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Research Article

# Exact Solutions of the Nonlinear Fin Problem with Temperature-dependent Coefficients

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**ABSTRACT**

The analytical solutions of a nonlinear fin problem with variable thermal conductivity and heat transfer coefficients are investigated by considering theory of Lie groups and its relations with $\lambda$-symmetries and Prelle-Singer procedure. Additionally, the classification problem with respect to different choices of thermal conductivity and heat transfer coefficient functions is carried out. In addition, Lagrangian and Hamiltonian forms related to the problem are investigated. Furthermore, the exact analytical solutions of boundary-value problems for the nonlinear fin equation are obtained and represented graphically.



## 1. INTRODUCTION

Fins are used in many engineering applications such as compressors, cooling of computer processor, and air-cooled craft engine in air conditioning in industry to increase the heat transfer from surfaces and they are frequently used to enable the heat loss from a heated wall. In the case of variable heat transfer coefficient and constant thermal conductivity, the analysis of the fin problem can be studied for the different cases of these dependent coefficients. If the heat transfer coefficient is a function on spatial coordinate and then it affects local temperature difference between the fin surface and the encompassing fluid. In some situations, the thermal conductivity may not be constant and it depends on temperature. This means that enormous temperature difference can be seen along the fin.

From the mathematical point of view, to obtain the exact solutions of nonlinear fin equation with temperature-dependent thermal conductivity and heat transfer coefficient is not straightforward because of the nonlinearity of the problem. The associated fin problem is given in the following dimensional form [19]

$$A_c \frac{d}{dX}\left(K(T)\frac{dT}{dX}\right) - PH(T)(T-T_a) = 0, \qquad 0 < X < L, \tag{1.1}$$

which is subjected to the boundary conditions

$$\left.\frac{dT}{dX}\right|_{X=0} = 0 \qquad \text{at the tip,} \tag{1.2}$$

$$T(L) = T_b \qquad \text{at the base,} \tag{1.3}$$

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where the heat transfer through the tip end face is assumed to be small. The conditions (1.2) and (1.3) are of mixed Neuman-Dirichlet type boundary conditions. In Eq. (1.1), $T$ is dimensional temperature of $X$, which is dimensional space coordinate, $T_a$ is the temperature of the environment into which the fin extends, $A_c$ is the cross-sectional area of the fin, $K(T)$ is the thermal conductivity dependent on the fin temperature, $H(T)$ is the heat transfer coefficient dependent on the temperature, $P$ is the fin parameter, $L$ is length of the fin, and $T_b$ is the temperature of the base to which the fin is attached. With the following dimensionless variables

$$\theta = \frac{T-T_a}{T_b-T_a}, \quad x=\frac{X}{L}, \quad k=\frac{K}{K_a}, \\ N^2=\frac{H_b P L^2}{K_a A_c}, \quad h = N^2 \frac{H}{H_b}\frac{T-T_a}{T_b-T_a}, \tag{1.4}$$

where $\theta$ is dimensionless temperature as a function of $x$, dimensionless space coordinate, $K_a = K(T_a)$, and $H_b = H(T_b)$, in dimensionless notation, the nonlinear fin equation (1.1) with variable coefficients takes the form

$$\frac{d}{dx}\left[k(\theta)\frac{d\theta}{dx}\right] = h(\theta), \quad \theta=\theta(x) \text{ and } 0<x<1. \tag{1.5}$$

Alternatively, one can rewrite Eq. (1.5) as

$$\ddot{\theta} + \frac{1}{k(\theta)}\frac{dk(\theta)}{d\theta}(\dot{\theta})^2 = \frac{h(\theta)}{k(\theta)}, \quad 0<x<1, \tag{1.6}$$

where "overdot" represents the ordinary derivative of $\theta$ function as the dependent variable with respect to $x$ as independent variable. The associated dimensionless boundary conditions related to the problem are

$$\left.\frac{d\theta}{dx}\right|_{x=0} = 0 \quad \text{at the tip,} \tag{1.7}$$

$$\theta(x)\,|_{x=1} = 1 \quad \text{at the base.} \tag{1.8}$$

In the literature, there are some studies devoted to the analysis of fin performance due to its important applications in engineering. Aziz [2] and Aziz and Enamul Hug [3] deals with perturbation method and obtains a closed-form solution for a straight convective fin with temperature-dependent thermal conductivity. Lesnic [21] Lesnic and Heggs [22] takes into account the Adomian's decomposition method to determine temperature distribution within a straight fin with temperature-dependent heat transfer coefficient. Dulkin and Garasko [11] obtain recurrent direct solution using the inversion of the closed form solution. Sohrabpour and Razani [39] consider a method of temperature correlated profiles to obtain the solution of the optimum convective fin when thermal conductivity and heat transfer coefficients are functions of temperature. They emphasize that the effect of temperature dependence of heat transfer coefficients is important in the design of optimum fins. Khani et al. [17] compares Adomian's decomposition method, homotopy perturbation method, and homotopy analysis method in optimizing fin performance. Zhou [43] takes into consideration differential transformation method, which based on the Taylor series expansion. Joneidi and Rashidi [15,8] present differential transformation method to obtain the efficiency of convective straight fins with temperature-dependent thermal conductivity. Lesnic [21] and Lesnic and Heggs [22] uses decomposition method, and Taylor series method used by Kim and Huang [18]. In addition, Kim et al. [19] consider the same problem by linearizing the fin equation and obtain an approximate solution.

It is a fact that theory of Lie symmetry groups is one of the most important solution methods for differential equations in the literature [16,13,12,33,32,41,42,20,31,30,27,40,14,1,35–37,34]. Specially, in the case of nonlinear differential equations, one may transform the differential equation into another equation with known solutions. Generally, transformed equations considered in the literature are linear equations. The first linearization problem for ordinary differential equation is carried out by Lie [23] indicating that a second-order ordinary differential equation is linearizable by a change of variables if and only if the equation is of the form

$$\ddot{\theta} + a_3(\theta,x)\dot{\theta}^3 + a_2(\theta,x)\dot{\theta}^2 + a_1(\theta,x)\dot{\theta} + a_0(\theta,x) = 0, \tag{1.9}$$

It is known that a second-order ordinary differential equation of the form (1.9) has first integrals, integrating factors and $\lambda$-symmetries developed by Muriel and Romero [24–26]. In recent years, some studies on the linearization through transformation involving nonlocal terms are carried out [10]. For the case of investigation of $\lambda$-symmetries of differential equations, it is possible to consider a feasible algorithm that can be applied to nonlinear differential equation of the form (1.9).

Alternatively, there is another approach called Prelle–Singer (PS) developed by Prelle and Singer [38] for analysis of first-order ordinary differential equations in the literature. PS method is based on the fact that if a given system of first-order ordinary differential equation has a solution in terms of elementary functions and then the method ensures that its solution can be found. Later Duarte et al. [9] reconstructs the technique and applies it to second-order ordinary differential equation. Their approach is based on the conjecture that if an elementary solution exists for the given second-order differential equation then there exists at least one elementary first integral. Recently, this theory is generalized to obtain general solutions without any integration and the generalized theory for the higher-order ordinary differential equations is introduced [5]. Additionally, first integrals of fin equation by linearization methods are studied and it is shown that a first integral of the associated differential equation can be obtained by using linearization methods [29]. Based on the Noether theorem, on the other hand, first integrals of an differential equation can be obtained with the method called $\lambda$-symmetry. In fact, there is a mathematical relation between $\lambda$-symmetries and PS procedure, which have been considered in literature [28,5–7]. In this study, we study novel exact invariant solutions and some solutions of boundary-value problem related to nonlinear fin equation by symmetry group-related approaches.

This study is organized as follows. In Section 2, some fundamental definitions and theorems related to the mathematical approaches are presented. In Section 3, $\lambda$-symmetries, associated first integrals, and integrating factors of the nonlinear fin equation are discussed. Additionally, exact solutions for some boundary-value problems are obtained by using the first integrals. This section also includes different cases corresponding to different choices of thermal conductivity and heat transfer coefficients. In Section 4, the generalized PS method is applied to fin equation and Lie point symmetry, first integral, $\lambda$-symmetry, integrating factor, and Lagrangian–Hamiltonian function are obtained and exact solutions for some boundary-value problems are introduced. Section 5 introduces novel solutions and results for boundary-value problems related to fin equation having physical meaning for a specific form of the thermal conductivity and some results are represented graphically.

## 2. PRELIMINARIES

Let us consider a second-order ordinary differential equation of the form [26]

$$\ddot{\theta} + a_2(\theta, x)\dot{\theta}^2 + a_1(\theta, x)\dot{\theta} + a_0(\theta, x) = 0, \tag{2.1}$$

admitting vector field $\upsilon = \partial_\theta$. For this equation, one can write a function called $\lambda$ in the following form

$$\lambda(x, \theta, \dot{\theta}) = \alpha(\theta, x)\dot{\theta} + \beta(\theta, x), \tag{2.2}$$

which functions $\alpha$ and $\beta$ satisfy following system of determining equations

$$\alpha_\theta + \alpha^2 + a_2\alpha + (a_2)_\theta = 0, \tag{2.3}$$

$$\beta_\theta + 2(a_2 + \alpha)\beta + (a_1)_\theta + \alpha_x = 0, \tag{2.4}$$

$$\beta_x + \beta^2 + a_1\beta + (a_0)_\theta - \alpha a_0 = 0. \tag{2.5}$$

First, solution of Eq. (2.3) gives $\alpha(\theta, x) = -a_2(\theta, x)$. By substituting $\alpha$ into Eqs. (2.4) and (2.5), one can obtain the following differential relations

$$\beta_\theta + (a_1)_\theta - (a_2)_x = 0, \tag{2.6}$$

$$\beta_x + \beta^2 + a_1\beta + (a_0)_\theta + a_0 a_2 = 0. \tag{2.7}$$

Hence, one can conclude that equations of the form (2.1) with the corresponding system (2.3)–(2.5) admit the solution for $(\alpha_0, \beta_0)$ such that $\alpha_0 = a_2$. It is a fact that the coefficients of Eq. (2.1) must satisfy one of the two following alternatives: if $S_1 = 0$, then $S_2$ should be zero which defining by following relations

$$S_1(\theta, x) = (a_1)_\theta - 2(a_2)_x, \tag{2.8}$$

$$S_2(\theta, x) = (a_0 a_2 + (a_0)_\theta)_\theta + ((a_2)_x - (a_1)_\theta)_x + (a_{2x} - (a_1)_\theta)a_1, \tag{2.9}$$

or, if $S_1 \neq 0$, $S_3 = 0$ and $S_4 = 0$, where

$$S_3(\theta, x) = \left(\frac{S_2}{S_1}\right)_\theta - ((a_2)_x - (a_1)_\theta), \tag{2.10}$$

$$S_4(\theta, x) = \left(\frac{S_2}{S_1}\right)_x + \left(\frac{S_2}{S_1}\right)^2 + a_0 a_2 + (a_0)_\theta. \tag{2.11}$$

Based on these new definitions [26], the following two different cases can be considered:

**Case 1:** If $S_1 = 0$, then $S_2$ must be equal to zero.

**Case 2:** If $S_1 \neq 0$, the functions $S_3$ and $S_4$ must be zero.

Thus, it means that one can obtain $\lambda$-symmetry for the equation of the form (2.1) by taking into account the functions $\alpha$ and $\beta$. Now we examine the solutions of function $\beta$ under the following different cases:

**Case I:** In this case, $S_1$ and $S_2$ are equal to zero. Then for this situation, we solve the following differential equation

$$\dot{\gamma}(x) + \gamma^2(x) + f(x) = 0, \tag{2.12}$$

where "overdot" represents ordinary derivative with respect to $x$ and $f(x)$ is defined as

$$f(\theta, x) = a_0 a_2 + (a_0)_\theta - \frac{1}{2}(a_1)_x - \frac{1}{4}a_1^2. \tag{2.13}$$

Then $\beta$ function is found as

$$\beta(\theta, x) = \gamma(x) - \frac{1}{2}a_1(\theta, x). \tag{2.14}$$

Finally, we find $\lambda$-symmetry in the following form

$$\lambda = -a_2(\theta, x)\dot{\theta} + \gamma(x) - \frac{1}{2}a_1(\theta, x). \tag{2.15}$$

**Case II:** For $S_1 \neq 0$, $S_3$ and $S_4$ must be zero. Then, we have $\beta = S_2/S_1$ and $\lambda$-symmetry is found by

$$\lambda = -a_2(\theta, x)\dot{\theta} + S_2 / S_1. \tag{2.16}$$

**Theorem 2.1.** *[26] We consider an equation of the form (2.1) and let $S_1$, $S_2$, $S_3$, and $S_4$ be the functions defined by (2.8)–(2.11). The equation is such that either $S_1 = S_2 = 0$ or $S_3 = S_4 = 0$ if and only if $\partial_\theta$ is a $\lambda$-symmetry of (2.1) for some $\lambda = -a_2(\theta, x)\dot{\theta} + \beta(\theta, x)$.*

## 2.1. Prelle–Singer Approach

To introduce the Prelle–Singer (PS) approach [9], let us consider the equation

$$\ddot{\theta} = \frac{P(x, \theta, \dot{\theta})}{Q(x, \theta, \dot{\theta})}, \tag{2.17}$$

and the associated differential form

$$\frac{P}{Q}dx - d\dot{\theta}. \tag{2.18}$$

By adding a differential form of $S(x, \theta, \dot{\theta})(\dot{\theta}dx - d\theta)$ to differential form (2.18) where $S$ is an unknown function called null function, it is possible to consider the following differential form

$$\left(\frac{P}{Q} + S\dot{\theta}\right)dx - (Sd\theta + d\dot{\theta}). \tag{2.19}$$

The PS method is based on finding a function $S$ such that the differential form (2.19) is proportional to the differential form $dI = I_x dx + I_\theta d\theta + I_{\dot{\theta}} d\dot{\theta}$ for some functions $I(x, \theta, \dot{\theta})$. This means that there exists a function $R$ called integrating factor such that

$$dI = R\left(\left(\frac{P}{Q} + S\dot{\theta}\right)dx - (Sd\theta + d\dot{\theta})\right). \tag{2.20}$$

The existence of the functions $S$, $I$, and $R$ satisfying (2.20) implies that

$$I_x = R\left(\frac{P}{Q} + S\dot{\theta}\right),\ I_\theta = -RS,\ I_{\dot{\theta}} = -R. \tag{2.21}$$

The compatibility conditions for (2.21) yield

$$A(S) = -\phi_\theta + S\phi_{\dot{\theta}} + S^2, \quad A(R) = -R(S + \phi_{\dot{\theta}}), \quad R_\theta = R_{\dot{\theta}} S + RS_{\dot{\theta}}, \tag{2.22}$$

where $\phi = \frac{P}{Q}$ and $A$ is the operator associated with Eq. (2.18). The first equation says that $v = \partial_\theta$ is a $\lambda$-symmetry for $\lambda = -S$. By writing the second and third equations of (2.22) in terms of $\lambda$, the relation $\mu = -R$ is obtained. Furthermore, by assuming the existence of a Hamiltonian, one can write

$$I(\theta, \dot{\theta}) = H(\theta,\ p) = p\dot{\theta} - L(\theta, \dot{\theta}), \tag{2.23}$$

where $L(\theta, \dot{\theta})$ is Lagrangian and $p$ is canonically conjugate momentum, thus we have

$$\frac{\partial I}{\partial \dot{\theta}} = \frac{\partial H}{\partial \dot{\theta}} = \frac{\partial p}{\partial \dot{\theta}}\dot{\theta} + p - \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial p}{\partial \dot{\theta}}\dot{\theta}. \tag{2.24}$$

From Eq. (2.24), we identify conjugate momentum as

$$p = \int \frac{I_{\dot{\theta}}}{\dot{\theta}} d\dot{\theta} + f(\theta), \tag{2.25}$$

where $f(\theta)$ is an arbitrary function.

## 3. $\lambda$-SYMMETRIES AND ASSOCIATED FIRST INTEGRALS OF THE FIN EQUATION

In the literature, the first integrals based on the linearization methods and Sundman transformation pair for nonlinear fin equation (1.6) are investigated in the study [29].

**Remark 3.1.** In the study [29], first integrals of the form $A\dot{x} + B$ and Sundman transformation pair are found based on the linearization methods which is defined by Duarte et al. [10] and reconstructed by the Muriel [26]. Then, invariant solutions are obtained by using these first integrals of the form $A\dot{x} + B$.

In this study, $\lambda$-symmetries are studied based on linearization methods. It is a fact that one can characterize a second-order ordinary differential equation that can be linearized by means of nonlocal transformations. This characterization is given in terms of the coefficients of the equation and determines a second-order ordinary differential equation that admits $\lambda$-symmetries. There is a systematic method to find the $\lambda$-symmetries, which are used to reduce the order of differential equation. Hence, a second-order ordinary differential equation can also be integrated by a unified procedure based on $\lambda$-symmetries. For this purpose, let us assume that the equation of the form (2.1) admiting the vector field $v = \partial_\theta$ by having $\lambda$-symmetry of the form

$$\lambda(x, \theta, \dot{\theta}) = \alpha(\theta, x)\dot{\theta} + \beta(\theta, x). \tag{3.1}$$

Then, we have the following propositions:

**Proposition 3.1.** *If $S_1 = S_2 = 0$, then there is a following relation between thermal conductivity coefficient $k(\theta)$ and heat transfer coefficient $h(\theta)$ of fin equation (1.6) of the form*

$$k(\theta) = -\frac{h'(\theta)}{\sigma}, \tag{3.2}$$

*where $h'(\theta) = \frac{dh(\theta)}{d\theta}$ and $\sigma$ is a constant [29].*

*Proof.* From the equation (1.6), we have

$$a_2(\theta, x) = \frac{k'(\theta)}{k(\theta)}, \qquad a_1(\theta, x) = 0, \qquad a_0(\theta, x) = -\frac{h(\theta)}{k(\theta)}, \tag{3.3}$$

where $k'(\theta) = \frac{dk(\theta)}{d\theta}$. The use of these coefficients yields $S_1 = 0$ given by (2.8). Hence, it is easy to see from Theorem 2.1 that $S_2$ must be zero if $S_1 = 0$. $S_2$ can be found from (2.9) and (3.3) as

$$S_2 = \left(\frac{-h'(\theta)}{k(\theta)}\right)_\theta, \tag{3.4}$$

and since $S_2$ must be zero for $S_1 = 0$ and then we have

$$\left(\frac{-h'(\theta)}{k(\theta)}\right)_\theta = 0, \tag{3.5}$$

Hence, the integration with respect to $\theta$ gives

$$\left(\frac{-h'(\theta)}{k(\theta)}\right) = \sigma. \tag{3.6}$$

**Proposition 3.2.** *Let's $S_1$, $S_2$, $S_3$, and $S_4$ be the functions defined by (2.8)–(2.11). The relation $S_1 = S_2 = 0$ is valid if and only if $\partial_\theta$ is a $\lambda$-symmetry of the form $\lambda = -a_2(\theta, x)\dot{\theta} + h(x) - \frac{1}{2}a_1(\theta, x)$. Thus, the function $\lambda = -\sqrt{\sigma}\tan(\sqrt{\sigma}x - \sqrt{\sigma}c_1) - \frac{h''(\theta)}{h'(\theta)}\dot{\theta}$ is a $\lambda$-symmetry of fin equation.*

*Proof.* First of all, the substitution of $f(x) = \sigma$ into (2.12) yields the following differential equation of the form

$$\dot{\gamma}(x) + \gamma^2(x) + \sigma = 0, \tag{3.7}$$

The solution of the equation gives

$$\gamma(x) = -\sqrt{\sigma}\tan(\sqrt{\sigma}x - \sqrt{\sigma}c_1), \tag{3.8}$$

where $c_1$ is an integration constant. From (2.14), the function $\beta$

$$\beta(\theta, x) = -\sqrt{\sigma}\tan(\sqrt{\sigma}x - \sqrt{\sigma}c_1) - \frac{h''(\theta)}{\dot{\theta}h'(\theta)}\dot{\theta}(x), \tag{3.9}$$

is determined. The $\lambda$-symmetry (2.15) of fin equation

$$\lambda(\theta, x) = -\sqrt{\sigma}\tan(\sqrt{\sigma}x - \sqrt{\sigma}c_1) - \frac{h''(\theta)}{h'(\theta)}\dot{\theta}(x), \tag{3.10}$$

is obtained.

It is a fact that there is an important relation among the $\lambda$-symmetries, the integrating factors, and the first integrals of second-order ordinary differential equation (3.10) [24,25]. To see this mathematical relation, let us consider a second-order differential equation of the form

$$\ddot{\theta} = \phi(x, \theta, \dot{\theta}), \tag{3.11}$$

and let vector field of equation (3.11) be in the form of

$$A = \partial_x + \dot{\theta}\partial_\theta + \phi(x, \theta, \dot{\theta})\partial_{\dot{\theta}}. \tag{3.12}$$

In terms of $A$, a first integral of (3.11) is any function in the form of $w(x, \theta, \dot{\theta})$ providing equality of $A(w) = 0$. Thus, the integrating factor of Eq. (3.11) is found from the solution of the equation

$$\mu[\ddot{\theta} - \phi(x, \theta, \dot{\theta})] = D_x w, \tag{3.13}$$

where $D_x$ is total derivative operator in the form

$$D_x = \partial_x + \dot{\theta}\partial_\theta + \ddot{\theta}\partial_{\dot{\theta}} + \ldots.$$

Hence, one can determine first integrals from $\lambda$-symmetries by following steps:

1. Find a first integral $w(x,\theta,\dot{\theta})$ of $v^{[\lambda,(1)]}$, that is a particular solution of the equation

$$w_\theta + \lambda w_{\dot{\theta}} = 0, \tag{3.14}$$

   where $v^{[\lambda,(1)]}$ is the first-order $\lambda$-prolongation of the vector field $v$.
2. The solution of (3.14) is in terms of first-order derivative of $\theta$. To write equation of (3.11) in terms of the reduced equation of $w$, one can obtain the first-order derivative the solution of (3.14) and write (3.11) equation in terms of $w$.
3. Let $G$ is an arbitrary constant of the solution of the reduced equation written in terms of $w$. Therefore,

$$\mu = G_w w_{\dot{\theta}}, \tag{3.15}$$

   is an integrating factor of (3.11).
4. The solution of $w(x,\theta,\dot{\theta})$ is the first integral of $v^{[\lambda,(1)]}$.
5. Finally, the invariant solution can be obtained by this first integral.

## 3.1. Invariant Solutions by using $\lambda$-Symmetries

In this part, we determine exact invariant solutions of nonlinear fin equation using $\lambda$-symmetries for different forms of heat transfer coefficient $h(\theta)$ as a classification.

**Case 1**: For the function $h(\theta) = e^\theta$, $\lambda$-symmetry of (1.6)

$$\lambda = -\sqrt{\sigma}\tan(\sqrt{\sigma}x - \sqrt{\sigma}c_1) - \dot{\theta}(x), \tag{3.16}$$

is determined by

$$k(\theta) = -\frac{e^\theta}{\sigma}. \tag{3.17}$$

To obtain an integrating factor associated to $\lambda$-symmetry, we find a first-order invariant $w(x,\theta,\dot{\theta})$ from solution of Eq. (3.14) as

$$w(x,\theta,\dot{\theta}) = \psi(x)e^\theta(\dot{\theta} + \sqrt{\sigma}\tan(\sqrt{\sigma}x - \sqrt{\sigma}c_1)), \tag{3.18}$$

where $c_1$ is a constant. For the simplicity by taking $\psi(x) = c_1 = 1$, we can express the following equality by using (3.18)

$$\dot{\theta} = e^{-\theta}\left(\sqrt{\sigma}e^\theta \tan(\sqrt{\sigma} - \sqrt{\sigma}\theta + w(x)\right). \tag{3.19}$$

And taking derivative of (3.19) with respect to $x$ gives

$$\ddot{\theta} = e^{-2\theta}\left(\sqrt{\sigma}e^\theta \tan(\sqrt{\sigma}x - \sqrt{\sigma})w(x) - w(x)^2 + e^\theta\left(-\sigma e^\theta \sec(\sqrt{\sigma}x - \sqrt{\sigma})^2) + \dot{w}(x)\right)\right). \tag{3.20}$$

The use of $\dot{\theta}$ and $\ddot{\theta}$ gives the reduced equation

$$e^{-\theta}\left(\dot{w}(x) - \sqrt{\sigma}w(x)\tan\left(\sqrt{\sigma}x - \sqrt{\sigma}\right)\right) = 0, \tag{3.21}$$

and the $w(x)$ function

$$w(x) = G\sec(\sqrt{\sigma}(x-1)), \quad G \in \mathbb{R}. \tag{3.22}$$

To find the integrating factor, above equation is written in terms of $G$ as

$$G = \cos(\sqrt{\sigma}(x-1))w, \tag{3.23}$$

and $\mu$ is obtained in the following form

$$\mu = e^{\theta}\cos(\sqrt{\sigma}(x-1)). \tag{3.24}$$

The conserved form of this equation is given by

$$e^{\theta}\left(\dot{\theta}(x) + \sqrt{\sigma}\tan\left(\sqrt{\sigma}(x-1)\right)\cos\left(\sqrt{\sigma}(x-1)\right)\right) - c_2 = 0, \tag{3.25}$$

where $c_2$ is an integration constant. Thus, the reduced equation is

$$\dot{\theta}(x) - \sqrt{\sigma}\tan(\sqrt{\sigma}(x-1)) - c_2 = 0. \tag{3.26}$$

The integration of the above equation yields the invariant solution

$$\theta(x) = \log\left(\frac{1}{2}c_2\cos\left(\sqrt{\sigma}(x-1)\right) + \frac{c_3\sin\left(\sqrt{\sigma}(x-1)\right)}{\sqrt{\sigma}}\right), \tag{3.27}$$

where $c_3$ is an integration constant.

**Case 2**: For the function $h(\theta) = \theta$, we have

$$\lambda = -\sqrt{\sigma}\tan(\sqrt{\sigma}x - \sqrt{\sigma}), \tag{3.28}$$

and corresponding thermal conductivity is determined as

$$k(\theta) = -\frac{1}{\sigma}. \tag{3.29}$$

Substituting $\lambda$ in (3.14), function $w$ is written as

$$w(x, \theta, \dot{\theta}) = \psi(x)(\dot{\theta} + \sqrt{\sigma}\theta\tan(\sqrt{\sigma}x - \sqrt{\sigma})). \tag{3.30}$$

Using same procedure, integrating factor becomes

$$\mu = \cos(\sqrt{\sigma}(x-1)). \tag{3.31}$$

Thus, the invariant solution is

$$\theta(x) = c_2\cos(\sqrt{\sigma}(x-1)) + \frac{c_3\sin(\sqrt{\sigma}(x-1))}{\sqrt{\sigma}}, \tag{3.32}$$

where $c_2$ and $c_3$ are integration constants.

**Case 3**: The function $h(\theta) = h\theta^{\beta}$, $\beta \neq 1$ gives $\lambda$-symmetry as

$$\lambda = -\sqrt{\sigma}\tan(\sqrt{\sigma}x - \sqrt{\sigma}) - \frac{(\beta-1)\dot{\theta}(x)}{\theta(x)}, \tag{3.33}$$

thus function $k(\theta)$ is

$$k(\theta) = -\frac{h\beta\theta^{\beta-1}}{\sigma}, \tag{3.34}$$

and function $w(x, \theta, \dot{\theta})$ is

$$w(x, \theta, \dot{\theta}) = \frac{\theta^{\beta-1}(\dot{\theta}\beta + \sqrt{\sigma}\theta\tan(\sqrt{\sigma}x - \sqrt{\sigma})}{\beta}, \tag{3.35}$$

and then integrating factor is found as

$$\mu = \cos(\sqrt{\sigma}(x-1))\theta^{\beta-1}. \tag{3.36}$$

Thus, the exact analytic solution of nonlinear fin equation (1.6)

$$\theta(x) = \left(c_2 \cos(\sqrt{\sigma}(x-1)) + \frac{c_3 \beta \sin(\sqrt{\sigma}(x-1))}{\sqrt{\sigma}}\right)^{\frac{1}{\beta}}, \tag{3.37}$$

is found with integration constants $c_2$ and $c_3$.

**Case 4**: Function $h(\theta) = h\theta^{\beta}$, $\beta \neq -1$ gives $\lambda$-symmetry as

$$\lambda = -\sqrt{\sigma}\tan(\sqrt{\sigma}x - \sqrt{\sigma}) + \frac{(\beta+1)\dot{\theta}(x)}{\theta(x)}, \tag{3.38}$$

then functions $k(\theta)$ is

$$k(\theta) = \frac{h\beta\theta^{-\beta-1}}{\sigma}. \tag{3.39}$$

and $w(x, \theta, \dot{\theta})$

$$w(x, \theta, \dot{\theta}) = -\frac{\theta^{-\beta-1}(-\dot{\theta}\beta + \sqrt{\sigma}\theta\tan(\sqrt{\sigma}x - \sqrt{\sigma}c_1)}{\beta}, \tag{3.40}$$

are determined, which gives associated integrating factor

$$\mu = \cos(\sqrt{\sigma}(x-1))\theta^{-\beta-1}. \tag{3.41}$$

Finally, the exact analytic solution of nonlinear fin equation (1.6) for this case is

$$\theta(x) = \left(c_2 \cos(\sqrt{\sigma}(x-1)) - \frac{c_3 \beta \sin(\sqrt{\sigma}(x-1))}{\sqrt{\sigma}}\right)^{-\frac{1}{\beta}}, \tag{3.42}$$

where $c_2$ and $c_3$ are integration constants.

**Case 5**: $h(\theta) = \dfrac{1}{m\theta + n}$; $m$, $n$ are constants. For this case, $\lambda$-symmetry

$$\lambda = -\sqrt{\sigma}\tan(\sqrt{\sigma}x - \sqrt{\sigma}) + \frac{2m\dot{\theta}(x)}{m\theta(x) + n}, \tag{3.43}$$

is obtained and then the associated thermal conductivity function $k(\theta)$ is

$$k(\theta) = -\frac{m}{\sigma(m\theta + n)^2}. \tag{3.44}$$

If one can write $\omega$ function as

$$w(x, \theta, \dot{\theta}) = \frac{(m\dot{\theta} - \sqrt{\sigma}n\tan(\sqrt{\sigma}x - \sqrt{\sigma}) - \sqrt{\sigma}m\theta\tan(\sqrt{\sigma}x - \sqrt{\sigma}))}{m(n + m\theta)^2}, \tag{3.45}$$

and then the integrating factor becomes

$$\mu = \frac{\cos(\sqrt{\sigma}(x-1))}{(n + m\theta)^2}. \tag{3.46}$$

Finally, the exact solution of the fin equation (1.6)

$$\theta(x) = -\frac{(\sqrt{-\sigma}e^{2\sqrt{-\sigma}\theta} + 2ac_2)\cos(\sqrt{\sigma}(x-1)) + (e^{2\sqrt{-\sigma}\theta} + 2\sqrt{-\sigma}c_2)(mnc_3 + \sqrt{\sigma}\sin(\sqrt{\sigma}(x-1)))}{c_3 m^2 (e^{2\sqrt{-\sigma}\theta} + 2\sqrt{-\sigma}c_2)}, \tag{3.47}$$

is obtained, where $c_2$ and $c_3$ are integration constants.

**Case 6**: As the last case, for $h(\theta)=\dfrac{h}{(\beta\theta+\gamma)^2}$, $\lambda$-symmetry

$$\lambda=-\sqrt{\sigma}\tan(\sqrt{\sigma}x-\sqrt{\sigma})+\frac{3\beta\dot{\theta}(x)}{\beta\theta(x)+\gamma}, \tag{3.48}$$

is found. For this case, function $k(\theta)$ is

$$k(\theta)=\frac{2h\beta}{\sigma(\beta\theta+\gamma)^3}, \tag{3.49}$$

and function $w$

$$w(x,\theta,\dot{\theta})=\frac{\theta^{\beta-1}(\dot{\theta}\beta+\theta\sqrt{\sigma}\tan(\sqrt{\sigma}x-\sqrt{\sigma})}{\beta}, \tag{3.50}$$

is determined. Corresponding integrating factor

$$\mu=\cos(\sqrt{\sigma}(x-1))\theta^{\beta-1}, \tag{3.51}$$

is computed and it gives the exact solution

$$\theta(x)=\left(c_2\cos(\sqrt{\sigma}(x-1))+\frac{c_3\beta\sin(\sqrt{\sigma}(x-1))}{\sqrt{\sigma}}\right)^{\frac{1}{\beta}}, \tag{3.52}$$

where $c_2$ and $c_3$ are integration constants.

## 3.2. Exact Solutions of Boundary-value Problems

The exact solutions of boundary-value problems can be obtained by using the tip and the base boundary conditions given by (1.7) and (1.8) related to the nonlinear fin problem (1.6). In the previous subsection, we obtain invariant solutions for fin equation including two arbitrary constant. Whereas by using boundary conditions, we determine these arbitrary constants and then exact solution of boundary-value equation for different cases as below.

**Case 1**: For the function $h(\theta)=e^{\theta}$, the exact solution is found in (3.27). To determine the integration constants, namely $c_1$ and $c_2$, firstly, the base condition (1.8) is applied to (3.27) to find $c_2=2e$. And then if the tip condition (1.7) is applied to (3.27) and then the other constant $c_3$ is determined as

$$c_3=-\sqrt{\sigma}e\tan(\sqrt{\sigma}), \tag{3.53}$$

As a result, the solution of boundary-value problem is given by

$$\theta(x)=\log\left(e\cos\left(x\sqrt{\sigma}\right)\sec\left(\sqrt{\sigma}\right)\right). \tag{3.54}$$

**Case 2**: For the function $h(\theta)=\theta$, the invariant solution is given in (3.32). Now, we determine these constants by using boundary conditions. The base and the tip conditions yield $c_2=1$ and

$$c_3=-\sqrt{\sigma}\tan(\sqrt{\sigma}), \tag{3.55}$$

respectively. Hence, the solution of boundary-value problem is

$$\theta(x)=\cos(x\sqrt{\sigma})\sec(\sqrt{\sigma}). \tag{3.56}$$

**Case 3**: The invariant solution is found in (3.37) for the function $h(\theta)=h\theta^{\beta}$, $\beta\neq 1$. Similarly, by using base condition (3.37), the constant $c_2$ is found as $c_2=1$. Then, the tip condition is applied to yield

$$c_3=-\frac{\sqrt{\sigma}\tan(\sqrt{\sigma})}{\beta}, \tag{3.57}$$

and the solution of boundary-value problem is

$$\theta(x) = (\cos(x\sqrt{\sigma})\sec(\sqrt{\sigma}))^{1/\beta}. \tag{3.58}$$

## 4. THE EXTENDED PRELLE–SINGER METHOD AND $\lambda$-SYMMETRY

In this section, we consider first integrals, corresponding Lagrangian and Hamiltonian forms and exact solutions of fin problem by considering the extended PS method [9] and its the mathematical relation with $\lambda$-symmetry [26] as a different approach from the mathematical point of view.

### 4.1. First Integrals and Associated Lagrangian and Hamiltonian

To obtain the first integral, Lagrangian, and Hamilton functions of fin equation, let us rewrite Eq. (1.6) by using the relation (3.11) as follows

$$\phi = -\frac{k'(\theta)}{k(\theta)}\dot{\theta}^2 + \frac{h(\theta)}{k(\theta)}. \tag{4.1}$$

If Eq. (4.1) has a first integral $I(x, \theta, \dot{\theta}) = C$, with a constant $C$ and then the total differential for the first integral can be written as

$$dI = I_x dx + I_\theta d\theta + I_{\dot{\theta}} d\dot{\theta} = 0. \tag{4.2}$$

The substitution of Eq. (4.2) into the formula $\phi dx - d\dot{\theta} = 0$ by adding a null term $S(x, \theta, \dot{\theta})\dot{\theta}dx - S(x, \theta, \dot{\theta})d\theta$ yields

$$(\phi + S\dot{\theta})dx - Sd\theta - d\dot{\theta} = 0, \tag{4.3}$$

and the multiplication of the relation (4.2) by integrating factor $R(x, \theta, \dot{\theta})$ gives

$$dI = R(\phi + S\dot{\theta})dx - RSd\theta - Rd\dot{\theta} = 0. \tag{4.4}$$

The comparison of the equations given by (4.2) and (4.4) provides the following relations

$$I_x = R(\phi + S\dot{\theta}), \quad I_\theta = -RS, \quad I_{\dot{\theta}} = -R. \tag{4.5}$$

Besides, the use of the compatibility conditions, namely $I_{x\theta} = I_{\theta}$, $I_{x\dot{\theta}} = I_{\dot{\theta}x}$, $I_{\theta\dot{\theta}} = I_{\dot{\theta}\theta}$ with the relations in (4.5) ensures the following system of coupled nonlinear differential equations in terms of $S$, $R$, and $\phi$ as

$$S_x + \dot{\theta}S_\theta + \phi S_{\dot{\theta}} = -\phi_\theta + \phi_{\dot{\theta}}S + S^2, \tag{4.6}$$

$$R_x + \dot{\theta}R_\theta + \phi R_{\dot{\theta}} = -(\phi_{\dot{\theta}} + S)R, \tag{4.7}$$

$$R_\theta - SR_{\dot{\theta}} - RS_{\dot{\theta}} = 0, \tag{4.8}$$

where the last Eq. (4.8) is called *compatibility equation*. In addition, one can determine the first integral $I$ by using functions $R$ and $S$ by following relations

$$I = r_1 - r_2 - \int\left[R + \frac{d}{d\dot{\theta}}(r_1 - r_2)\right]d\dot{\theta}, \tag{4.9}$$

where

$$r_1 = \int R(\phi + \dot{\theta}s)dx, \quad r_2 = \int\left(RS + \frac{d}{d\theta}r_1\right)d\theta. \tag{4.10}$$

First of all, let us consider time-independent first integral case, that is $I_x = 0$. Accordingly, one can easily find null function $S$ from the first equation in (4.5) of the form

$$S = \frac{-\phi}{\dot{\theta}} = -\frac{\dot{\theta}^2 k'(\theta) - h(\theta)}{\dot{\theta} k(\theta)}. \tag{4.11}$$

By substituting $S$ into Eq. (4.7), we get

$$R_x + R_\theta \dot{\theta} + R_{\dot{\theta}} \left( \frac{h(\theta) - \dot{\theta}^2 k'(\theta)}{k(\theta)} \right) + R \left( \frac{-2\dot{\theta} k'(\theta)}{k(\theta)} - \frac{h(\theta) - \dot{\theta}^2 k'(\theta)}{\dot{\theta} k(\theta)} \right). \tag{4.12}$$

In fact, Eq. (4.12) is a first-order linear partial differential equation and in order to search an explicit solution for Eq. (4.12), one can assume the integrating factor $R$ of the form

$$R = \frac{\dot{\theta}}{(A(\theta) + B(\theta)\dot{\theta} + C(\theta)\dot{\theta}^2)^r}, \tag{4.13}$$

where $A(\theta)$, $B(\theta)$, and $C(\theta)$ are functions of $\theta$ and $r$ is a constant. If we substitute (4.13) into Eq. (4.12), then we obtain a set of equations in terms of $\dot{\theta}$ and its powers. From the solutions of these equations, we have

$$A(\theta) = c_1 k(\theta)^{-\frac{2}{r}} - 2c_2 \left( \int h(\theta)k(\theta)d\theta \right) k(\theta)^{-\frac{2}{r}}, \tag{4.14}$$

$$B(\theta) = c_3 k(\theta)^{1-\frac{2}{r}}, \tag{4.15}$$

$$C(\theta) = c_2 k(\theta)^{2-\frac{2}{r}}, \tag{4.16}$$

where $c_1$, $c_2$, and $c_3$ are arbitrary constants. Thus, the integration factor $R$ is written as

$$R = \dot{\theta} \left( k(\theta)^{-\frac{2}{r}} (c_1 - 2c_2 \int h(\theta)k(\theta)d\theta + c_2 \dot{\theta}^2 k(\theta)^2) \right)^{-r}. \tag{4.17}$$

One can easily check that the functions $R$ (4.17) and $S$ (4.11) satisfy Eqs. (4.6)–(4.8). The associated first integral from Eq. (4.9)

$$\begin{aligned} I = &-\int \dot{\theta}(-2\int k(\theta)^{-\frac{2+r}{r}} (k(\theta)^{-\frac{2}{r}} (c_1 - 2c_2 \int h(\theta)k(\theta)d\theta + c_2 \dot{\theta}^2 k(\theta)^2)^{-1-r} \\ &(c_2 r h(\theta)k(\theta)^2 + (c_1 - 2c_2 \int h(\theta)k(\theta)d\theta - c_2(-1+r)\dot{\theta}^2 k(\theta)^2)k'(\theta))d\theta \\ &+(k(\theta)^{-\frac{2}{r}} (c_1 - 2c_2 \int h(\theta)k(\theta)d\theta + c_2 \dot{\theta}^2 k(\theta)^2)^{-r} d\dot{\theta} \\ &-\frac{\int k(\theta)^{-\frac{2}{r}} (c_1 - 2c_2 \int h(\theta)k(\theta)d\theta + c_2 \dot{\theta}^2 k(\theta)^2)^{-r} (-h(\theta) + \dot{\theta}^2 k'(\theta))}{k(\theta)} d\theta, \end{aligned} \tag{4.18}$$

is written. As an example, for $r = -2$, and the linear functions $k(\theta) = \theta$ and $h(\theta) = \theta$, the first integral of the fin equation

$$I = \frac{1}{162}\theta^2(2\theta - 3\dot{\theta}^2)(27c_1^2 - 9c_1c_2\theta^2(2\theta - 3\dot{\theta}^2) + c_2^2\theta^4(2\theta - 3\dot{\theta}^2)^2), \tag{4.19}$$

is obtained. Furthermore, one can determine the corresponding conjugate momentum related with the first integral (4.19) as below

$$p = \frac{1}{9}(-\theta^2(-3c_1 + 2c_2\theta^3)^2 \dot{\theta} + 2c_2\theta^4(-3c_1 + 2c_2\theta^3)\dot{\theta}^3 - \frac{9}{5}c_2^2\theta^6 \dot{\theta}^5). \tag{4.20}$$

Then the corresponding Lagrangian is

$$L=\frac{1}{9}\dot{\theta}(-\theta^2(-3c_1+2c_2\theta^3)^2\dot{\theta}+2c_2\theta^4(-3c_1+2c_2\theta^3)\dot{\theta}^3-\frac{9}{5}c_2^2\theta^6\dot{\theta}^5)-\frac{1}{162}\theta^2(2\theta-3\dot{\theta}^2)(27c_1^2-9c_1c_2\theta^2(2\theta-3\dot{\theta}^2)+c_2^2\theta^4(2\theta-3\dot{\theta}^2)^2), \tag{4.21}$$

and Hamiltonian form related with first integral (4.18) is determined as

$$H=\frac{1}{162}\theta^2(2\theta-3\dot{\theta}^2)(27c_1^2-9c_1c_2\theta^2(2\theta-3\dot{\theta}^2)+c_2^2\theta^4(2\theta-3\dot{\theta}^2)^2). \tag{4.22}$$

## 4.2. $\lambda$-Symmetries Determined from Lie Point Symmetries

The relation between $\lambda$-symmetries and Lie point symmetries for determination of integrating factors and first integrals of a second-order ordinary differential equation is important from mathematical point of view, based on fact that if Lie point symmetries of an equation are known and then one can construct $\lambda$-symmetries from them. For this purpose, firstly, we suppose that an equation has Lie symmetries, then determine $\lambda$-symmetries and find corresponding first integrals by using relation between the null function $S$ and $\lambda$-symmetries.

**Remark 4.1.** In the study [33], Lie symmetries and associated $\lambda$-symmetries are investigated and then the invariant solutions are determined. However, in this current study, we consider not only Lie point symmetries and related $\lambda$-symmetries but also the mathematical relation between $\lambda$-symmetries and PS method to determine invariant solutions.

**Proposition 4.1.** *If $\lambda$-symmetries are obtained from Lie symmetries and then the integrating factors of the equation can be determined from these $\lambda$-symmetries.*

*Proof.* Now, we first follow an algorithm that gives $\lambda$-symmetries from Lie symmetries. For this purpose, let us consider the following second-order ordinary differential equation

$$\ddot{\theta}=\phi(x,\theta,\dot{\theta}). \tag{4.23}$$

Let the vector field of Eq. (4.23) be in the form of

$$A=\partial_x+\dot{\theta}\partial_\theta+\phi(x,\theta,\dot{\theta})\partial_{\dot{\theta}}, \tag{4.24}$$

then in terms of $A$, a first integral of (4.23) is any function in the form of $I(x,\theta,\dot{\theta})$ providing equality of $A(I)=0$. Additionally, an integrating factor of Eq. (4.23) is any function satisfying the following equation

$$\mu[\ddot{\theta}-\phi(x,\theta,\dot{\theta})]=D_xI, \tag{4.25}$$

where $D_x$ is total derivative operator in the form of

$$D_x=\partial_x+\dot{\theta}\partial_\theta+\ddot{\theta}\partial_{\dot{\theta}}. \tag{4.26}$$

Thus, $\lambda$-symmetries of second-order differential equation (4.23) can be obtained directly by using Lie symmetries of the same equation. Secondly, let

$$\upsilon=\xi(\theta,x)\frac{\partial}{\partial x}+\eta(\theta,x)\frac{\partial}{\partial\theta}, \tag{4.27}$$

be a Lie point symmetry of (4.23), where $\xi(\theta,x)$ and $\eta(\theta,x)$ are functions of their arguments. If the characteristic function of the infinitesimal operator $\upsilon$ is written as

$$Ş=\eta-\xi\dot{\theta}, \tag{4.28}$$

and then $\lambda$-symmetry is given by

$$\lambda=\frac{A(Ş)}{Ş}. \tag{4.29}$$

If $\partial_\theta$ is assumed to be a $\lambda$-symmetry of (4.23) and $\omega(x,\theta,\dot{\theta})$ is a first-order invariant of $\theta^{[\lambda,(1)]}$, namely any particular solution of the equation

$$\omega_\theta + \lambda(x,\theta,\dot{\theta})\omega_{\dot{\theta}} = 0, \tag{4.30}$$

and then one can evaluate $A(w)$ and express $A(w)$ in terms of $(x, w)$ as $A(w) = F(x, w)$ and obtain a first integral $G$ as $I(x,\theta,\dot{\theta}) = G(x, w(x,\theta,\dot{\theta}))$. It is clear that $D_t(G(x,\omega(x,\theta,\dot{\theta}))=0$ is an equivalent form of (4.23).

Consequently,

$$\mu(x,\theta,\dot{\theta}) = G_\omega(x,\theta,\omega(x,\theta,\dot{\theta}))\omega_t(x,\theta,\dot{\theta}), \tag{4.31}$$

is an integrating factor of (4.23).

**Proposition 4.2.** *Suppose that $v = \partial_\theta$ is a $\lambda$-symmetry. If $I$ is a first integral of (4.23), then $\mu = I_{\dot{\theta}}$ is an integrating factor of (4.23) and $-\mu\phi = I_x + \dot{\theta} I_\theta$. The compatibility conditions for system (4.5) imply*

$$A(S) = -\phi_\theta + S\phi_{\dot{\theta}} + S^2, \qquad A(R) = -R(S+\phi_{\dot{\theta}}), \qquad R_\theta = R_{\dot{\theta}}S + RS_{\dot{\theta}}. \tag{4.32}$$

*The first equation in (4.32) says that $v = \partial_\theta$ is a $\lambda$-symmetry for $\lambda = -S$. By writing the second and third equations of $\lambda = -S$ (4.32) in terms of $\lambda$ we obtain $\mu = -R$. This reveals a role of $\lambda = -S$ on the integration of Eq. (4.23), if we add $-\lambda(\dot{\theta}dx - d\theta)$ to the differential form $\phi dx - d\dot{\theta}$, then the resulting differential form $\phi dx - d\dot{\theta} - \lambda(\dot{\theta}dx - d\theta)$ admits an integrating factor $\mu$. Thus, it means that the function $S$ corresponds to Lie point symmetries of an equation, which is given by $S = -\frac{A(Q)}{Q}$.*

Now, we apply this algorithm to the fin equation to obtain nontrivial $\lambda$-symmetries.

**Proposition 4.3.** *Using compatibility condition, one-parameter Lie point symmetry of fin equation such that $v = \frac{\partial}{\partial x}$ can be obtained and the use of Lie point symmetry yields associated $\lambda$-symmetry, first integral, Lagrangian, conjugate momentum, and Hamiltonian function.*

*Proof.* For the function $\phi$ defined by (4.1), the operator $A$ (4.24)

$$A = \frac{\partial}{\partial x} + \dot{\theta}\frac{\partial}{\partial\theta} - \left(\frac{h(\theta)}{k(\theta)} - \frac{k'(\theta)}{k(\theta)}\dot{\theta}^2\right)\frac{\partial}{\partial\dot{\theta}}, \tag{4.33}$$

is computed and from Eq. (4.29), $\lambda$-symmetry

$$\lambda = \frac{h(\theta)\xi - \dot{\theta}^2\xi k'(\theta) + k(\theta)(-\eta_\theta\dot{\theta} + \dot{\theta}^2\xi_\theta - \eta_x + \dot{\theta}\xi_x}{k(\theta)(\eta - \dot{\theta}\xi)}. \tag{4.34}$$

is found. Using the relation between $S$ and $\lambda$, that is $S = -\lambda$, one can determine null function $S$ satisfying Eq. (4.6), then if we substitute this form of $S$ into (4.6) and then a system of over-determined partial differential equations is obtained. From solutions of this system, one can determine infinitesimal functions $\xi$ and $\eta$, which correspond to Lie point symmetries of fin equation as

$$\xi = c_1 = \text{constant and } \eta = 0, \tag{4.35}$$

or equivalently in the operator form

$$v = \frac{\partial}{\partial x}. \tag{4.36}$$

It means that fin equation has only one-parameter of symmetry group. Thus, if we substitute functions $\xi$ and $\eta$ (4.35) into (4.34), the expression

$$S = -\lambda = -\frac{h(\theta)\xi - \dot{\theta}^2\xi k'(\theta) + k(\theta)(-\eta_\theta\dot{\theta} + \dot{\theta}^2\xi_\theta - \eta_x + \dot{\theta}\xi_x}{k(\theta)(\eta - \dot{\theta}\xi)}. \tag{4.37}$$

is obtained. To determine a first integral $w(x, \theta, \dot{\theta})$ of $v^{[\lambda,(1)]}$, we consider a particular solution of the equation

$$w_\theta + \lambda w_{\dot{\theta}} = 0, \tag{4.38}$$

where $v^{[\lambda,(1)]}$ is the first-order $\lambda$-prolongation of the vector field $v$ [26] and [24]. The solution of (4.38) can be written in terms of first-order derivative of $\theta$ function. In order to write reduced equation of (4.23) in terms of $w$, one can obtain the first-order derivative for the solution of (4.38) and then write (4.23) equation in terms of $w$ function. Hence, if we substitute the associated $\lambda$ function into differential equation (4.38) then function $w$

$$w = -2\int h(\theta)k(\theta)d\theta + 2\dot{\theta}^2 k(\theta)^2, \tag{4.39}$$

is obtained. It is easy to see that we find a first integral as below

$$I = -2\int h(\theta)k(\theta)d\theta + 2\dot{\theta}^2 k(\theta)^2, \tag{4.40}$$

and the corresponding integrating factor is determined as

$$\mu = 2\dot{\theta}k(\theta)^2, \tag{4.41}$$

and since $\mu = -R$, then we have

$$R = -2\dot{\theta}k(\theta)^2. \tag{4.42}$$

In fact, it is possible to show that $S$ (4.37) and $R$ (4.42) satisfy Eqs. (4.6)–(4.8). Furthermore, one can determine canonical conjugate momentum related with first integral (4.40)

$$p = 2\dot{\theta}k(\theta)^2. \tag{4.43}$$

And the corresponding Lagrangian

$$L = 2\int h(\theta)k(\theta)d\theta + \dot{\theta}^2 k(\theta)^2, \tag{4.44}$$

and Hamiltonian

$$H = 2\int h(\theta)k(\theta)d\theta, \tag{4.45}$$

are obtained in terms of temperature-dependent thermal conductivity and heat transfer coefficients.

## 5. A SPECIAL APPLICATION

In this section, the thermal conductivity in nonlinear fin equation is considered as a linear function of temperature as a special application having physical meaning in the literature (see [19] and [18]) as

$$K(T) = K_a[1 + \gamma(T - T_a)] \text{ or } k(\theta) = 1 + \beta\theta, \tag{5.1}$$

where $\gamma$ is the parameter to describe the dependence on temperature, $K_a$ is the thermal conductivity and $\beta = \gamma(T_a - T_b)$ is its corresponding nondimensionalized parameter. Firstly, for an arbitrary thermal conductivity function $k(\theta)$, if fin equation (1.6) is substituted in the first compability condition (2.22) and then one can write the following equation

$$\begin{aligned} &\eta k'(\theta)(-h(\theta) + \dot{\theta}^2 k'(\theta)) + k(\theta)(\eta(h'(\theta) - \dot{\theta}^2 k''(\theta)) - \dot{\theta}k'(\theta)(\dot{\theta}\eta_\theta + \dot{\theta}\xi_\theta) + 2\eta_x \\ &+ h\ (\theta)(-\eta_\theta + 3\dot{\theta}\xi_\theta + 2\xi_x)) + k(\theta)^2(-\eta_{xx} + \dot{\theta}(-2\eta_{x\theta} + \dot{\theta}(-2\eta_{\theta\theta} + \dot{\theta}\xi_{\theta\theta} + 2\xi_{t\theta}) + \xi_{xx})) = 0\ . \end{aligned} \tag{5.2}$$

In addition, Eq. (5.2) is separated according to the power of derivatives with respect to $\theta$ to obtain determining equations in the following form

$$\xi_{\theta\theta} - \frac{k'(\theta)}{k(\theta)^2}\xi_\theta = 0, \tag{5.3}$$

$$\eta_{\theta\theta} + \frac{k'(\theta)}{k(\theta)}\eta_\theta + \frac{k''(\theta)}{k(\theta)}\eta - \frac{k'(\theta)^2}{k(\theta)^2}\eta - 2k(\theta)^3 a'(x) = 0, \tag{5.4}$$

$$3a(x)h(\theta) - 2c'(x) - 3(a''(x)\int k(\theta)d\theta + b''(x)) = 0, \tag{5.5}$$

where $a(x)$, $b(x)$, and $c(x)$ are arbitrary functions. First, the integration of Eq. (5.3) yields

$$\xi = a(x)\int k(\theta)d\theta + b(x), \tag{5.6}$$

The solution of the second determining equations gives

$$\eta = \frac{c(x)\int k(\theta)d\theta + (\int k(\theta)d\theta)^2 a'(x)}{k(\theta)} + \frac{d(x)}{k(\theta)}, \tag{5.7}$$

where $d(x)$ is an arbitrary function. The infinitesimal functions $\xi$ and $\eta$ are inserted to the last determining equation (5.5) and then the heat transfer coefficient is determined as below

$$h(\theta) = \frac{3a''(x)\int k(\theta)d\theta + 2c'(x) - b''(x)}{3a(x)}. \tag{5.8}$$

Secondly, by taking the thermal conductivity $k(\theta)$ as a linear function of temperature as in (5.1) and by substituting (5.8) into (5.2), all coefficients $a(x)$, $b(x)$, $c(x)$, and $d(x)$ are determined. As a result, the heat transfer coefficient as a power function of temperature and associated infinitesimal functions will be as below

$$h(\theta) = e^{2c_2}(\theta + \frac{\theta^2\beta}{2}), \tag{5.9}$$

$$\xi(\theta, x) = c_4 + \frac{1}{4}e^{-2c_2 - e^{c_2}(c_3+x)}(c_1e^{2c_2} + e^{2e^{c_2}(c_3+x)})\theta(2+\theta\beta), \tag{5.10}$$

$$\begin{aligned}\eta(\theta, x) = &\frac{1}{8(1+\theta\beta)}\Big(e^{-c_2 - e^{c_2}(c_3+x)}(8c_6e^{c_2 + c_3e^{c_2}} + 8c_5e^{c_2 + e^{c_2}(c_3+2x)} \\ &+\theta(2+\theta\beta)\Big(4e^{3c_2 + e^{c_2}(c_3+x)} - c_1e^{2c_2}\theta(2+\theta\beta) + e^{2e^{c_2}(c_3+x)}\theta(2+\theta\beta)\Big)\Big),\end{aligned} \tag{5.11}$$

where $c_1$, $c_2$, $c_3$, $c_4$, $c_5$, and $c_6$ are group parameters and $e$ is a exponential function. To determine a specific solution, let us take $c_1 = 1$ and other constants are zero. Hence, the infinitesimal functions become

$$\xi(\theta, x) = \frac{1}{2}e^{-x}(1+e^{2x})\left(\theta + \frac{\theta^2\beta}{2}\right). \tag{5.12}$$

$$\eta(\theta, x) = \frac{e^{-x}\theta(2+\theta\beta)(4e^x - \theta(2+\theta\beta) + e^{2x}\theta(2+\theta\beta))}{8(1+\theta\beta)},$$

For this case, the thermal conductivity and the heat transfer coefficient are

$$k(\theta) = 1 + \beta\theta \text{ and } h(\theta) = \left(\theta + \frac{\theta^2\beta}{2}\right). \tag{5.13}$$

First, the characteristic equation

$$Q(x, \theta, \dot{\theta}) = \frac{1}{4+4\theta\beta}\theta(2+\theta\beta)(2 - 2\dot{\theta}(1+\theta\beta)\cosh x + \theta(2+\theta\beta)\sinh x), \tag{5.14}$$

is determined and then $\lambda$-symmetry

$$\lambda=\frac{\dot{\theta}(2+\theta\beta(2+\theta\beta))}{\theta(1+\theta\beta)(2+\theta\beta)}, \tag{5.15}$$

is found by using the characteristic equation (5.14). The null function $S$ is determined from relation $S=-\lambda$. Thus, the corresponding function $w$ is

$$w(x,\theta,\dot{\theta})=\frac{\dot{\theta}(1+\theta\beta)}{\theta(2+\theta\beta)}, \tag{5.16}$$

By evaluating $A(w)$, one can have

$$A(w)=\frac{1}{2}-2w^2=F(x,w), \tag{5.17}$$

and

$$G(x,w)=\log\left(-\frac{(e^{2x}-2e^{2x}w)^{1/4}}{(1+2w)^{1/4}}\right). \tag{5.18}$$

If $G(x,w)$ is expressed in terms of $(x,\theta,\dot{\theta})$ and then first integral

$$I(x,\theta,\dot{\theta})=\log\left(-\frac{\left(e^{2x}\left(1-\dfrac{2\dot{\theta}(1+\theta\beta}{\theta(2+\theta\beta)}\right)\right)^{\frac{1}{4}}}{\left(1+\dfrac{2\dot{\theta}(1+\theta\beta}{\theta(2+\theta\beta)}\right)^{\frac{1}{4}}}\right), \tag{5.19}$$

is obtained. The integration of first integral with respect to $\dot{\theta}$ by using the relation $R=-\mu$ yields

$$R=\frac{\theta(1+\theta\beta)(2+\theta\beta)}{\left(2(\theta+\dot{\theta})+\theta(\theta+2\dot{\theta})\beta\right)\left(\theta(2+\theta\beta-2\dot{\theta}\beta)-2\dot{\theta}\right)}. \tag{5.20}$$

If the functions $R$ and $S$ are substituted into Eqs. (4.6)–(4.8), one can check that $S$ and $R$ also satisfy these equations. First, from the solution of Eq. (5.19), the invariant solution

$$\theta(x)=\frac{e^{-x}\left(-e^{x}+\sqrt{e^{2x}+e^{x+2c_1}(e^{4c_2}+e^{2x})\beta}\right)}{\beta}, \tag{5.21}$$

is found and then the use of tip $\dot{\theta}(0)=0$ (1.7) and base $\theta(1)=1$ (1.8) conditions for fin problem yields

$$c_1=\log\left(\frac{e}{1+e^2}\sqrt{2+\beta}\right) \text{ and } c_2=\frac{1}{4}\left(1-2c_1+\log(2-e^{1+2c_1}+\beta)\right). \tag{5.22}$$

Finally, the solution of boundary-value problem is written as

$$\theta(x)=\frac{e^{-x}\left(-e^{x}\sqrt{1+e^2}+\sqrt{e^{x}\left(e^{x}+e^{x+2}+e\beta(2+\beta)+e^{1+2x}\beta(2+\beta)\right)}\right)}{\beta\sqrt{1+e^2}}. \tag{5.23}$$

Figure 1 and Figure 2 show the evaluation of the effects of design parameters on temperature for some values of $\beta$ parameter.

As another case, we consider that $c_2 = 1$ and other constants are zero. For this case, the infinitesimals take the form

$$\xi(\theta, x) = \frac{1}{4}e^{(ex-2)}\theta(2+\theta\beta), \tag{5.24}$$
$$\eta(\theta, x) = \frac{\theta(2+\theta\beta)(4e^3+\theta e^{ex}(2+\theta\beta))}{8(e+e\theta\beta)}.$$

It is clear that the associated thermal conductivity and the heat transfer coefficients are

$$k(\theta) = 1+\beta\theta \text{ and } h(\theta) = e^2\left(\theta+\frac{\theta^2\beta}{2}\right). \tag{5.25}$$

And the characteristic equation is

$$Q(x, \theta, \dot{\theta}) = \frac{e^{(-1-ex)}\theta(2+\theta\beta)\left(4e^{(3+ex)}+\theta e^{2ex}(2+\theta\beta)\right)}{8(1+\theta\beta)} - \frac{1}{4}e^{ex-2}\theta\dot{\theta}(2+\theta\beta) \tag{5.26}$$

The $\lambda$-symmetry is obtained as

$$\lambda = \frac{\dot{\theta}(2+\theta\beta(2+\theta\beta))}{\theta(1+\theta\beta)(2+\theta\beta)}. \tag{5.27}$$

The null function $S$, which is defined by PS method, is

$$S = -\frac{\dot{\theta}(2+\theta\beta(2+\theta\beta))}{\theta(1+\theta\beta)(2+\theta\beta)}. \tag{5.28}$$

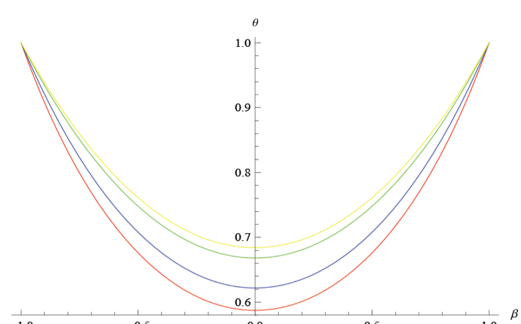


**Figure 1** | Plot of the temperature (5.23) of the fin for different values of $\beta$.

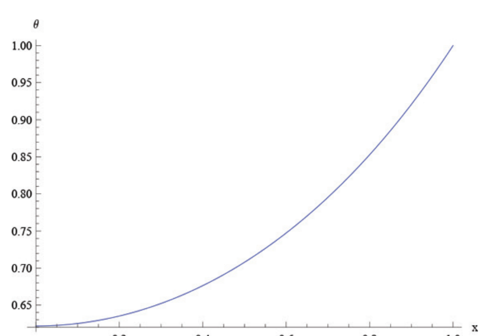


**Figure 2** | The change of the temperature (5.23) of the fin for $\beta = -0.2$.

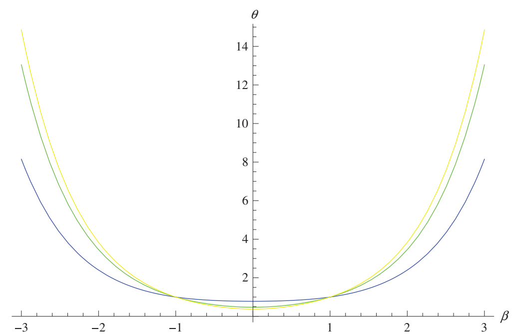


**Figure 3** | Plot of the temperature (5.32) of the fin for different values of $\beta$.

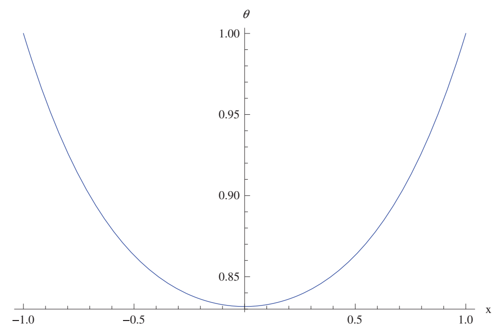


**Figure 4** | The change of the temperature (5.32) of the fin for $\beta = -2.5$.

The use of previous same procedure yields the integrating factor as

$$R = -\frac{\theta(1+\theta\beta)(2+\theta\beta)}{\left(e\theta(2+\theta\beta)-2(\dot{\theta}+\beta\theta\dot{\theta})\right)\left(e\theta(2+\theta\beta)+2(\dot{\theta}+\beta\theta\dot{\theta})\right)}. \tag{5.29}$$

It can be shown that $R$ and $S$ also satisfy compatibility condition. The corresponding exact invariant solution can be determined as

$$\theta(x) = \frac{-1+e^{-ex}\sqrt{e^{2ex}+\beta e^{(ex+2c_1)}+\beta e^{(4c_2e+3ex+2c_1)}}}{\beta}. \tag{5.30}$$

In addition, the application of boundary conditions gives

$$c_1 = \log\left(-\frac{e^{e/2}\sqrt{2+\beta}}{\sqrt{1+e^{2e}}}\right) \text{ and } c_2 = \frac{-2(c_1+e)+\log(-e^{2c_1}+e^e(2+\beta))}{4e}. \tag{5.31}$$

Finally, one can write the analytical solution of boundary-value problem as below

$$\theta(x) = -\frac{e^{-ex}\left(e^{ex}+e^{e(2+x)}+\sqrt{1+e^{2e}}\sqrt{e^{ex}\left(e^{ex}+e^{e(2+x)}+e^{e}\beta(2+\beta)+e^{e+2ex}\beta(2+\beta)\right)}\right)}{(1+e^{2e})\beta}. \tag{5.32}$$

Figure 3 and Figure 4 show the evaluation of the effects of design parameters on temperature for some values of $\beta$ parameter.

## 6. CONCLUSION

In this study, a nonlinear fin problem in a one-dimensional model describing heat transfer with temperature dependent thermal conductivity and heat transfer coefficient is investigated by using the symmetry-transformation approach including $\lambda$-symmetries and Lie point symmetries. First, we analyze first integrals, integrating factor, and exact solutions of nonlinear fin equation by considering $\lambda$-symmetries and Lie point symmetries. Here, we suppose thermal conductivity and heat transfer coefficient as variable functions of temperature. From the mathematical point of view, it can be said that this problem is highly nonlinear. For different heat transfer coefficient and thermal conductivity functions, we obtain first integrals, $\lambda$-symmetries, and integrating factors. Finally, we determine the corresponding invariant solutions for each case. In addition, we determine time-independent first integrals by using the modified Prelle–Singer approach and present the mathematical relations between these approaches. Moreover, we construct Lagrangian and Hamiltonian functions from time independent first integrals and transformed corresponding Hamiltonian forms to standard Hamiltonian forms. Via the Prelle–Singer procedure, the explicit solutions satisfying all three determining equations (4.6)–(4.8) are determined. In this study, a specific ansatz forms to determine the null forms *S*, and integrating factor *R* is considered. The results related to this ansatz validate with the solution for the special case and a satisfactory accuracy is observed.

Furthermore, we investigate not only invariant solutions but also solutions of boundary-value problems related to fin problem. After obtaining the invariant analytical solutions, the boundary conditions at the tip and at the base of the fin are applied. For special choices of thermal conductivity and heat transfer coefficient functions, the solutions of boundary-value problem are determined for different cases. Furthermore, a special application with physical meaning is considered. In this case, the thermal conductivity in the nonlinear fin equation is taken as a linear function of temperature and the heat transfer coefficient is determined as a power function of temperature, which are functions considered usually for analysis of fin problem and for these special types of functions, numerical and approximate analytical solutions are carried out in the literature. However, in this current study, we represent the exact analytical solutions for the same problem. Further, to obtain the exact invariant solution and associated solution of boundary-value problem, the cases $c_1 = 1$ and $c_2 = 1$ are only considered. It is clear that the other types of analytical solutions can be investigated for other group parameters. In addition, the graphical interpretations of the solution as temperature function with respect to different values of a nondimensionalized parameter $\beta$ and space variable $x$ are represented.

## CONFLICTS OF INTEREST

The authors declare they have no conflicts of interest.

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