is identically zero. The conversion relation (6.6) gives

$$\rho^2 u_x v_x = u_y v_y = \rho - \rho^2. \tag{6.8}$$

Note that from the conversion relation (6.5), Eq. (6.2) can be rewritten into

$$\rho\partial_x(u_t - uvu_x) = (2\rho - 1)u. \tag{6.9}$$

Eliminating ρ by use of (6.8) follows

$$u_{xt} = u + v(u_x^2 + uu_{xx}). \tag{6.10}$$

Similarly, from Eq. (6.3) we can derive

$$v_{xt} = v + u(v_x^2 + vv_{xx}). \tag{6.11}$$

Thus, the two-component mSP equations are derived from the K–K general CD equations. In the degenerated case $u = v$, we obtain the mSP equation from (6.1)–(6.3) by the hodograph transformation (6.5).

6.2. From Complex Generalized CD Equations to Complex mSP Equation in Focusing Type

By setting the relation $v = u^*$ in the K–K general CD equations (6.1)–(6.3), we get the complex generalized CD equations in focusing type

$$\rho_s + (|u|^2)_y = 0, \tag{6.12}$$

$$u_{ys} = (2\rho - 1)u. \tag{6.13}$$

From the first equation (6.12) of the complex generalized CD equations, which stands for a conservation law, we can define a hodograph transformation

$$dx = \rho dy - |u|^2 ds, \qquad dt = ds, \tag{6.14}$$

and it follows

$$\partial_y = \rho\partial_x, \qquad \partial_s = -|u|^2\,\partial_x + \partial_t. \tag{6.15}$$

Through above hodograph transformation, the complex focusing mSP equation (1.10) can be derived from the complex generalized CD equations (6.12) and (6.13).

6.3. From the Complex Generalized CD Equations to Complex mSP Equation in Defocusing Type

By setting $v = -u^*$ in K–K general CD equations (6.1)–(6.3), we have complex generalized CD equations in defocusing type

$$\rho_s - (|u|^2)_y = 0, \tag{6.16}$$

$$u_{ys} = (2\rho - 1)u. \tag{6.17}$$

Similar as the focusing case, we define the hodograph transformation

$$dx = \rho dy + |u|^2\, ds, \qquad dt = ds, \tag{6.18}$$

and it follows

$$\partial_y = \rho\partial_x, \qquad \partial_s = |u|^2\,\partial_x + \partial_t. \tag{6.19}$$

Through above hodograph transformation, the complex defocusing mSP equation (1.11) is obtained from the complex generalized CD equations (6.16) and (6.17).

7. FROM THE MULTI-COMPONENT REAL AND COMPLEX CD TO THE MULTI-COMPONENT REAL AND COMPLEX SP/mSP

Matsuno proposed the multi-component SP equation [11]

$$u_{i,xt} + u_i + \frac{1}{2}\left[\left(\sum_{1\le j<k\le n} c_{jk}u_ju_k\right)u_{i,x}\right]_x = 0, \quad i = 1, 2, \cdots, n, \tag{7.1}$$

where c_{jk} are arbitrary constants with the symmetry $c_{jk} = c_{kj}$. When the matrix $(c_{jk})_{j,k=1}^n$ is positive definite, we can recast (7.1) into

$$u_{i,xt} + u_i + \frac{1}{2}\left[\left(\sum_{k=1}^{n} u_k^2\right)u_{i,x}\right]_x = 0, \quad i = 1, 2, \cdots, n. \tag{7.2}$$

Feng proposed the multi-component complex SP equation [4]

$$u_{j,xt} + u_j + \frac{\sigma}{2}\left(|\mathbf{u}|^2\, u_{j,x}\right)_x = 0, \quad j = 1, 2, \cdots, n, \tag{7.3}$$

where the parameter σ is a constant and $|\mathbf{u}|^2 = \sum_{j=1}^{n}|u_j|^2$. From the complex CD equations

$$\rho_y + \frac{\sigma}{2}\left(|\mathbf{u}|^2\right)_s = 0, \tag{7.4}$$

$$u_{j,ys} = \rho u_j, \qquad j = 1, 2, \cdots, n, \tag{7.5}$$

we define a hodograph transformation

$$dx = \rho ds - \frac{\sigma}{2}|\mathbf{u}|^2\, dy, \qquad dt = -dy,$$

or equivalently,

$$\partial_s = \rho\partial_x, \qquad \partial_y = -\partial_t - \frac{\sigma}{2}|\mathbf{u}|^2\,\partial_x.$$

Under the hodograph transformation, Eq. (7.5) was transformed into

$$\rho\partial_x\left(-u_{j,t} - \frac{\sigma}{2}|\mathbf{u}|^2\,u_{j,x}\right) = \rho u_j, \tag{7.6}$$

which gives the multi-component complex SP equation (7.3). When u is real, we can modify our deduction and derive the real multi-component SP equation (7.2).

Based on the second form of the multi-component complex CD equations

$$\rho_y = -\left(|\mathbf{u}|^2\right)_s, \tag{7.7}$$

$$u_{j,ys} = (2\rho - 1)u_j, \qquad j = 1, 2, \cdots, n, \tag{7.8}$$

we define the hodograph transformation

$$\partial_s = -|\mathbf{u}|^2\,\partial_x + \partial_t, \qquad \partial_y = \rho\partial_x.$$

Thus we have

$$-\rho_s = \left(\sum_{k=1}^{n}|u_k|^2\right)_y = \frac{1}{2\rho - 1}\left(\sum_{k=1}^{n}u_{k,y}u^*_{k,ys} + u^*_{k,y}u_{ys}\right), \tag{7.9}$$

that is

$$\partial_s\left(\rho^2 - \rho + \sum_{k=1}^{n}u_{k,y}u^*_{k,y}\right) = 0.$$

Under the zero boundary condition and the hodograph transformation, we obtain

$$\rho - \rho^2 = \sum_{k=1}^{n}u_{k,y}u^*_{k,y} = \rho^2\sum_{k=1}^{n}u_{k,x}u^*_{k,x}. \tag{7.10}$$

By use of Eq. (7.8), we get

$$\rho\partial_x\left[\left(\partial_t - |\mathbf{u}|^2\,\partial_x\right)u_j\right] = (2\rho - 1)u_j. \tag{7.11}$$

Eliminating ρ from (7.10) and (7.11) results in the multi-component complex modified SP equation

$$u_{j,xt} = u_j + \left(|\mathbf{u}|^2\,u_{j,x}\right)_x - \left(\sum_{k=1}^{n}u_{k,x}u^*_{k,x}\right)u_j. \tag{7.12}$$

When potentials u_j are real, the system (7.12) becomes the multi-component mSP proposed by Matsuno [12].

Remark: Matrix AKNS(−1) was given in Chen et al. [1], and the links between the vector AKNS(−1) and SP equation were presented therein.

8. DETERMINANT EXPRESSION FOR MULTI-SOLITON SOLUTIONS OF THE mSP AND THE TWO-COMPONENT mSP SYSTEM

By setting $u = v$ in the K–K general CD equations (6.1)–(6.3), we obtain

$$\rho_s + (u^2)_y = 0, \tag{8.1}$$

$$u_{ys} = (2\rho - 1)u. \tag{8.2}$$

The bilinearization of the mCD equations is established by the following proposition.

Proposition 8.1. *By means of the dependent variable transformations*

$$u=\frac{g}{f},\qquad \rho=1-(\ln f)_{ys}, \tag{8.3}$$

Equations (8.1) and (8.2) are transformed into the bilinear equations

$$D_s^2 f\cdot f=2g^2, \tag{8.4}$$

$$D_yD_s g\cdot f=fg, \tag{8.5}$$

where D is the Hirota D-operator defined by

$$D_s^m D_y^n f\cdot g=\frac{\partial^m}{\partial\varepsilon^m}\frac{\partial^n}{\partial\delta^n}f(s+\varepsilon,y+\delta)g(s-\varepsilon,y-\delta)\big|_{\varepsilon=0,\delta=0}. \tag{8.6}$$

The proof can be found in Zhang et al. [25].

Proposition 8.2. *By means of the dependent variable transformation*

$$u=\frac{g}{f}, \tag{8.7}$$

and the hodograph transformation

$$x=y-(\ln f)_s,\quad t=s, \tag{8.8}$$

the bilinear equations (8.4) and (8.5) derive the mSP equation (1.7).

Proof. From the hodograph transformation and bilinear equations, we have

$$\frac{\partial x}{\partial y}=1-(\ln f)_{ys}=\rho,\quad \frac{\partial x}{\partial s}=-(\ln f)_{ss}=-u^2, \tag{8.9}$$

which implies

$$\partial_y=\rho\partial_x,\qquad \partial_s=-u^2\partial_x+\partial_t. \tag{8.10}$$

Thus, the mSP equation is derived from the K–K general CD system based on the discussion in Section 6.1.

Remark: We emphasize that the SP and mSP are connected with the same bilinear form but different hodograph links.

Based on the reduction of the two-component KP hierarchy [17], we have the Gram determinant solution for the bilinear equations (8.4) and (8.5).

Theorem 8.1. *The bilinear equations (8.4) and (8.5) admit the following determinant solution:*

$$f=\begin{vmatrix}A & I\\ -I & B\end{vmatrix},\quad g=\begin{vmatrix}A & I & \mathbf{\Phi}^T\\ -I & B & \mathbf{0}^T\\ \mathbf{0} & C & 0\end{vmatrix}, \tag{8.11}$$

*where I is an N × N identity matrix, **0** is an N-component zero row vector, A and B are N × N matrices, **Φ**, C are N-component row vectors whose elements are defined as*

$$a_{ij}=\frac{1}{p_i+p_j}e^{\xi_i+\xi_j},\quad b_{ij}=\frac{\alpha_i\alpha_j}{p_i+p_j}, \tag{8.12}$$

$$\mathbf{\Phi}=(e^{\xi_1},e^{\xi_2},\cdots,e^{\xi_N}),\quad C=(\alpha_1,\alpha_2,\cdots,\alpha_N), \tag{8.13}$$

with $\xi_i=p_i^{-1}y+p_is+\xi_{i0}$, p_i, α_i *and* ξ_{i0} *are constants.*

The proof can be found in Shen et al. [17]. So we can express solutions of the mSP equation in the determinant form.

Proposition 8.3. *By means of the dependent variable transformations*

$$u=\frac{g}{f},\qquad v=\frac{h}{f},\qquad \rho=1-(\ln f)_{ys}, \tag{8.14}$$

the K–K general CD equations (6.1)–(6.3) are transformed into the following bilinear equations [21]

$$D_s^2 f\cdot f=2gh, \tag{8.15}$$
$$D_yD_s f\cdot g=fg, \tag{8.16}$$
$$D_yD_s f\cdot h=fh. \tag{8.17}$$

Proposition 8.4. *With the dependent variable transformation*

$$u=\frac{g}{f},\quad v=\frac{h}{f}, \tag{8.18}$$

and the hodograph (reciprocal) transformation

$$x=y-(\ln f)_s,\quad t=s, \tag{8.19}$$

the two-component mSP equation shares the same bilinear equations (8.15)–(8.17).

The proof is similar as that of Proposition 2.

Theorem 8.2. *The bilinear equations (8.15)–(8.17) admit the determinant solution*

$$f=\begin{vmatrix} A & I\\ -I & B\end{vmatrix},\quad g=\begin{vmatrix} A & I & \Phi^T\\ -I & B & \mathbf{0}^T\\ \mathbf{0} & -\bar{\Psi} & 0\end{vmatrix},\quad h=\begin{vmatrix} A & I & \mathbf{0}^T\\ -I & B & \Psi^T\\ -\bar{\Phi} & \mathbf{0} & 0\end{vmatrix}, \tag{8.20}$$

where I is an $N\times N$ identity matrix, $\mathbf{0}$ is an N-component zero row vector, A and B are $N\times N$ matrices with entries

$$a_{ij}=\frac{1}{p_i+\bar{p}_j}e^{\xi_i+\bar{\xi}_j},\quad b_{ij}=\frac{\alpha_i\bar{\alpha}_j}{\bar{p}_i+p_j}, \tag{8.21}$$

where

$$\xi_i=\frac{1}{p_i}y+p_is+\xi_{i0},\quad \bar{\xi}_j=\frac{1}{\bar{p}_j}y+\bar{p}_js+\bar{\xi}_{j0},$$

and Φ, Ψ, Φ, $\bar{\Psi}$ are N-component row vectors defined by

$$\Phi=(e^{\xi_1},e^{\xi_2},\cdots,e^{\xi_N}),\quad \Psi=(\alpha_1,\alpha_2,\cdots,\alpha_N),$$
$$\bar{\Phi}=(e^{\bar{\xi}_1},e^{\bar{\xi}_2},\cdots,e^{\bar{\xi}_N}),\quad \bar{\Psi}=(\bar{\alpha}_1,\bar{\alpha}_2,\cdots,\bar{\alpha}_N).$$

The proof can be completed by using the reduction technique from the extended KP hierarchy (see [17] for reference). So solutions of the two-component mSP system can be expressed in the compact determinant form. We note that Theorem 3.1 in Matsuno [11] gives different determinant form of the multi-soliton solutions to the bilinear equations (8.15)–(8.17).

9. CONCLUSION AND DISCUSSION

The correspondence between CD- and SP-type equations is established by using appropriate hodograph transformation. Especially, we present the link of the motions of space curves to the modified CD equations and mSP equation. Determinant solutions of the mSP and the two-component mSP equations are expressed by using Hirota method.

The so-called derivative CD equations

$$q_t + \sum_{1\le k<l\le M} c_{kl}(r_{k,t}r_l - r_k r_{l,t}) = 0, \tag{9.1}$$

$$r_{k,xt} - 2q_x r_k = 0, \quad k = 1,2,\cdots,M \tag{9.2}$$

were proposed [23]. Through the variable transformation

$$q = x - (\ln F)_t, \quad r_k = G_k / F, \tag{9.3}$$

Equations (9.1) and (9.2) are transformed into the system of bilinear equations

$$D_t^2 F\cdot F - 2\sum_{1\le k<l\le M} c_{kl} D_t G_k \cdot G_l = 0, \tag{9.4}$$

$$(D_x D_t - 2)F\cdot G_i = 0, \qquad i = 1,2,\cdots,M. \tag{9.5}$$

It would be interesting to find the correspondence to the SP-type equations. When considering Sp(m) invariant systems, the coupled system

$$u_{i,xt} = u_i - \sum_{1\le j<k\le n} c_{jk}(u_{j,x}u_k - u_j u_{k,x})u_i, \qquad i = 1,2,\cdots,n, \tag{9.6}$$

was proposed [20] with skew-symmetric coupling constants $c_{jk} = -c_{kj}$ By the variable transformation $u_i = g_i/f$, the coupled PDEs (9.6) are transformed to the bilinear equations

$$D_x D_t f\cdot g_i = fg_i, \qquad (i = 1,2,\cdots,n), \tag{9.7}$$

$$D_x D_t f\cdot f = \sum_{1\le j<k\le n} c_{jk} D_x g_j \cdot g_k. \tag{9.8}$$

Note that bilinear equations (9.7) and (9.8) are similar to (9.4) and (9.5). One question is whether there exists some link between the Sp(m) invariant system (9.6) and the derivative CD system (9.1) and (9.2). Besides, the integrable discretization procedure has been applied to CD [21], GCD [24] and mCD equations [26]. Since there exists the correspondence between CD-type equations and SP-type equations, it is possible to construct integrable discrete versions of the modified SP and multi-component mSP equations. We shall investigate these topics in future.

CONFLICTS OF INTEREST

The authors declare they have no conflicts of interest.

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