Inverse Scattering Transformation for the Fokas–Lenells Equation with Nonzero Boundary Conditions

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1. INTRODUCTION

The Nonlinear Schrödinger (NLS) equation

\[ iu_t + u_{xx} - 2\nu|u|^2 u = 0 \]  \hspace{1cm} (1.1)

with \( \nu = \pm 1 \) is a significant mathematical and physical model for the optical fibers, deep water waves and plasma physics [3]. The NLS equation is a completely integrable system which admits Lax pair and bi-Hamiltonian formulation [21, 22].

In the late 70s, standard bi-Hamiltonian formulation was used to obtain an integrable generalization of a given equation [7]. For instance, the Camassa–Holm equation is derived from the KdV equation via the bi-Hamiltonian structure [6], and the same mathematical trick can be applied to the NLS equation yields Fokas–Lenells (FL) equation [8]

\[ iu_t + \nu u_{xx} + \gamma u_{xx} + \sigma |u|^2 (u + i\nu u_x) = 0, \sigma = \pm 1. \]  \hspace{1cm} (1.2)

If let \( \alpha = \gamma/\nu > 0 \) and \( \beta = \frac{1}{\nu} \) and

\[ u \rightarrow \beta \sqrt{\alpha} e^{i\beta} u, \sigma \rightarrow -\sigma, \]

then equation (1.2) can be converted into

\[ u_{xx} + \alpha \beta^2 u - 2\alpha \beta \nu u_x - \alpha u_{xx} + \sigma i\alpha \beta^2 |u|^2 u_x = 0, \]  \hspace{1cm} (1.3)

where \( \alpha, \beta > 0 \). In this paper, we consider the focusing case with \( \sigma = -1. \)
In recent years, much work has been done on the FL equation (1.3). For example, Lax pair for FL equation was obtained via the bi-Hamiltonian structure by Fokas and Lenells [8]. They further considered the initial-boundary problem for the FL equation on the half-line by using the Fokas unified method [9]. The dressing method is applied to obtain an explicit formula for bright-soliton and dark soliton solutions for Eq. (1.3) [11,15]. The bilinear method was used to obtain bright and dark soliton solution are obtained [12,13]. The Darboux transformation is used to obtain rogue waves of the FL equation [10,19]. The Deift–Zhou nonlinear steepest decedent method was used to analyze long-time asymptotic behavior for FL equation with decaying initial value [18]. Riemann–Hilbert (RH) approach was adopted to construct explicit soliton solutions under zero boundary conditions [1]. As far as we know, the soliton solutions for the FL equation (1.3) with Nonzero Boundary Conditions (NZBCs) have not been reported. In this paper, we apply RH approach to study the inverse transformation of the FL equation (1.3) with the following NZBCs

\[ u(x, t) \sim q e^{-i \beta x + 2ia \beta t}, \quad x \to \pm \infty, \]  

where \( |q| = \sqrt{\frac{2}{\beta}} \) and assume that \( u(x, t) - q \in L^1(\mathbb{R} \pm); \) with respect to \( x \) for all \( t \geq 0 \). In next section, we will see that this kind of NZBCs (1.4) avoids the discussion on a multi-valued function as the case of the NLS equation [3].

The inverse scattering transform is an important method to study important nonlinear wave equations with Lax pair such as the NLS equation, the modified KdV equation, Sine-Gordan equation [5,14]. As an improved version of inverse scattering transform, the RH method has been widely adopted to solve nonlinear integrable systems [2,4,16,17,20,23].

The paper is organized as follows. In Section 2, by introducing appropriate transformations, we change the asymptotic boundary conditions (1.4) into constant boundary conditions. Furthermore, we analyze the analyticity, symmetry and asymptotic behavior of eigenfunctions and scattering matrix associated with the Lax pair. In Section 3, a generalized RH problem for the FL equation is constructed, and the distribution of discrete spectrum and residue conditions associated with RH problem are discussed. Based on these results, we reconstruct the potential function from the solution of the RH problem. In Section 4, we give the \( N \)-soliton solutions via solving RH problem under reflectionless case.

2. THE DIRECT SCATTERING WITH NZBCs

2.1. Jost Solutions

It is well-known that the FL equation (1.3) admits a Lax pair

\[ \begin{align*}
\psi_x + ik^2 \sigma_3 \psi &= kU \psi, \\
\psi_t + i\eta^2 \sigma_3 \psi &= \left[ \alpha kU_x + \frac{i\alpha \beta^2}{2} \sigma_3 \left( \frac{1}{k} U - U^\dagger \right) \right] \psi,
\end{align*} \]  

where

\[ U = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \eta = \sqrt{\alpha} \left( k - \frac{\beta}{2k} \right), \quad v = -u^\dagger. \]

In order to invert the nonzero boundary conditions into the constant boundary conditions, we introduce a transformation for eigenfunctions and potentials.

**Theorem 2.1.** By transformation

\[ \begin{align*}
u &= q e^{-i \beta x + 2ia \beta t}, \\
\eta &= e^{\frac{1}{2} i (\beta x + a \beta t)} \phi, \end{align*} \]  

the FL equation (1.3) becomes

\[ q_{xx} - i \beta q_x + 2ia \beta q_x - aq_{xx} + (2a \beta^2 - a \beta^2 |q|^2) q - i a \beta^2 |q|^2 q_x = 0 \]

with corresponding boundary conditions

\[ q \to q_0, \quad x \to \pm \infty. \]

And Eq. (2.3) is the compatibility condition of the Lax pair

\[ \phi_x = X \phi, \quad \phi_t = T \phi. \]
where
\[
X = -i k^2 \sigma_x + \frac{1}{2} i \beta \sigma_z - i \beta k \sigma_y Q + k Q_y,
\]
\[
T = -i \eta^2 \sigma_y - \frac{1}{2} i \alpha \beta^2 \sigma_x Q^2 - \frac{1}{2} i \alpha \beta^2 \sigma_y Q^3 - i \alpha \beta k \sigma_y Q + \alpha k Q_y + \frac{i \alpha \beta^2}{2k} \sigma_y Q,
\]
with \(Q = \begin{pmatrix} 0 & q \\ -q & 0 \end{pmatrix}\).

**Proof.** Suppose the transformation is
\[
u = q e^{i A x + B y}, \quad \varphi = e^{(C \sigma_y + D \sigma_y) q} \varphi.
\]
Substituting it into (1.3) yields
\[
q_i A_i q_i + i(B - 2 \alpha \beta - 2 \alpha \lambda) q_i + (2 \alpha \beta A + \alpha \beta^2 + \alpha \lambda^2 - AB) q - \alpha q_i + A \alpha \beta^2 |q|^2 q - i \alpha \beta^2 |q|^2 = 0.
\]
And the Lax pair becomes
\[
\varphi_x = e^{\left[\left(\frac{a}{2} + \left(\frac{b}{2} - i \eta \right)\right)\sigma_y \right]} \begin{pmatrix}
-ik^2 - iC & k(Aiq + q_i) \\
-k(Aiq' - q'_i) & ik^2 + iC
\end{pmatrix} \varphi,
\]
\[
\varphi_1 = e^{\left[\left(\frac{a}{2} + \left(\frac{b}{2} + i \eta \right)\right)\sigma_y \right]} \begin{pmatrix}
-in\eta^2 + \frac{1}{2} i \alpha \beta^2 |q|^2 - iD & \alpha k(Aiq + q_i) + \frac{i \alpha \beta^2}{2k} q_i \\
-\alpha k(-Aiq' + q'_i) + \frac{i \alpha \beta^2}{2k} q'_i & in\eta^2 - \frac{1}{2} i \alpha \beta^2 |q|^2 + iD
\end{pmatrix} \varphi.
\]

Take \(A = 2C\) and \(B = 2D\) in the above equation and consider the limit as \(x \to \pm \infty\), one gets
\[
X_x = \begin{pmatrix}
-ik^2 - \frac{A}{2} i & kAiq_x \\
kAiq'_x & ik^2 + \frac{A}{2} i
\end{pmatrix},
\]
\[
T_x = \begin{pmatrix}
-in\eta^2 + i \alpha \beta - iD & \alpha kAiq_x + \frac{i \alpha \beta^2}{2k} q_x \\
\alpha kAiq'_x + \frac{i \alpha \beta^2}{2k} q'_x & in\eta^2 - i \alpha \beta + iD
\end{pmatrix}.
\]
These two matrices are proportional if
\[
A = -\beta, \quad B = 2 \alpha \beta.
\]
Moreover, \(T_x\) and \(X_x\) are proportional
\[
T_x = \frac{\sqrt{\alpha \eta}}{k} X_x.
\]
which implies that their eigenvalues are proportional and they share the same eigenvector matrices.
To diagonalize the matrices \(X_x\) and \(T_x\) and further obtain the Jost solutions, we need to get the eigenvalues and eigenvector matrices of them.
Direct calculation shows that the eigenvalues of the matrices \(X_x\) are \(\pm \frac{ik}{\sqrt{\alpha}} \lambda\); and the eigenvalues of the matrices \(T_x\) are \(\pm i \eta \lambda\), where
\[
\lambda = \sqrt{\frac{\alpha \eta}{k + \beta}}.
\]
and the corresponding eigenvector matrices are
\[
Y_x(k) = 1 - \frac{\beta}{2k} Q_x, \quad \det Y_x(k) = 1 + \frac{\beta}{2k} \Delta \gamma(k),
\]
\[
with \quad Q_x = \begin{pmatrix} 0 & q_x \\ -q_x & 0 \end{pmatrix}.
\]

With the above results, the matrices can be diagonalized as:
\[
X_x = Y_x \left( -\frac{ik\lambda\sigma_1}{\sqrt{\alpha}} \right) Y_x^{-1}, \\
T_x = Y_x (-i\eta\lambda\sigma_3) Y_x^{-1}.
\] (2.8)

Thus, the asymptotic Lax pair
\[
\tilde{\varphi}_x = X_x \tilde{\varphi}, \quad \tilde{\varphi}_t = T_x \tilde{\varphi}
\] (2.9)
can be written as
\[
(Y_x^{-1} \tilde{\varphi})_x = -\frac{ik\lambda\sigma_1}{\sqrt{\alpha}} (Y_x^{-1} \tilde{\varphi}), \quad (Y_x^{-1} \tilde{\varphi})_t = -i\eta\lambda\sigma_3 (Y_x^{-1} \tilde{\varphi}),
\]
which has a solution
\[
\tilde{\varphi} = Y_x (k)e^{-i(\lambda x + \lambda\eta t)}.
\] (2.10)

Therefore, the asymptotic of the eigenfunctions \( \varphi_x \) are
\[
\varphi_x \sim Y_x (k)e^{-i(\lambda x + \lambda\eta t)}, \quad x \to \pm \infty,
\] (2.11)
where
\[
\theta(x, t, k) = \frac{k\lambda}{\sqrt{\alpha}} + \lambda\eta.
\]

Define the Jost solutions as
\[
J_x (x, t, k) = \varphi_x (x, t, k)e^{i\theta},
\] (2.12)
then the Lax pair (2.5) is changed to
\[
J_{x,x} - \frac{ik\lambda}{\sqrt{\alpha}} J \sigma_3 = XJ_x, \\
J_{x,t} - i\lambda\eta J \sigma_3 = TJ_x,
\] (2.13)
and
\[
J_x \sim Y_x (k), \quad x \to \pm \infty.
\] (2.14)

2.2. Scattering Matrix

The functions \( \varphi_x \) are both fundamental matrix solutions of the Lax pair (2.5), thus there exists a matrix \( S(k) \) that only depends on \( k \), such that
\[
\varphi_x (x, t, k) = \varphi_x (x, t, k) S(k),
\] (2.15)
where \( S(k) \) is called scattering matrix. Columnwise, Eq. (2.15) reads
\[
\varphi_{x,1} = s_{11} \varphi_{x,-1} + s_{12} \varphi_{x,-2}, \quad \varphi_{x,2} = s_{21} \varphi_{x,-1} + s_{22} \varphi_{x,-2}.
\] (2.16)

Since \( \text{tr}(X) = \text{tr}(T) = 0 \), according to the Abel's formula, we have
\[
(\det \varphi_x)_x = (\det \varphi_x)_t = 0,
\]
which implies
\[
\det \varphi_x = \lim_{x \to \pm \infty} \det \varphi_x = \gamma(k),
\]
then the scattering coefficients can be expressed as Wronskians of columns \( \varphi \) in the following way

\[
\begin{align*}
    s_{11}(k) &= \frac{\text{Wr}(\varphi_{1-}, \varphi_{2+})}{\gamma(k)}, \\
    s_{12}(k) &= \frac{\text{Wr}(\varphi_{1-}, \varphi_{2-})}{\gamma(k)}, \\
    s_{21}(k) &= \frac{\text{Wr}(\varphi_{2+}, \varphi_{2-})}{\gamma(k)}, \\
    s_{22}(k) &= \frac{\text{Wr}(\varphi_{2+}, \varphi_{2+})}{\gamma(k)}.
\end{align*}
\] (2.17)

2.3. Asymptotic Analysis

To properly construct the Riemann–Hilbert problem, we need to consider the asymptotic behavior of eigenfunctions and scattering matrix as \( k \to \infty \) and \( k \to 0 \).

2.3.1. Asymptotic as \( k \to \infty \)

Consider a solution of (2.13) of the form

\[
    J = J^{(0)} + \frac{J^{(1)}}{k} + \frac{J^{(2)}}{k^2} + \cdots,
\]

then substituting the above expansion into (2.13) and comparing the coefficients of the same order of \( k \), we get the comparison results:

x-part:

\[
    O(k^0): J^{(0)}(\sigma, J^{(2)}) = i\beta \sigma J^{(0)}_r - i\beta \sigma Q_1^{(1)} + Q_2 J^{(1)},
\]

(2.18)

(2.19)

\[
    O(k^2): J^{(0)}_x = \sigma J^{(0)}_x.
\]

(2.19)

\[
    O(k^3): J^{(0)}_{x_1} + i\sigma J^{(2)} = i\alpha \beta \sigma J^{(0)}_{r_1} - \frac{1}{2} i\alpha \beta^2 Q^2 \sigma J^{(0)}_x - \frac{1}{2} i\alpha \beta^2 q_1 \sigma J^{(0)}_x - i\alpha \beta \sigma Q_1^{(1)} + \sigma Q_2 J^{(1)}.
\]

(2.21)

Based on these results, we derive that \( J^{(0)}_{x_1} \) is diagonal and satisfies

\[
    J^{(0)}_{x_1} = i\nu_1 \sigma J^{(0)}_{x_1},
\]

(2.22)

\[
    J^{(0)}_{x_2} = -i\nu_1 \sigma J^{(0)}_{x_2},
\]

(2.23)

where

\[
    \nu_1(x, t) = \beta + \frac{1}{2} \beta^2 q_1 + i\beta q_1 r + \frac{1}{2} q_1 r,
\]

\[
    \nu_2(x, t) = \alpha \beta - \frac{1}{2} \alpha \beta^2 q_1^2 + i\alpha \beta (q_1 r - q_1 r) + \frac{1}{2} \alpha q_1 r.
\]

Note that the FL equation (2.3) admits the conservation law

\[
    (q_1 r - i\beta q_1 r + i\beta q_1 r + \beta^2 q_1 r)_t = (\alpha q_1 r + i\alpha \beta (q_1 r - q_1 r))_t,
\]

thus Eqs. (2.22) and (2.23) are consistent and are both satisfied if we define

\[
    J^{(0)}_{x} = e^{i\nu_1} \psi = \int_{-\infty}^{\infty} \left( \beta + \frac{1}{2} \beta^2 q_1 r + i\beta q_1 r + \frac{1}{2} q_1 r \right) dx.
\]

(2.24)

Therefore, we obtain the limit of the Jost solutions as \( k \to \infty \):

\[
    J_s \sim J^{(0)}_s, \quad k \to \infty.
\]

(2.25)

Define

\[
    J_s = J^{(0)}_s \mu_s,
\]

(2.26)
then we have

\[ \mu_\pm \to I, \quad k \to \infty. \]  

(2.27)

In addition, the asymptotic property of \( \varphi_\pm \) follows:

\[ \varphi_\pm \sim e^{i(\sigma \pm)\xi}, \quad k \to \infty. \]

Consider the wronskians expression of the scattering coefficients (2.17), we find that

\[ s_{11}(k) \to 1, \ s_{22}(k) \to 1, \quad k \to \infty. \]

(2.28)

### 2.3.2. Asymptotic as \( k \to 0 \)

We first assume that \( J_\pm \) admit a Lorentz expansion

\[ J_\pm = \sum_{n=-\infty}^{\infty} k^n D^{(n)}_{\pm}. \]

Substituting into (2.13) and comparing the coefficients of the same order of \( k \) gives that \( D^{(n)}_{\pm} = 0, \ n = -2, -3, \ldots. \) Thus we expend solution of (2.13) of the form

\[ J_\pm = \frac{D^{(-1)}_{\pm}}{k} + D^{(0)}_{\pm} + D^{(1)}_{\pm}k + \cdots, \]

then we obtain the comparison results:

**x-part:**

\[ O(k^{-1}): \ D^{(-1)}_{\pm} - \frac{1}{2} i \beta \sigma \sigma_{\pm} = \frac{1}{2} i \beta \sigma_{\pm} D_{\pm}^{-1}, \]

(2.29)

**t-part:**

\[ O(k^{-1}): \ D_{\pm}^{-1} \sigma_{\pm} = - \sigma_{\pm} D_{\pm}^{-1}. \]

(2.30)

It is easy to check that

\[ D^{(-1)}_{\pm} = 0, \]

(2.31)

which implies \( D^{(-1)}_{\pm} \) is a constant independent on \( x \). This implies that the following limit

\[ \lim_{k \to 0} k J_\pm = D^{(-1)}_{\pm} \]

(2.32)

is uniformly convergent for \( x \in \mathbb{R} \). While the following limit

\[ \lim_{x \to \infty} k J_\pm = kI - \frac{\beta}{2} Q_\pm \]

(2.33)

exists for every fixed \( k \in \mathbb{C} \). Thus the following limit is commutative, and using (2.32) and (2.33) yields

\[ D^{(-1)}_{\pm} = \lim_{x \to \infty} \lim_{k \to 0} k J_\pm = \lim_{k \to 0} \lim_{x \to \infty} k J_\pm = - \frac{\beta}{2} Q_\pm, \]

which leads to

\[ J_\pm = - \frac{\beta Q_\pm}{2k} + O(1), \quad k \to 0. \]

(2.34)

The asymptotics of functions \( \varphi_\pm \) and \( \mu_\pm \) can be obtained by transformations (2.12) and (2.26). Consider the wronskian expressions of scattering coefficients and note the boundary conditions, we find that

\[ s_{11} = \text{Wr}(J_1 e^{-\theta}, J_\pm e^\theta) = \frac{q}{Q_\pm} + O(1), \quad k \to 0, \]

\[ s_{22} = \text{Wr}(J_\pm e^{-\theta}, J_\pm e^\theta) = \frac{q}{Q_\pm} + O(1), \quad k \to 0. \]
2.4. Analyticity

Noticing that
\[ \text{Im} \left( \frac{k \lambda}{\sqrt{\alpha}} \right) = 4 \text{Re} \text{Im} k, \]
we define two domains and their boundary
\[ D^+ = \{ k \mid \text{Re} \text{Im} k > 0 \} = \left\{ k \mid \arg k \in \left( 0, \frac{\pi}{2} \right) \cup \left( \frac{\pi}{2}, \frac{3\pi}{2} \right) \right\}, \]
\[ D^- = \{ k \mid \text{Re} \text{Im} k < 0 \} = \left\{ k \mid \arg k \in \left( \frac{\pi}{2}, \frac{\pi}{2} \right) \cup \left( \frac{3\pi}{2}, 2\pi \right) \right\}, \]
\[ \Sigma = \{ k \mid \text{Re} \text{Im} k = 0 \} = \mathbb{R} \cup i\mathbb{R}. \]

which are shown in Figure 1.

Under the transformation (2.26), we write the Lax pair as
\[ \mu_{as} - \frac{i k \lambda}{\sqrt{\alpha}} [\mu_s, \sigma_3] = \hat{X} \mu_s, \]
\[ \mu_{as} - i \lambda \eta [\mu_s, \sigma_3] = \hat{T} \mu_s, \]
where
\[ \hat{X} = e^{-i \sigma_3} \left( X + \frac{i k \lambda \sigma_3}{\sqrt{\alpha}} - iv \sigma_3 \right), \]
\[ \hat{T} = e^{-i \sigma_3} \left( T + i \lambda \eta \sigma_3 - iv \sigma_3 \right). \]

We rewrite the Eq. (2.35) into a full differential form
\[ d(e^{i \sigma_3} \mu_s) = e^{i \sigma_3} (\hat{X}dx + \hat{T}dt) \mu_s, \]
which implies \( \mu_s \) can be formally integrated to obtain the integral equations for the eigenfunctions:
\[ \mu_- = \mu_-^0 + \int_{-\infty}^{0} e^{\frac{ik \lambda}{\sqrt{\alpha}} (y-y') \sigma_3} \hat{X}(y, t, k) \mu_-(y, t, k) \, dy, \]
\[ \mu_+ = \mu_+^0 - \int_{0}^{\infty} e^{\frac{ik \lambda}{\sqrt{\alpha}} (y-y') \sigma_3} \hat{X}(y, t, k) \mu_+(y, t, k) \, dy, \]
where
\[ \mu_- = Y_-, \quad \mu_+ = e^{\int_{0}^{\infty} \gamma(s) dx' \sigma_3} Y_+. \]
By using a similar way to Appendix A in Biondini and Kovačič [1], under mild integrability conditions on the potential, the eigenfunctions (2.37) and (2.38) can be analytically extended in the complex $k$-plane into the following regions:

$$\mu_{+,1}, \mu_{+,2}: D^+ \cup \mu_{-,1}, \mu_{-,2}: D^-,$$

(2.39)

where $\mu_i = (\mu_{+,i}, \mu_{-,i})$, the subscript 1, 2 denote the first and second column of $\mu_i$.

Apparently, the functions $\varphi_j$ and $\mu_i$ share the same analyticity, hence from the wronskians expression of the scattering coefficients, we know that $s_{11}$ is analytic in $D^+$, and $s_{22}$ is analytic in $D^-$.

### 2.5. Symmetry

To investigate the discrete spectrum and residue conditions in the Riemann–Hilbert problem, one needs to analyze the symmetric property for the solutions $\varphi_j$ and the scattering matrix $S(k)$.

**Theorem 2.2.** The Jost eigenfunctions satisfy the following symmetric relations

$$\sigma \varphi_j^\ast(k^\ast) \sigma = -\varphi_j(k),$$

(2.40)

$$\sigma \varphi_j^\ast(-k^\ast) \sigma = \varphi_j(k),$$

(2.41)

and the scattering matrix satisfies

$$-\sigma S(k) \sigma = S^\ast(k^\ast),$$

(2.42)

$$\sigma S^\ast(-k^\ast) \sigma = S(k).$$

(2.43)

where

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

**Proof.** We only prove (2.40) and (2.42), (2.41) and (2.43) can be shown in a similar way. The functions $\varphi_j$ are the solutions of the spectral problem

$$\varphi_{\pm,j}(k) = X(k) \varphi_j(k).$$

(2.44)

Conjugating and multiplying left by $\sigma$ on both sides of the equation gives

$$\left(\sigma \varphi_j^\ast(k^\ast) \sigma \right)_j = \sigma X^\ast(k^\ast) \varphi_j^\ast(k^\ast) \sigma.$$

Note that $\sigma X^\ast(k^\ast) = X(k) \sigma$, hence $\sigma \varphi_j^\ast(k^\ast) \sigma$ is also a solution of (2.44). By using relations $\sigma Y_j^\ast(k^\ast) = Y_j(k) \sigma$ and $\sigma e^{i\sigma \xi} \sigma = -e^{-i\sigma \xi}$, we obtain

$$\sigma \varphi_j^\ast(k^\ast) \sigma \sim -Y_j(k) e^{-i\sigma \xi}, x \to \pm \infty,$$

(2.45)

which leads to (2.40).

For the scattering matrix, conjugating on the both sides of Eq. (2.15) leads to

$$\varphi_j^\ast(k^\ast) = \varphi_j^\ast(k^\ast) S^\ast(k^\ast),$$

substituting (2.40) into the above formula yields

$$\varphi_j^\ast(k) = -\varphi_j(k) \sigma S^\ast(k^\ast) \sigma,$$

comparing with (2.15) gives

$$S^\ast(k^\ast) = -\sigma S(k) \sigma.$$

Elementwise, Eqs. (2.40)–(2.43) read as

$$s_{11}^\ast(k^\ast) = s_{22}(k), \quad s_{22}^\ast(k^\ast) = -s_{12}(k),$$

$$s_{11}^\ast(-k^\ast) = s_{22}(k), \quad s_{22}^\ast(-k^\ast) = s_{12}(k).$$

(2.46)
\[ \varphi_{s,1}(k) = \sigma \varphi_{s,1}'(k), \quad \varphi_{s,2}(k) = -\sigma \varphi_{s,2}'(k), \]
\[ \varphi_{s,1}(k) = \sigma \varphi_{s,1}'(-k), \quad \varphi_{s,2}(k) = \sigma \varphi_{s,2}'(-k), \]
(2.47)

which implies
\[ \varphi_{s,1}(k) = -\sigma \varphi_{s,2}(-k), \quad \varphi_{s,2}(k) = \sigma \varphi_{s,1}(-k), \]
(2.48)

note the relation (2.12) and (2.26), there exists
\[ \varphi_s = e^{\alpha \tau - \beta \sigma}, \]
(2.49)

thus for the eigenfunctions \( \mu_s \), we derive
\[ \mu_{s,2}(k) = -\sigma \mu_{s,1}(-k), \quad \mu_{s,1}(k) = \sigma \mu_{s,1}(-k). \]
(2.50)

3. THE INVERSE SCATTERING WITH NZBCs

3.1. Generalized Riemann–Hilbert Problem

We define the two matrices
\[ M^+ = \begin{pmatrix} \mu_{s,1} & \mu_{s,2} \\ -s_{21} & \mu_{s,2} \end{pmatrix}, \quad k \in D^+, \]
\[ M^- = \begin{pmatrix} \mu_{s,1} & \mu_{s,1} \\ s_{12} & -\mu_{s,1} \end{pmatrix}, \quad k \in D^-, \]

which are analytical in \( D^+, D^- \) respectively, and admit asymptotic
\[ M^+ = I + O(1/k), \quad k \to \infty, \]
\[ M^- = \frac{1}{k} e^{-\alpha \tau} \overline{Q}_+ + O(1), \quad k \to 0, \]
(3.1)

where \( \overline{Q}_+ = \begin{pmatrix} 0 & -\frac{1}{q_-} \\ \frac{1}{q_-} & 0 \end{pmatrix} \).

By using (2.49) and (3.1), we rewrite (2.16) and get a generalized Riemann–Hilbert problem

\[ M(x, t, k) \text{ is meromorphic in } C \setminus \Sigma, \]
(3.2)
\[ M^+(x, t, k) = M^-(x, t, k)(I - G(x, t, k)), \quad k \in \Sigma, \]
(3.3)
\[ M(x, t, k) \text{ satisfies residue conditions at zeros } \{ k : s_{11}(k) = s_{12}(k) = 0 \}, \]
(3.4)
\[ M^+ = I + O(1/k), \quad k \to \infty, \]
\[ M^- = \frac{1}{k} e^{-i\omega} \tilde{Q}_+ + O(1), \quad k \to 0, \]

where the jump matrix
\[
G = \begin{pmatrix}
0 & -e^{-2i\theta} \rho(k) \\
e^{2i\theta} \rho(k) & \rho(k) \rho(k)
\end{pmatrix}, \quad \rho(k) = \frac{s_{11}}{s_{11}}, \quad \tilde{\rho}(k) = \frac{s_{12}}{s_{22}}.
\]

and \( \Sigma = \mathbb{R} \cup i\mathbb{R} \) denotes the jump contour in Figure 1.

### 3.2. Discrete Spectrum and Residue Conditions

Suppose that \( s_{11}(k) \) has \( N \) simple zeroes in \( D^+ \cap \{ \text{Im } k < 0 \} \) denoted by \( k_n, n = 1, 2, \ldots, N. \) Owing to the symmetries in (2.46), there exists
\[
s_{11}(k_n) = 0 \Leftrightarrow s_{21}(-k^*_n) = 0 \Leftrightarrow s_{22}(k^*_n) = 0 \Leftrightarrow s_{11}(-k^*_n) = 0,
\]
thus the discrete spectrum is the set
\[
\{ \pm k_n, \pm k^*_n \},
\]
which distribute in the \( k \)-plane as shown in Figure 2.

Next we derive the residue conditions will be needed for the RH problem. If \( s_{11}(k) = 0 \) at \( k = k_n \) the eigenfunctions \( \varphi_{+1} \) and \( \varphi_{-2} \) at \( k = k_n \) must be proportional
\[
\varphi_{+1}(k_n) = b_n \varphi_{-2}(k_n),
\]
where \( b_n \) is an arbitrary constant independent on \( x, t. \) Under the transformation (2.49), there exists linear relation for \( \mu_k \)
\[
\mu_{+1}(k_n) = b_n e^{2i\theta(k_n)} \mu_{-2}(k_n),
\]
Thus we get
\[
\text{Res}_{k=k_n} \left[ \frac{\mu_{+1}}{s_{11}} \right] = \frac{\mu_{+1}(k_n)}{s_{11}'(k_n)} = C_n e^{2i\theta(k_n)} \mu_{-2}(k_n),
\]
(3.10)

where \( C_n = \frac{b_n}{s_{11}'(k_n)}. \) As for \( k = -k_n \), substituting (2.48) into (3.8) leads to
\[
\varphi_{+1}(-k_n) = -b_n \varphi_{-2}(-k_n),
\]
(3.11)
then applying the relation (2.49) yields
\[
\mu_{+1}(-k_n) = -b_n e^{2i\theta(k_n)} \mu_{-2}(-k_n),
\]
(3.12)
thus we obtain
\[
\text{Res}_{k=-k_n} \left[ \frac{\mu_{+1}}{s_{11}} \right] = \frac{\mu_{+1}(-k_n)}{s_{11}'(-k_n)} = -C_n e^{2i\theta(k_n)} \sigma_{1} \mu_{-2}(k_n),
\]
(3.13)
Similarly, the residue conditions at \( k = \pm k^*_n \) are
\[
\text{Res}_{k=k^*_n} \left[ \frac{\mu_{+1}}{s_{22}} \right] = \frac{\mu_{+1}(k^*_n)}{s_{22}'(k^*_n)} = C_n e^{-2i\theta(k^*_n)} \mu_{-2}(k^*_n),
\]
(3.14)
\[
\text{Res}_{k=-k^*_n} \left[ \frac{\mu_{+1}}{s_{22}} \right] = \frac{\mu_{+1}(-k^*_n)}{s_{22}'(-k^*_n)} = -C_n e^{-2i\theta(k^*_n)} \sigma_{2} \mu_{-2}(k^*_n),
\]
(3.15)
where \( C_n = -C_n. \)
Recall the definition of \( M^* \), there follows
\[
\text{Res}_{k \to k^*} M^* = \left\{ 0, \tilde{C} e^{-i\theta(k^*_n)} \mu_{-\infty}(k^*_n) \right\}, \quad \text{Res}_{k \to k^*} M^* = \left\{ 0, \tilde{C} e^{-i\theta(k^*_n)} \sigma_{\infty}(k^*_n) \right\},
\]
(3.16)

3.3. Reconstruction Formula for the Potential

To solve the Riemann–Hilbert problem (3.3), one needs to regularize it by subtracting out the asymptotic behaviour and the pole contribution. Hence, we rewrite Eq. (3.3) as
\[
M^* - I - \frac{1}{k} e^{-i\omega(\xi)} \tilde{Q} - \sum_{n=1}^{N} \text{Res}_{k \to k^*_n} M^* - \sum_{m=1}^{N} \text{Res}_{k \to k^*_m} M^* = M^* - I - \frac{1}{k} e^{-i\omega(\xi)} \tilde{Q} - \sum_{n=1}^{N} \text{Res}_{k \to k^*_n} M^* - \sum_{m=1}^{N} \text{Res}_{k \to k^*_m} M^* - \sum_{i=1}^{N} \text{Res}_{k \to k^*_i} M^* - M^* G,
\]
(3.17)
where \( \tilde{Q} = \begin{pmatrix} 0 & -1/q_n \\ \frac{1}{q_n} & 0 \end{pmatrix} \). Then the Plemelj’s formula shows
\[
M(x,t,k) = I + \frac{1}{k} e^{-i\omega(\xi)} \tilde{Q} + \sum_{n=1}^{N} \text{Res}_{k \to k^*_n} M^* + \sum_{m=1}^{N} \text{Res}_{k \to k^*_m} M^* + \sum_{n=1}^{N} \text{Res}_{k \to k^*_n} M^* + \sum_{m=1}^{N} \text{Res}_{k \to k^*_m} M^* + \sum_{i=1}^{N} \text{Res}_{k \to k^*_i} M^* + \frac{1}{2\pi i} \int_{\gamma} \frac{M^* G(x,t,\xi)}{\xi - k} \, d\xi,
\]
(3.18)
and the (1, 2)-element of \( M \) is
\[
M_{12} = \frac{1}{k} \left\{ e^{-i\omega(\xi)} \tilde{Q} + \sum_{n=1}^{N} \left( \text{Res}_{k \to k^*_n} M^* + \text{Res}_{k \to k^*_m} M^* \right) - \frac{1}{2\pi i} \int_{\gamma} \frac{M^* G(x,t,\xi)}{\xi - k} \, d\xi \right\}_{12} + O(1/k^2).
\]
(3.19)
Comparing the (1, 2)-element on the both sides of Eq. (2.20) yields
\[
q_i - i\beta q = 2ie^{\tau} f_{12}.
\]
(3.20)
Recall the transformation (2.2), we know
\[
u e^{i\tau(x,t)} = q_i - i\beta q,
\]
where \( \tau(x,t) = \beta x + \alpha t \beta q^2 t \). Thus we can write (3.20) as
\[
u e^{i\tau(x,t)} = q_i - i\beta q
\]
(3.21)
where
\[
M^* = M_\infty + \frac{M^*}{k} + \cdots.
\]
Substituting (3.19) and (3.16) into (3.21), we obtain the reconstruction formula for potential
\[
\nu = 2ie^{i\tau} \left\{ e^{-i\omega(q)} q_+ + \sum_{n=1}^{N} \tilde{C} e^{-2i\theta(k^*_n)} \mu_{-\infty}(k^*_n) - \frac{1}{2\pi i} \int_{\gamma} \left( M^* G \right)_{12}(\xi) d\xi \right\},
\]
(3.22)
4. REFLECTIONLESS POTENTIALS

Now we consider the potential \( u(x,t) \) for which the reflection coefficient \( \rho(k) \) vanishes identically, that is, \( G = 0 \). In this case, Eq. (3.22) reads as
\[
u = 2ie^{i\tau} \left\{ \sum_{n=1}^{N} 2\tilde{C} e^{-2i\theta(k^*_n)} \mu_{-\infty}(k^*_n) - \frac{e^{i\omega(q)}}{q_+} \right\}.
\]
(4.1)
To obtain the expression of the term $\mu_{-1,1}(k'_n)$, we consider the first and second column of (3.18) respectively under reflectionless case:

$$
\mu_{-2}(k_n) = \begin{cases} 
\frac{e^{-ir}}{k_q} + \sum_{j=1}^{N} \frac{\tilde{C}_j e^{-2i\theta(k'_j)}}{k_n - k'_j}, & n = 1, 2, \ldots, N \\
\frac{1}{k_q}, & n = N + 1, \ldots, 2N
\end{cases}
$$

$$
\mu_{-1}(k'_n) = \begin{cases} 
\frac{1}{k'_q}, & n = 1, 2, \ldots, N \\
\frac{e^{ir}}{k'_q} + \sum_{j=1}^{N} \frac{C_j e^{2i\theta(k'_j)}}{k'_n - k'_j}, & n = N + 1, \ldots, 2N
\end{cases}
$$

which can be further written as

$$
\mu_{-2}(k_n) = \frac{e^{-ir}}{k_q} + 2 \sum_{j=1}^{N} \frac{\tilde{C}_j e^{-2i\theta(k'_j)}}{k_n^2 - (k'_j)^2} K_1 \mu_{-1}(k'_j),
$$

$$
\mu_{-1}(k'_n) = \frac{1}{k'_q} + 2 \sum_{j=1}^{N} \frac{C_j e^{2i\theta(k'_j)}}{k'_n(k'_n - k'_j)} K_2 \mu_{-2}(k'_j),
$$

where

$$
K_1 = \begin{pmatrix} k_n & 0 \\
0 & k'_n \end{pmatrix}, \quad K_2 = \begin{pmatrix} k'_n & 0 \\
0 & k_n \end{pmatrix}.
$$

Define

$$c_j(x, t, k) = \frac{C_j}{(k^2 - (k'_j)^2)} e^{-2i(x, t, k)},$$

whose conjugate gives

$$c'_j(k) = -\frac{\tilde{C}_j}{k^2 - (k'_j)^2} e^{-2i(k'_j)}.$$

Then (4.3) reduces to

$$
\mu_{-1,1}(k'_1) = 1 + 2 \sum_{j=1}^{N} k_j c_j(k'_j) \mu_{-1,2}(k'_j),
$$

$$
\mu_{-1,2}(k'_n) = \frac{e^{ir}}{k'_q} - 2 \sum_{m=1}^{N} k'_m c'_m(k'_n) \mu_{-1,1}(k'_n),
$$

substituting (4.5) into (4.4) yields

$$
\mu_{-1,1}(k'_n) = 1 - \frac{2e^{ir}}{q'_n} \sum_{j=1}^{N} c_j(k'_n) - 4 \sum_{j=1}^{N} \sum_{m=1}^{N} k'_{j} c'_m(k'_n) \mu_{-1,1}(k'_n), \quad n = 1, 2, \ldots, N.
$$

Introducing notations

$$X = (X_1, X_2, \ldots, X_N)^\top, \quad A = (A_{n,m})_{1 \leq n,m \leq N}, \quad B = (B_1, B_2, \ldots, B_N)^\top$$

with components being

$$X_n = \mu_{-1,1}(k'_n), \quad A_{n,m} = \sum_{j=1}^{N} 4k'_j c'_j(k'_n) c'_m(k'_n), \quad B_n = 1 - \frac{2e^{ir}}{q'_n} \sum_{j=1}^{N} c'_j(k'_n),$$

then the system (4.6) can be written as matrix form

$$HX = B,$$

where

$$H = I + A = (H_1, H_2, \ldots, H_N).$$
By standard Cramer rule, the system (4.7) is the solution of

\[
\mu_{n+1}(k') = X_n = \frac{\det H'_{\text{out}}}{\det H},
\]

where

\[
H'_{\text{out}} = (H_1, \ldots, H_{n+1}, B, \ldots, H_N).
\]

Note that

\[
u = \int (\beta + \frac{1}{2} u, v_r)(x', t) \, dx',
\]

then Eq. (2.24) reduces to

\[
u = \int (\beta + \frac{1}{2} u, v_r)(x', t) \, dx'.
\]

Therefore, we obtain a compact solution:

\[
u = 2i e^{-i(v, v_r)} \left( \frac{\det H_{	ext{aug}}}{\det H} e^\nu - \frac{1}{q_r} \right),
\]

where the augmented \((N+1) \times (N+1)\) matrix \(H_{\text{aug}}\) is

\[
H_{\text{aug}} = \begin{pmatrix} 0 & Y' \\ B & H \end{pmatrix}, \quad Y = (Y_1, \ldots, Y_N)',
\]

and \(Y_n = 2C_{\text{e}} e^{-2i(v, v_r)} = -2C_{\text{e}} e^{-2i(v, v_r)}\).

5. TRACE FORMULA AND THETA CONDITION

Define

\[
\beta^+ = s_{11}(k) \prod_{n=1}^{N} \frac{k^2 - (k_n')^2}{k^2 - k_n^2}, \quad \beta^- = s_{22}(k) \prod_{n=1}^{N} \frac{k^2 - k_n^2}{k^2 - (k_n')^2},
\]

we see that they are analytic and no-zeros in \(D^-\) and \(D^+\), respectively. Moreover, \(\beta^+ \beta^- = s_{11}(k)s_{22}(k)\). Note that \(\det S(k) = s_{11}s_{22} - s_{12}s_{12} = 1\), this implies

\[
\frac{1}{s_{11}s_{22}} = 1 - \rho(k) \rho(k) = 1 + \rho(k) \rho'(k'),
\]

thus

\[
\beta^+ \beta^- = s_{11}s_{22} = \frac{1}{1 + \rho(k) \rho'(k')}, \quad k \in \Sigma.
\]

Taking logarithms leads to

\[
\log \beta^+ - (-\log \beta^-) = -\log[1 + \rho(k) \rho(k')], \quad k \in \Sigma,
\]

then Applying Plemelj formula, we have

\[
\log \beta^+ = \mp \frac{1}{2\pi i} \int_{k} \frac{[\log[1 + \rho(s) \rho'(s')])}{s - k} \, ds, \quad k \in D^+.
\]

Substituting into (5.1), we obtain the trace formula

\[
s_{11}(k) = \exp \left[ \frac{1}{2\pi i} \int_{\Sigma} \frac{\log[1 + \rho(s) \rho'(s')]}{s - k} \, ds \right] \prod_{n=1}^{N} \frac{k^2 - (k_n')^2}{k^2 - k_n^2}, \quad k \in D^-,
\]

\[
s_{22}(k) = \exp \left[ -\frac{1}{2\pi i} \int_{\Sigma} \frac{\log[1 + \rho(s) \rho'(s')]}{s - k} \, ds \right] \prod_{n=1}^{N} \frac{k^2 - k_n^2}{k^2 - (k_n')^2}, \quad k \in D^+.
\]
Under reflectionless condition, they reduce to
\[ s_{11}(k) = \prod_{n=1}^{N} \frac{k^2 - k_n^2}{k^2 - (k_n^*)^2}, \quad k \in D^-; \quad s_{22}(k) = \prod_{n=1}^{N} \frac{k^2 - (k_n^*)^2}{k^2 - k_n^2}, \quad k \in D^+. \tag{5.4} \]

Taking limit as \( k \to 0 \) for (5.4) leads to
\[ \frac{q_-}{\beta q^*} = \exp \left[ - \frac{i}{2\pi} \int \frac{\log(1 + \rho(s) \rho^*(s^*))}{s} ds \right] \exp \left[ 4i \sum_{n=1}^{N} \arg(k_n) \right], \quad k \in D^-, \tag{5.5} \]

note that \( \beta \) is a positive constant, then we obtain the theta condition
\[ \arg \left( \frac{q_-}{q^*} \right) = - \frac{1}{2\pi} \int \frac{\log(1 + \rho(s) \rho^*(s^*))}{s} ds + 4 \sum_{n=1}^{N} \arg(k_n). \tag{5.6} \]

under reflectionless condition, we have
\[ \arg \left( \frac{q_-}{q^*} \right) = 4 \sum_{n=1}^{N} \arg(k_n). \tag{5.7} \]

6. ONE-SOLITON SOLUTION

As an application of the formula (4.10) of \( N \)-soliton solution, we construct one-soliton solution for the FL equation, which corresponds to \( N = 1 \). Then Eq. (4.10) becomes
\[ u_x = 2ie^{-i\varphi} e^{i\varphi} e^{i\beta (k_1^* x + \lambda x + \lambda_1 \eta y)}, \quad \lambda_1 = \sqrt{\alpha} \left( k_1 + \frac{\beta}{2k_1} \right), \tag{6.1} \]

where \( k_1 \) is an eigenvalue, \( C_1 \) is an arbitrary constant and
\[ \varphi(k_1) = \frac{k_1 \lambda_1}{\sqrt{\alpha}} x + \lambda_1 \eta y, \]
\[ \eta = \sqrt{\alpha} \left( k_1 - \frac{\beta}{2k_1} \right), \quad c_1(k_1) = \frac{C_1}{(k_1^*)^2 - k_1^2} e^{2i\beta k_1}. \tag{6.2} \]

CONFLICTS OF INTEREST

The authors declare they have no conflicts of interest.

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