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Research Article

Orbits and Lagrangian Symmetries on the Phase Space

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ABSTRACT

In this article, given a regular Lagrangian system L on the phase space TM of the configuration manifold M and a 1-parameter group G of transformations of M whose lifting to TM preserve the canonical symplectic dynamics associated to L, we determine conditions so that its infinitesimal generator produces conservation laws, in terms of the orbits of G in TM.

1. INTRODUCTION

Variational geometry is a central theme in mathematics and in mathematical physics. In its development, it uses methods of the calculus of variations, differential geometry and global analysis and is based on different geometric structures: tangent spaces, fibered manifolds, etc. See for example the great work [13] for an intrinsic characterization of variational problems (variation formulas, symmetries and conserved quantities, Dirac theory of constraints, etc.) on fibered manifolds.

In the time independent case, Lagrangian theory, is a theory on the tangent bundle of the configuration space. Therefore, it is the geometry itself of the tangent bundle that offers its richness in the construction of a Lagrangian system. Thus, it is a well known fact that to every Lagrangian system on the phase space $L : TM \to \mathbb{R}$ it is possible to associate an exact 2-form w_L on which Poisson algebra of functions on TM it is established the essential dynamical quantities of the system.

When the function L defines a regular Lagrangian system, the metric w_L is irreducible, establishing in this way the base for a dynamical theory. We believe that this splendid construction of autonomous Variational Calculus starts with J. Klein [11,12] and Grifone [9], providing a fundamental structure capable to place the theory in a unifying position in many local and global concepts of the Calculus of Variations (physical quantities, Noether invariants, infinitesimal symmetries, dynamical variables, etc.); it even allows to see its reflection in the multiphase construction of the Classical Field Theories (see, by example, [10]).

It is a remarkable fact that in this framework, the group of Noether symmetries of the Lagrangian system L, beyond being a group of point transformations it claims a deep geometrical meaning in terms of symplectic symmetries of the Hamiltonian dynamical system $i_{\zeta_L} w_L = dE_L$.

It is not an easy task to find new invariants on the critical locus of a Variational problem. For Lagrangian systems some significant advances are Prince [15] Crampin [4], Marmo and Mukunda [14], de León and Martín [5]. Here, roughly speaking, we prove that for a 1-parameter group G of transformations of M whose action in (TM, w_L) is symplectic, the corresponding infinitesimal generator produce first integrals of the motion for the ζ_L-dynamics whenever the *Legendre function* of the system is constant on the orbits of G.

2. SYMPLECTIC DYNAMICS OF LAGRANGIAN SYSTEMS

Let M be an n-dimensional differentiable manifold and TM its tangent bundle. Let us denote by $\pi_M : TM \to M$ the canonical projection. Given a coordinate neighborhood $U \subset M$ with local coordinates (x_i), let us denote with TU the corresponding neighborhood in TM coordinated by (x_i, v_j).

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Given $x \in M$, and $y \in T_xM$, a vector $v \in T_y(TN)$ such that $\pi_{M_*}(v) = 0$ is called a vertical vector. A vector field X on TN is called vertical if X_y is a vertical vector for each $y \in TN$. An element $u \in T_xM$ determines a vertical vector at any point y in the π_M-fiber over x, as the tangent vector at $t = 0$ to the curve $t \to y + tu$. Naturally, we can define the vertical lift X^v of a vector field X on N. Locally, if $X = X_i\partial/\partial x_i$, then

$$X^v = X_i\partial/\partial v_i.$$

On the other hand, if ϕ_t is a local 1-parameter group of transformations of M with the vector field X as its generator, then it has a natural lift as a local 1-parameter group $T\phi_t$ of transformations of TM whose generator is called the complete lift of X to TN and is denoted by X^c. In local coordinates, if $X = X_i\partial/\partial x_i$, then

$$X^c = X_i\partial/\partial x_i + v_j\partial X_i/\partial x_j\partial/\partial v_i$$

(one may consult Crampin [4] or de Leon-Rodrigues [6]).

Let us consider the canonical almost tangent structure J on TM locally defined by the $(1, 1)$ tensor field

$$J = \frac{\partial}{\partial v_i} \otimes dx_i.$$

We define the *Liouville* vector field C on TM as the one given in local coordinates by the expression

$$C = v_i\frac{\partial}{\partial v_i}.$$

Let us consider a regular Lagrangian function $L : TM \to \mathbb{R}$, that is, the Hessian matrix in every coordinate neighborhood (x_i, v_j)

$$\left(\frac{\partial^2 L}{\partial v_i\partial v_i}\right),$$

is invertible. As a consequence, the 2-form on TM

$$w_L = -d\theta_L$$

where $\theta_L = dL \circ J$, is nondegenerate (see, for example, [7]).

In this way, we define the symplectic dynamics associated to the Lagrangian system L as the determined by the Hamilton equation

$$i_{\zeta_L} w_L = dE_L, \tag{2.1}$$

where $E_L = CL - L$ is the energy function corresponding to L.

It is useful to consider the expression in local coordinates of the *Legendre* 1-form θ_L,

$$\theta_L = \sum_i \frac{\partial L}{\partial v_i} dx_i.$$

The key fact is that the Hamiltonian vector field in (1) has the local coordinate expression

$$\zeta_L = v_i\frac{\partial}{\partial x_i} + a_k(x_i, v_j)\frac{\partial}{\partial v_k} \tag{2.2}$$

where a_k are smooth functions $(1 \le k \le n)$. Consult Crampin [3] or Gotay and Nester [8]; also the monograph [6] goes through all the theory in its Chapter 7. This fact defines ζ_L as a second-order differential equation field; the projections of its integral curves onto M are the solutions of the system

$$\ddot{x}_k = a_k(x_i, \dot{x}_j)$$

which comprise the Euler-Lagrange equations for the extremals of the Lagrangian L.

Finally, it easily follows from (2.2) the following expression for the *Legendre function* $\theta_L(\zeta_L)$ of the Lagrangian system L,

$$\theta_L(\zeta_L) = CL. \tag{2.3}$$

3. *G*-ORBITS vs. ζ_L-ORBITS: CONSERVATIONS LAWS

The search of constants of motion corresponding to 1-parameter groups of symmetries in classical mechanics, has had since Emmy Noether, a fruitful history. We turn to the problem of symmetry groups in mechanical Lagrangian systems by considering groups of evolution of the Hamiltonian dynamical system $i_{\zeta_L} w_L = dE_L$. One may consult [6] for a modern version of the classical Noether theorem in our very context.

First of all we consider the Poisson structure associated to the symplectic phase space manifold (TM, w_L). For every $f \in C^\infty(TM)$, its Hamiltonian vector field is the unique vector field X_f on TM such that $i_{X_f} w_L = df$. In this way, the Poisson bracket on $C^\infty(TM)$ is given by

$$\{f, g\} = X_f g = -X_g f = -w(X_f, X_g),\ f,\ g \in C^\infty(TM).$$

It satisfies Leibniz's rule

$$\{f, gh\} = \{f, g\}h + g\{f, h\}$$

and the Jacobi's identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

We say that $f \in C^\infty(TM)$ is a *conservation law* for the symplectic dynamics $i_{\zeta_L} w_L = dE_L$ if it belongs to the centralizer of E_L for the Poisson bracket. We denote by S the set of all conservation laws

$$S = \{f \in C^\infty(TM) : \{f, E_L\} = 0\}.$$

It is clear, by the Jacobi's identity, that S is stable under Poisson bracket.

On account of this structure, we can settle this work on the essential consideration of 1-parameter groups of transformations which permute the classical trajectories of the Lagrangian system defined by $L : TM \to \mathbb{R}$. These symmetries, denoted *dynamical symmetries* by Prince [15] and Crampin [4], are crystallized by vector fields $X \in \mathfrak{X}(M)$ that act as infinitesimal symmetries of the second order differential equation field ζ_L,

$$\left[X^c, \zeta_L\right] = 0. \tag{3.1}$$

Under these assumptions, Prince and Crampin (one may also consult the beautiful presentation on the subject in de Leon-Rodrigues [6]) with an effective combination of the vertical lift X^v and the complete lift X^c of the vector field X on M, raise a theory of conserved quantities of a Lagrangian system, under the condition that $L_{X^c}\theta_L$ is an exact 1-form df on TM and that $X^c E_L = 0$. Let us take as an example the conservation law

$$\left\{f - X^v L, E_L\right\} = 0.$$

Our goal here is to find other conditions on the 1-parameter invariance groups, that submitted to the assumption (3.1) are able to generate first integrals on the trajectories of the Lagrangian system.

A natural condition, that comes from considering those transformations by diffeomorphisms of the Lagrangian L that lead to the same symplectic dynamics (see, for example, [1] or [2]), is

$$L_{X^c}\theta_L = 0. \tag{3.2}$$

In such a case, we say that $X \in \mathfrak{X}(M)$ conforms a vector field on TM of Crampin-Prince symplectic symmetries.

As a direct consequence of this condition, we have

$$i_{X^c} w_L = -i_{X^c} d\theta_L = d\theta_L(X^c). \tag{3.3}$$

Theorem 3.1. *Let us consider a regular Lagrangian system $L : TM \to \mathbb{R}$ and X a vector field on M whose complete lift X^c to TM is constituted in a group of Crampin-Prince symmetries of the symplectic dynamics $i_{\zeta_L} w_L = dE_L$ canonically associated to L. Then*

$$\theta_L(X^c) \text{ is a conservation law } \Leftrightarrow \begin{cases} X^c(L) = 0 \\ \theta_L(\zeta_L) \text{ is constant on the } 1 - \text{parameter orbits of } X^c. \end{cases}$$

Proof. The assumption that the function $\theta_L(X^c)$ is a conservation law, means

$$\zeta_L\left\{\theta_L(X^c)\right\} = 0,$$

In this way, by (3.1) we have

$$\zeta_L\left\{\theta_L(X^c)\right\} = L_{\zeta_L}(i_{X^c}\theta_L) = i_{X^c}(L_{\zeta_L}\theta_L).$$

Now, by the Cartan formula $i_{\zeta_L} d + di_{\zeta_L}$ for the Lie derivative L_{ζ_L} along the field ζ_L, and the expression (2.3) above, we obtain

$$L_{\zeta_L}\theta_L = dL,$$

consequently

$$\zeta_L\left\{\theta_L(X^c)\right\} = i_{X^c}(dL) = X^c L = 0.$$

On the other hand, since by (3.3), the vector field X^c is the Hamiltonian vector corresponding to the function $\theta_L(X^c)$, the condition $\zeta_L\{\theta_L(X^c)\} = 0$ is equivalent to $X^c(E_L) = 0$. In this way,

$$0 = X^c(E_L) = X^c(L - \theta_L(\zeta_L)) = -X^c\theta_L(\zeta_L).$$

Conversely, if $X^c\{\theta_L(\zeta_L)\} = 0$ and $X^cL = 0$, then

$$X^c(L - \theta_L(\zeta_L)) = 0$$

which means $\zeta_L\{\theta_L(X^c)\} = 0$. □

We shall finish this article with the following unexpected result.

Proposition 3.1. *With the previous notations, let us consider the Hamiltonian vector field X^c on TM,*

$$i_{X^c}w_L = d\left\{\theta_L(X^c)\right\}.$$

Let Y be the unique vector field on TM such that $i_Y w_L = -\theta_L$. Then

$$\theta_L(X^c) = Y\left\{\theta_L(X^c)\right\}.$$

Proof. We shall prove, in a more general manner, that for a Hamiltonian vector field X_f, that is $i_{X_f}w_L = df$ for some smooth function f on TM, we have

$$\theta_L(X_f) = Yf.$$

Now

$$\begin{aligned} -\theta_L(X_f) &= i_{X_f}(i_Y w_L) = -i_Y\left(i_{X_f}w_L\right) \\ &= -i_Y df = -Yf. \end{aligned}$$ □

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