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Classical and Bayesian Inference for the Burr Type XII Distribution Under Generalized Progressive Type I Hybrid Censored Sample

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ABSTRACT

This paper describes the classical and Bayesian estimation for the parameters of the Burr Type XII distribution based on generalized progressive Type I hybrid censored sample. We first discuss the maximum likelihood estimators of unknown parameters using the expectation-maximization (EM) algorithm and associated interval estimates using Fisher information matrix. We then derive the Bayes estimators with respect to different symmetric and asymmetric loss functions. In this regard, we use Lindley's approximation and importance sampling methods. Highest posterior density (HPD) intervals of unknown parameters are constructed as well. The results of simulation studies and real data analysis are conducted to compare the performance of the proposed point and interval estimators.

1. INTRODUCTION

The Burr Type XII (BXII) distribution first appears as part of the Burr system of distributions which was introduced by Burr (Burr [1]). The BXII distribution is becoming increasingly used in the contexts of lifetime data analysis, reliability analysis, quality control, insurance risk and actuarial science in order to reduce the likelihood of failure. In the recent past few years, this distribution has gained some attention among researchers, see for example, Gunasekera [2] and Panahi [3]. The probability density function (PDF) and cumulative distribution function (CDF) of the BXII distribution are given by, respectively,

$$f_{BXII}(x;\alpha,\beta) = \alpha\beta x^{\beta-1}(1+x^{\beta})^{-(\alpha+1)};\quad x>0, \tag{1}$$

and

$$F_{BXII}(x;\alpha,\beta) = 1-(1+x^{\beta})^{-\alpha};\ \ x>0.$$

The hazard rate function is

$$HR_{BXII}(t;\alpha,\beta) = \alpha\beta t^{\beta-1}(1+t^{\beta})^{-1},$$

where, $\alpha > 0, \beta > 0$ are the shape parameters. The shape of the hazard function of the BXII distribution depends only on the parameter β as

- For $\beta > 0$, *the hazard function is eventually decreasing.*
- For $\beta > 1$, *the hazard function is unimodal.*
- For $\beta \leq 1$, *the hazard function is decreasing.*

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Moreover, There are many situations including reliability and life-testing experiments where observed data are censored in nature. Type I and Type II censoring schemes are the two most commonly used censoring schemes. The hybrid censoring scheme is a mixture of the Type I and Type II censoring schemes which first introduced by Epstin [4]. Some recent studies on hybrid censoring scheme have been carried out by many authors including Gupta and Singh [5] and Panahi and Sayyareh [6]. One of the traditional defects in the Type I, Type II or hybrid censoring schemes is that they do not allow for removal of units at points other than the final point of the experiment. To deal with this problem, a more general censoring scheme called progressive Type II censoring (PC) has been introduced. Moreover, Kundu and Joarder [7] proposed the progressive hybrid censoring (PHC) scheme. Many authors have discussed the estimation procedure under PC and PHC schemes. See, for example, Lin *et al.* [8], Panahi [9].

One limitation of the PHC scheme is that it cannot be applied when very few failures may occur before time T. So, Cho *et al.* [10] introduced generalized progressive hybrid censoring scheme (GPHCS) which not only control the experiment within a proper testing period, but also guarantee certain number of failures in testing procedure. It can be described as follows. Suppose that n independent items are put on a life test and the integers $k, m \in \{1, 2, ..., n\}$ is previously fixed such that $k < m$. Also, T ($T \in (0, \infty)$) is a prefixed time point and $(R_1, R_2, ..., R_m)$ is also prefixed integers satisfying $\sum_{j=1}^{m} R_j + m = n$. At the first failure time, say $X_{1:m:n}$, R_1 surviving units are randomly selected and removed from the experiment. Similarly, at the time of the second failure, say $X_{2:m:n}$, R_2 surviving units are removed, and so on. This process continues until, immediately following the terminated time $T^{*} = \max\{X_{k:m:n}, \min(X_{m:m:n}, T)\}$, at this time all the remaining units are removed from the experiment. Therefore, under GPHCS, we have one of the following types of observations:

$$Case1 : X_{1:m:n}, X_{2:m:n}, \cdots, X_{k:m:n} \qquad if \quad T < X_{k:m:n} < X_{m:m:n},$$

$$Case2 : X_{1:m:n}, \cdots, X_{k:m:n}, \cdots, X_{\nu:m:n} \quad if \quad X_{k:m:n} < T < X_{m:m:n},$$

$$Case3 : X_{1:m:n}, \cdots, X_{k:m:n}, \cdots, X_{m:m:n} \quad if \quad X_{k:m:n} < X_{m:m:n} < T.$$

Note that for Case 2, $X_{\nu:m:n} < T < X_{\nu+1:m:n}$ and $X_{\nu+1:m:n}, ..., X_{m:m:n}$ are not observed. For Case 3, $T < X_{k:m:n} < X_{m:m:n}$ and $X_{k+1:m:n}, ..., X_{m:m:n}$ are not observed.

The generalized progressive hybrid censored samples have been investigated for instance by Cho *et al.* [11], Gorny and Cramer [12], Koley and Kundu [13] and Mohie El-Din *et al.* [14]. In this paper, we consider the analysis of generalized progressive hybrid censored lifetime data when the lifetime of each experimental unit follows a BXII distribution, and we try to compute the MLE's and Bayesian estimates of the unknown parameters. The rest of the paper is organized as follows: In Section 2, we obtain the maximum likelihood estimators of the unknown parameters of the BXII distribution using the EM algorithm. Using missing information principle, the asymptotic confidence intervals are also constructed. In Section 3, the Bayesian estimates are computed using Lindley's and Markov Chain Monte Carlo (MCMC) techniques. The highest posterior density (HPD) credible intervals with some calculations are also constructed as well. Simulation results of the different methods are presented in Section 4. A real set of data is analyzed in Section 5, and in Section 6, we conclude the paper.

2. EM ALGORITHM

Let $X_{1:n}, X_{2:n}, ..., X_{n:n}$ be an ordered sample of n independent units obtained from a BXII as defined in Equation (1). Based on the Cases 1, 2 and 3 of the GPHCS, the likelihood function is

$$\mathcal{L}(\alpha, \beta) = \prod_{j=1}^{\omega}\sum_{k=j}^{m}(1 + R_k)(\alpha\beta)^{\omega}\prod_{j=1}^{\omega}x_j^{\beta-1}(1 + x_j^{\beta})^{-\alpha(1+R_j)-1}\eta_1. \tag{2}$$

Here, $X_j = X_{j:m:n}$ and also, $\omega = k, \eta_1 = 1$ for Case 1, $\omega = \nu, \eta_1 = (1 + T^{\beta})^{-\alpha R^{*}_{\nu+1}}$ for Case 2 and $\omega = m, \eta_1 = 1$ for Case 3. Based on the observed data, the log-likelihood function for combined Cases 1, 2 and 3 can be written as

$$l(\alpha, \beta) = ln\mathcal{L}(\alpha, \beta) = \omega(ln\alpha + ln\beta) + (\beta - 1)\sum_{j=1}^{\omega} ln(x_j) - \sum_{j=1}^{\omega}\{\alpha(1 + R_j) + 1\}ln(1 + x_j^{\beta}) - \eta_2 \tag{3}$$

Here $\eta_2 = (0,\ \alpha R^{*}_{\nu+1} ln(1 + T^{\beta}),\ 0)$ for (Cases 1, 2 and 3). Note that the maximum likelihood estimators of the unknown parameters α and β can be obtained by solving the following likelihood equations:

$$\frac{\partial l(\alpha, \beta)}{\partial\alpha} = \frac{\omega}{\alpha} - \sum_{j=1}^{\omega}(1 + R_j)ln(1 + x^{\beta}) - \eta_3 = 0 \tag{4}$$

$$\frac{\partial l(\alpha, \beta)}{\partial\beta} = \frac{\omega}{\beta} + \sum_{j=1}^{\omega}ln(x_j) - \sum_{j=1}^{\omega}\{\alpha(1 + R_j) + 1\}\frac{ln(x_j)}{1 + x_j^{-\beta}} - \eta_4 = 0 \tag{5}$$

Here $\eta_3 = (0,\ R^{\divideontimes}_{\nu+1} ln(1+T^{\beta}),\ 0)$ and $\eta_4 = (0,\ \alpha R^{\divideontimes}_{\nu+1}\frac{lnT}{1+T^{-\beta}},\ 0)$ for (Cases 1, 2 and 3).

It is observed that it is difficult to solve likelihood equations analytically due to the associated form of likelihood function. So, we use the EM algorithm (Dempster *et al.* [15]) to compute them. Suppose that $\mathbf{X} = (X_1, X_2, \cdots, X_\omega)$ denotes the observed and $(\mathbf{Z}_j, \mathbf{Z}^c)$ represent the censored data, where, $\mathbf{Z}_j = (Z_{j_1}, Z_{j_2}, \cdots, Z_{jR_j})$ and $Z^c = (Z_1, Z_2, \cdots, R^{\divideontimes}_{\nu+1})$. The log-likelihood function of (α, β) given the complete sample is

$$l_C(\alpha,\beta) = \begin{cases} \nabla & Case1 \\ \nabla + \nabla^c & Case2 \\ \nabla & Case3, \end{cases} \tag{6}$$

where,

$$\begin{aligned}\nabla &= nln\alpha + nln\beta + (\beta-1)\sum_{j=1}^{\omega} lnx_j - (\alpha+1)\sum_{j=1}^{\omega} ln(1+x_j^{\beta}) \\ &\quad +(\beta-1)\sum_{j=1}^{\omega}\sum_{k=1}^{R_j} E[lnZ_{jk}|Z_{jk} > x_j] - (\alpha+1)\sum_{j=1}^{\omega}\sum_{k=1}^{R_j} E[ln(1+Z_{jk}^{\beta})|z_{jk} > x_j] \\ \nabla^c &= (\beta-1)\sum_{p=1}^{R^{\divideontimes}_{\nu+1}} E[ln(Z_p^c)|Z_p^c > T] - (\alpha+1)\sum_{p=1}^{R^{\divideontimes}_{\nu+1}} E[ln(1+(Z_p^c)^{\beta})|Z_p^c > T]\end{aligned}$$

The E-step of the EM-iteration needs the following conditional expectations:

$$\flat_1(c,\alpha,\beta) = E[lnZ_{jk}|Z_{jk} > c] = (S_{BXII}(c))^{-1}\int_c^{\infty} lnx f_{BXII}(x)dx$$

$$\flat_2(c,\alpha,\beta) = E[ln(1+Z_{jk}^{\beta})|Z_{jk} > c] = (S_{BXII}(c))^{-1}\int_c^{\infty} ln(1+x^{\beta}) f_{BXII}(x)dx$$

$$\flat_3(T,\alpha,\beta) = E[lnZ'_p|Z'_p > T] = (S_{BXII}(T))^{-1}\int_T^{\infty} lnx f_{BXII}(x)dx$$

$$\flat_4(T,\alpha,\beta) = E[ln(1+(Z'_p)^{\beta})|Z'_p > T] = (S_{BXII}(T))^{-1}\int_T^{\infty} ln(1+x^{\beta}) f_{BXII}(x)dx$$

The M-step in a EM-iteration is maximizing the likelihood function based on complete sample over (α, β), with the missing values replaced by their conditional expectations.

Further, we show the existence and uniqueness of the ML estimates of the parameters of the BXII distribution based on GPHCS data using the graphical method (Ateya [16]) as follows:

◇ Choose certain Case of censored data as $n = 20, m = 18, k = 16, T = 2, R_9 = 2, R_j = 0, j \neq 9$).

◇ Plot the curves of the equations $\partial(l(\alpha,\beta))/\partial(\alpha) = 0$ and $\partial(l(\alpha,\beta))/\partial(\beta) = 0$ (Figure 1).

◇ Figure 1 indicates that there exist one intersection point (1.0029, 1.1202). So, we can say that the solution of the $\partial(l(\alpha,\beta))/\partial(\alpha) = 0$ and $\partial(l(\alpha,\beta))/\partial(\beta) = 0$ exists and is unique. We observed a similar pattern in other Cases of the GPHCS as well.

2.1. Approximate Confidence Interval

In this subsection, we obtain the observed Fisher information matrix $I_{\mathbf{X}}(\alpha,\beta)$ using the missing information principle. It can be used to construct the approximate confidence intervals (ACIs). Therefore,

$$I_{\mathbf{X}}(\alpha,\beta) = I_{\mathbf{W}}(\alpha,\beta) - I_{\mathbf{Z}|\mathbf{X}}(\alpha,\beta), \tag{7}$$

where, $I_{\mathbf{W}}(\alpha,\beta)$, $I_{\mathbf{X}}(\alpha,\beta)$ and $I_{\mathbf{Z}|\mathbf{X}}(\alpha,\beta)$ are the complete, observed and missing information matrix respectively. It is to be noted that we have

$$I_{\mathbf{W}}(\alpha,\beta) = -E_{(\alpha,\beta)}\begin{bmatrix} \frac{\partial^2 l_C(\alpha,\beta)}{\partial\alpha^2} & \frac{\partial^2 l_C(\alpha,\beta)}{\partial\alpha\partial\beta} \\ \frac{\partial^2 l_C(\alpha,\beta)}{\partial\beta\partial\alpha} & \frac{\partial^2 l_C(\alpha,\beta)}{\partial\beta^2}. \end{bmatrix} \tag{8}$$

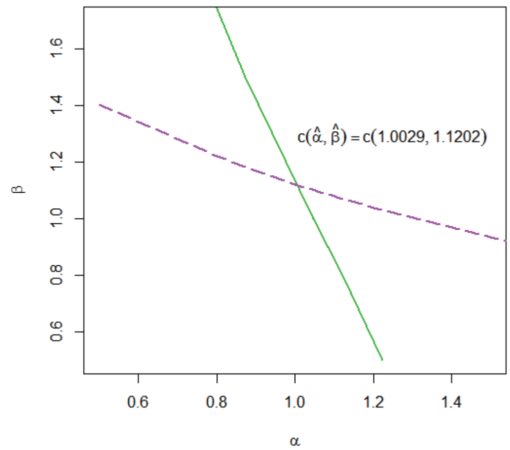

Figure 1 maximum likelihood estimator (MLEs) of α and β graphically.

Further, we have

$$I^{(j)}_{\mathbf{Z}|\mathbf{X}}(\alpha,\beta) = -E_{(\alpha,\beta)}\begin{bmatrix} \frac{\partial^2}{\partial\alpha^2} ln(f(z_{jk}|x_{j:m:n},\alpha,\beta)) & \frac{\partial^2}{\partial\alpha\partial\beta} ln(f(z_{jk}|x_{j:m:n},\alpha,\beta)) \\ \frac{\partial^2}{\partial\beta\partial\alpha} ln(f(z_{jk}|x_{j:m:n},\alpha,\beta)) & \frac{\partial^2}{\partial\beta^2} ln(f(z_{jk}|x_{j:m:n},\alpha,\beta)) \end{bmatrix}$$

Therefore, the expected missing information can then be computed as

$$I_{\mathbf{Z}|\mathbf{X}}(\alpha,\beta) = \begin{cases} \sum_{j=1}^{k} R_i I^{(j)}_{\mathbf{Z}|\mathbf{X}}(\alpha,\beta), & Case1 \\ \sum_{j=1}^{\nu} R_i I^{(j)}_{\mathbf{Z}|\mathbf{X}}(\alpha,\beta) + R^{*}_{\nu+1} I^{*}_{\mathbf{Z}|\mathbf{X}}(\alpha,\beta), & Case2 \\ \sum_{j=1}^{m} R_i I^{(j)}_{\mathbf{Z}|\mathbf{X}}(\alpha,\beta) & Case3. \end{cases}$$

where $I^{(j)}_{\mathbf{Z}|\mathbf{X}}(\alpha,\beta)$ and $I^{*}_{\mathbf{Z}|\mathbf{X}}(\alpha,\beta)$ are the information matrix of a single observation for the truncated BXII distribution with left truncation at, x_j and T, respectively. Therefore, using the asymptotic normality of the MLE, the ACIs for the parameters α and β are given by

$$\left(\hat{\alpha} - Z_{\gamma/2}\sqrt{Var(\hat{\alpha})}\ ,\ \hat{\alpha} + Z_{\gamma/2}\sqrt{Var(\hat{\alpha})}\right), \text{and} \left(\hat{\beta} - Z_{\gamma/2}\sqrt{Var(\hat{\beta})}\ ,\ \hat{\beta} + Z_{\gamma/2}\sqrt{Var(\hat{\beta})}\right).$$

3. BAYESIAN ETIMATION

3.1. The Prior and Posterior Distributions

In this section, we discuss Bayesian estimates of the unknown parameters of the BXII distribution under GPHCS using the squared error (SE) and linex (LI) loss functions. These loss functions are defined as, respectively,

$$L_{SE}(G(\alpha,\beta),\hat{G}(\alpha,\beta)) = (\hat{G}(\alpha,\beta) - G(\alpha,\beta))^2$$

$$L_{LI}(G(\alpha,\beta),\hat{G}(\alpha,\beta)) = e^{h\,(\hat{G}(\alpha,\beta)-G(\alpha,\beta))} - h\left(\hat{G}(\alpha,\beta) - G(\alpha,\beta)\right) - 1; \qquad h \neq 0.$$

Here $\hat{G}(\alpha,\beta)$ denotes an estimate of some parametric function $G(\alpha,\beta)$. We assume that α and β are independently distributed as $\gamma_\alpha(a_1,b_1)$ and $\gamma_\beta(a_2,b_2)$ priors, respectivelty. The join prior density function is

$$\pi(\alpha,\beta) = \gamma_\alpha(a_1,b_1)\gamma_\beta(a_2,b_2),$$

where $\gamma_\alpha(a_1, b_1)$ and $\gamma_\beta(a_2, b_2)$ are the gamma distributions. So, the posterior density function of (α, β) given data can be written as

$$\pi(\alpha, \beta|\mathbf{X}) \propto \mathcal{L}(\alpha, \beta|\mathbf{X})\pi(\alpha, \beta),$$

Therefore,

$$\pi(\alpha, \beta\,|\mathbf{X}) \propto \begin{cases} \zeta\alpha^{k+a_1-1}\beta^{k+a_2-1}e^{-\alpha b_1-\beta b_2}\prod_{j=1}^{k}x_j^{\beta-1}(1+x_j^{\beta})^{-\alpha(1+R_j)-1} & Case1 \\ \zeta\alpha^{\nu+a_1-1}\beta^{\nu+a_2-1}e^{-\alpha b_1-\beta b_2}\prod_{j=1}^{\nu}x_j^{\beta-1}(1+x_j^{\beta})^{-\alpha(1+R_j)-1} & Case2 \\ \zeta\alpha^{m+a_1-1}\beta^{m+a_2-1}e^{-\alpha b_1-\beta b_2}\prod_{j=1}^{m}x_j^{\beta-1}(1+x_j^{\beta})^{-\alpha(1+R_j)-1} & Case3 \end{cases} \tag{9}$$

where ζ is $\prod_{j=1}^{k}\sum_{k=j}^{m}(1+R_k)$, $\prod_{j=1}^{\nu}\sum_{k=j}^{m}(1+R_k)(1+T^{\beta})^{-\alpha R^{*}_{\nu+1}}$ and $\prod_{j=1}^{m}\sum_{k=j}^{m}(1+R_k)$ for Cases 1, 2 and 3 respectively. Thus, the Bayesian estimate of $G(\alpha, \beta)$ under SE loss function is evaluated as

$$\hat{G}_{SE}(\alpha, \beta) = E[G(\alpha, \beta)|\mathbf{X}] = \iint G(\alpha, \beta)\pi(\alpha, \beta|\mathbf{X})d\alpha\, d\beta. \tag{10}$$

Similarly for the LI loss function, we have

$$\begin{aligned} \hat{G}_{LI}(\alpha, \beta) &= -\frac{1}{h}lnE[e^{-hG(\alpha,\beta)}|\mathbf{X}] \\ &= -\frac{1}{h}ln[\iint e^{-hG(\alpha,\beta)}\pi(\alpha, \beta|\mathbf{X})d\alpha\, d\beta]. \end{aligned} \tag{11}$$

Unfortunately, we cannot obtain (10) and (11) analytically. So, we propose two approximation methods for evaluating the Bayes estimates of α and β, namely, Lindley's approximation and MCMC method.

3.2. Lindley's Approximation

In this section, Lindley's approximation (Lindley [17]) is applied to gain Bayes estimates of α and β . Based on the Lindley's approximation, the Bayesian estimates of α and β under SE and LI loss functions are

$$\begin{aligned} \hat{G}(\alpha, \beta) \approx\ & G(\hat{\alpha},\hat{\beta}) + .5[(\hat{\hbar}_{11} + 2\hat{\mathfrak{R}}_1\hat{\hbar}_1)\hat{\sigma}_{11} + (\hat{\hbar}_{21} + 2\hat{\mathfrak{R}}_2\hat{\hbar}_1)\hat{\sigma}_{21} \\ & +(\hat{\hbar}_{12} + 2\hat{\mathfrak{R}}_1\hat{\hbar}_2)\hat{\sigma}_{12} + (\hat{\hbar}_{22} + 2\hat{\mathfrak{R}}_2\hat{\hbar}_2)\hat{\sigma}_{22} + \S_1(\hat{\hbar}_1\sigma_{11} + \hat{\hbar}_2\hat{\sigma}_{12}) \\ & +\S_2(\hat{\hbar}_1\hat{\sigma}_{21} + \hat{\hbar}_2\hat{\sigma}_{22})] \end{aligned} \tag{12}$$

and

$\S_1 = \hat{l}_{111}\hat{\sigma}_{11} + \hat{l}_{121}\hat{\sigma}_{12} + \hat{l}_{211}\hat{\sigma}_{21} + \hat{l}_{221}\hat{\sigma}_{22}$, $\S_2 = \hat{l}_{211}\hat{\sigma}_{11} + \hat{l}_{122}\hat{\sigma}_{12} + \hat{l}_{212}\hat{\sigma}_{21} + \hat{l}_{222}\hat{\sigma}_{22}$ respectively. Where, $\hat{\mathfrak{R}}_1 = \frac{b_1-1}{\hat{\alpha}} - a_1$, $\hat{\mathfrak{R}}_2 = \frac{b_2-1}{\hat{\beta}} - a_2$, and $\hat{\sigma}_{ij}$ are the $(ij)^{th}$ elements of matrix $[-\partial^2 l(\alpha, \beta)/\partial\alpha\partial\beta\,]^{-1}$; $i, j = 1, 2$. For the SE loss function, we get that

$$G(\alpha, \beta) = \alpha \ \ , \hat{\hbar}_1 = 1 \ \ , \hat{\hbar}_{11} = 0 \ \ , \hat{\hbar}_2 = 0, \ \ \hat{\hbar}_{22} = 0 \ \ \hat{\hbar}_{12} = 0$$

and the corresponding Bayesian estimate of α is $\widehat{\alpha_{SE}} = E[\alpha|X]$. Also, the Bayesian estimate of α under LI loss function is obtained as

$$\widehat{\alpha_{LI}} = -\frac{1}{h}\ln\left\{E\left(e^{-h\,\alpha}|\mathbf{X}\right)\right\}$$

where $G(\alpha, \beta) = e^{-h\,\alpha}$, $\hat{\hbar}_1 = -he^{-h\,\alpha}$, $\hat{\hbar}_{11} = h^2e^{-h\,\alpha}$, $\hat{\hbar}_2 = \hat{\hbar}_{22} = \hat{\hbar}_{12} = \hat{\hbar}_{21} = 0$. Proceeding similarly, the Bayesian estimate of β under SE loss function can be obtained as $\widehat{\beta_{SE}} = E[\beta|X]$. Where, $G(\alpha, \beta) = \beta$, $\quad \hat{\hbar}_2 = 1, \hat{\hbar}_1 = \hat{\hbar}_{22} = \hat{\hbar}_{12} = \hat{\hbar}_{21} = 0$.

Also, the Bayesian estimate of β under LI loss function is obtained as

$$\widehat{\beta_{LI}} = -\frac{1}{h}\ln\left\{E\left(e^{-h\,\beta}|\mathbf{X}\right)\right\}; \qquad h \neq 0$$

$$G(\alpha,\beta)=e^{-h\beta},\ \hat{\hbar}_2=-he^{-h\beta},\ \hat{\hbar}_{22}=h^2e^{-h\beta},\ \hat{\hbar}_1=\hat{\hbar}_{11}=\hat{\hbar}_{12}=\hat{\hbar}_{21}=0$$

Moreover, $\hat{l}_{11}=\frac{\partial^2 l(\alpha,\beta)}{\partial\alpha^2}|_{\alpha=\hat{\alpha},\beta=\hat{\beta}}$, $\hat{l}_{22}=\frac{\partial^2 l(\alpha,\beta)}{\partial\beta^2}|_{\alpha=\hat{\alpha},\beta=\hat{\beta}}, \cdots$

3.3. MCMC Method and HPD Credible Intervals

The MCMC methodology serves as an effective tool for generating random samples on complex Bayesian models. The importance sampling method provides commonly used in MCMC methods. So, we use this method to compute the Bayes estimates and construct the HPD credible intervals of unknown parameters. Based on the independent proposed priors, the posterior density functions of α and β can be rewritten as

$$\pi(\alpha,\beta|\mathbf{X})\propto\gamma_{\alpha|\beta}(\omega+a_1,\sum_{j=1}^{\omega}(1+R_j)ln(1+x_j^{\beta})+b_1+\eta_3)$$
$$\times\ \gamma_{\beta}(\omega+a_2,b_2-\sum_{j=1}^{\omega}ln(x_j))H(\alpha,\beta),$$

where

$$H(\alpha,\beta)=e^{-\sum_{j=1}^{\omega}ln(1+x_j^{\beta})}(\sum_{j=1}^{\omega}(1+R_j)ln(1+x_j^{\beta})+b_1+\eta_3)^{-(\omega+a_1)}.$$

We propose the following algorithm along the line of Kundu and Pradhan [18] to compute the Bayes estimate of $G(\alpha,\beta)$, say $\hat{G}(\alpha,\beta)$ and also to construct the associated HPD credible interval.

Step I : Generate $\beta_1\sim\gamma_{\beta}(\omega+a_2,b_2-\sum_{j=1}^{\omega}lnx_j)$.

Step II : Given β_1 generated in step I, generate α_1 from $\gamma_{\alpha|\beta}(\omega+b_1,\sum_{j=1}^{\omega}(1+R_j)ln(1+x_j^{\beta})+b_1+\eta_3)$.

Step III : Repeat Steps I and II, M times to obtain the importance sample $(\alpha_1,\beta_1),(\alpha_2,\beta_2),(\alpha_M,\beta_M)$.

Step IV: The Bayes estimate of $G(\alpha,\beta)$under SE and LI loss functions can be approximated as

$$\widehat{G_{SE}}(\alpha,\beta)=\frac{\frac{1}{M}\sum_{j=1}^{M}G(\alpha_j,\beta_j)H(\alpha_j,\beta_j)}{\frac{1}{M}\sum_{j=1}^{M}H(\alpha_j,\beta_j)} \tag{13}$$

and

$$\widehat{G_{LI}}(\alpha,\beta)=\frac{-1}{h}ln\left\{\frac{\frac{1}{M}\sum_{j=1}^{M}e^{-hG(\alpha_j,\beta_j)}H(\alpha_j,\beta_j)}{\frac{1}{M}\sum_{j=1}^{M}H(\alpha_j,\beta_j)}\right\} \tag{14}$$

We also construct the HPD intervals of α and β by using the method of Chen and Shao [19] (1999).

4. SIMULATION STUDY

In this section, we present simulation study to compare the performance of the classical and Bayesian estimation procedures under different GPHCS. Extensive computations were performed using statistical software *R*. We simulate GPHCS for different combinations of (n,m,k,T) from the BXII (α,β) distribution. For convenience we consider the true values of unknown parameters as $\alpha=1.2$ and $\beta=1.5$.

By employing an EM algorithm, the maximum likelihood estimates have been computed. Approximate expressions for the Bayesian estimators have been computed using the Lindley's approximation and importance sampling algorithm. The Bayes estimates are obtained by assuming that α and β have $\gamma(a_1,b_1)$ and $\gamma(a_2,b_2)$ priors, respectively with $a_1=15, a_2=12, b_1=10, b_2=10$. The %95 ACIs and HPD credible intervals for the parameters are also constructed. The HPD credible intervals are computed based on 5000 MCMC samples. We take three different censoring schemes as follows:

Scheme 1: $R_1=R_m=(n-m)/2$ and $R_j=0$ for $j\neq 1,m$

Scheme 2: $R_1=n-m$ and $R_j=0$ for $j\neq 1$.

Scheme 3: $R_m=(n-m)$ and $R_j=0$ for $j\neq m$.

Table 1 The mean values of MLEs and Bayesian estimates along with associated MSEs.

				Lindely		MCMC	
	Scheme	**(m,k)**	$\hat{\alpha}_{EM}$	$\hat{\alpha}_{SE}$	$\hat{\alpha}_{LI}$	$\hat{\alpha}_{SE}$	$\hat{\alpha}_{LI}$
$T=.9$	R_1	(24, 12)	1.7870 (.6999)	1.6439 (.6644)	1.4997 (.5844)	1.5651 (.3525)	1.4912 (.3414)
$n=30$	R_2		1.8548 (.7231)	1.7212 (.6538)	1.5618 (.4471)	1.7412 (.4444)	1.6368 (.3961)
	R_3		1.7021 (.5263)	1.5774 (.3862)	1.4312 (.3060)	1.6041 (.3052)	1.5357 (.2329)
	R_1	(28, 12)	1.5119 (.6467)	1.3923 (.4457)	1.2678 (.3655)	1.4384 (.2432)	1.3575 (.2370)
	R_2		1.5663 (.5489)	1.4121 (.4812)	1.2939 (.3922)	1.4603 (.3255)	1.3788 (.2373)
	R_3		1.4927 (.4066)	1.3777 (.3473)	1.2556 (.2514)	1.4246 (.2321)	1.3451 (.1389)
$T=1$	R_1	(24, 12)	1.7182 (.5769)	1.6418 (.4570)	1.4806 (.3513)	1.6236 (.3370)	1.5475 (.2446)
$n=30$	R_2		1.8393 (.6847)	1.7239 (.4616)	1.5615 (.3465)	1.6882 (.3277)	1.5584 (.2440)
	R_3		1.7029 (.5123)	1.5905 (.2933)	1.4414 (.1989)	1.7067 (.1768)	1.6324 (.1248)
	R_1	(28, 12)	1.4220 (.4306)	1.3540 (.3243)	1.2144 (.2201)	1.2542 (.1841)	1.2018 (.1704)
	R_2		1.4568 (.3791)	1.3747 (.2902)	1.2320 (.2531)	1.2941 (.2176)	1.2418 (.1925)
	R_3		1.3962 (.3855)	1.3402 (.2150)	1.2017 (.1682)	1.2780 (.0751)	1.2260 (.0602)
$T=.9$	R_1	(32, 16)	1.7247 (.5141)	1.6220 (.3250)	1.4906 (.2150)	1.5993 (.1362)	1.5580 (.1257)
$n=40$	R_2		1.8944 (.9081)	1.7145 (.4315)	1.5908 (.3708)	1.6325 (.2787)	1.5780 (.2133)
	R_3		1.6594 (.4087)	1.5770 (.2637)	1.4568 (.1863)	1.5077 (.1503)	1.4599 (.1230)
	R_1	(36,16)	1.5045 (.3809)	1.4157 (.2590)	1.3130 (.2038)	1.4165 (.0977)	1.3797 (.0828)
	R_2		1.5684 (.5176)	1.4512 (.3918)	1.3496 (.3094)	1.3804 (.2075)	1.3391 (.1642)
	R_3		1.4471 (.2380)	1.3869 (.1339)	1.2790 (.1103)	1.3823 (.0712)	1.3459 (.0582)
$T=1$	R_1	(32, 16)	1.7346 (.4949)	1.6435 (.3156)	1.5095 (.2161)	1.6531 (.1681)	1.6097 (.1308)
$n=40$	R_2		1.8297 (.6762)	1.7060 (.4281)	1.5606 (.2977)	1.7220 (.2417)	1.6638 (.2021)
	R_3		1.6586 (.3726)	1.5899 (.2311)	1.4659 (.1769)	1.5375 (.1456)	1.4897 (.1215)
	R_1	(36, 16)	1.4447 (.1892)	1.3963 (.1309)	1.2798 (.0990)	1.4240 (.0926)	1.3856 (.0750)
	R_2		1.7434 (.2320)	1.4154 (.1499)	1.2941 (.1057)	1.4356 (.0947)	1.3925 (.0752)
	R_3		1.4352 (.1791)	1.3904 (.1226)	1.2795 (.0865)	1.3913 (.0621)	1.3537 (.0478)

MSE, mean squared error; MCMC, Markov Chain Monte Carlo.

Table 2 The mean values of MLEs and Bayesian estimates along with associated MSEs.

				Lindely		MCMC	
	Scheme	**(m,k)**	$\widehat{\beta}_{EM}$	$\widehat{\beta}_{SE}$	$\widehat{\beta}_{LI}$	$\widehat{\beta}_{SE}$	$\widehat{\beta}_{LI}$
$T=.9$	R_1	(24, 12)	1.6759 (.2715)	1.5907 (.2510)	1.4981 (.2375)	1.4024 (.2239)	1.2732 (.1902)
$n=30$	R_2		1.0707 (.2578)	1.6108 (.1995)	1.5180 (.0925)	1.3765 (.0869)	1.3553 (.0710)
	R_3		1.6516 (.1828)	1.6078 (.1733)	1.4826 (.1543)	1.5411 (.1422)	1.5069 (.1396)
	R_1	(28, 12)	1.6335 (.2568)	1.5948 (.2397)	1.4439 (.2090)	1.3297 (.1964)	1.3054 (.1701)
	R_2		1.6472 (.1701)	1.5892 (.1557)	1.4527 (.0975)	1.3607 (.0634)	1.3343 (.0590)
	R_3		1.6274 (.1691)	1.5961 (.1596)	1.4416 (.0900)	1.3271 (.0820)	1.3022 (.0785)
$T=1$	R_1	(24, 12)	1.6759 (.1499)	1.6347 (.1437)	1.4997 (.1128)	1.4887 (.1060)	1.4560 (.1030)
$n=30$	R_2		1.7098 (.1768)	1.6212 (.1589)	1.5205 (.0914)	1.6106 (.0683)	1.3868 (.0606)
	R_3		1.6565 (.1315)	1.6272 (.1211)	1.4879 (.0854)	1.5331 (.0726)	1.5059 (.0697)
	R_1	(28, 12)	1.6153 (.1418)	1.6314 (.1403)	1.4325 (.1011)	1.4021 (.0937)	1.3570 (.0834)
	R_2		1.6227 (.1563)	1.6302 (.1457)	1.4356 (.0831)	1.3666 (.0552)	1.3383 (.0517)
	R_3		1.6085 (.1259)	1.6317 (.1176)	1.4297 (.0708)	1.4309 (.0648)	1.3846 (.0530)
$T=.9$	R_1	(32, 16)	1.6361 (.1052)	1.5989 (.1946)	1.5019 (.0855)	1.4099 (.0725)	1.3994 (.0636)
$n=40$	R_2		1.6684 (.1300)	1.5743 (.1286)	1.5233 (.0750)	1.5387 (.0359)	1.5216 (.0337)
	R_3		1.6260 (.0945)	1.6009 (.0873)	1.4992 (.0796)	1.4178 (.0694)	1.4044 (.0611)
	R_1	(36, 16)	1.6022 (.1007)	1.5775 (.0993)	1.4227 (.0673)	1.4507 (.0667)	1.4338 (.0557)

(*continued*)

Table 2 The mean values of MLEs and Bayesian estimates along with associated MSEs. *(Continued)*

	Scheme	(m, k)	$\widehat{\beta}_{EM}$	Lindely $\widehat{\beta}_{SE}$	Lindely $\widehat{\beta}_{LI}$	MCMC $\widehat{\beta}_{SE}$	MCMC $\widehat{\beta}_{LI}$
	R_2		1.6145 (.1124)	1.5697 (.1051)	1.4694 (.0718)	1.5238 (.0278)	1.5028 (.0258)
	R_3		1.5905 (.0921)	1.5868 (.0825)	1.4559 (.0638)	1.4478 (.0642)	1.4293 (.0624)
$T = 1$	R_1	(32, 16)	1.6500 (.1037)	1.6274 (.0965)	1.5168 (.0618)	1.4669 (.0614)	1.4548 (.0508)
$n = 40$	R_2		1.6672 (.1186)	1.6251 (.1044)	1.5247 (.0684)	1.5429 (.0510)	1.5209 (.0419)
	R_3		1.6326 (.0886)	1.6235 (.0872)	1.5076 (.0645)	1.4682 (.0545)	1.4515 (.0529)
	R_1	(36, 16)	1.6064 (.0917)	1.5931 (.0874)	1.4575 (.0596)	1.4894 (.0434)	1.4692 (.0334)
	R_2		1.6003 (.0982)	1.6093 (.0959)	1.4600 (.0622)	1.5184 (.0244)	1.4916 (.0214)
	R_3		1.5942 (.0852)	1.6074 (.0787)	1.4625 (.0616)	1.4916 (.0488)	1.4678 (.0425)

MSE, mean squared error; MCMC, Markov Chain Monte Carlo.

Table 3 The ACIs and HPD credible intervals for α and β.

T	Scheme	(m, k)	α ACI	α HPD	β ACI	β HPD
$T = .9$	R_1	(24, 12)	(.9344, 2.7197)	(.8375, 2.3375)	(.9800, 2.2939)	(.4902, 1.4964)
$n = 30$	R_2		(.9654, 2.9061)	(.9122, 2.2947)	(1.0448, 2.3448)	(.4103, 1.3928)
	R_3		(.9091, 2.4950)	(.7390, 1.8992)	(.8581, 2.2480)	(.5074, 1.5712)
	R_1	(28, 12)	(.7567, 2.2823)	(.5909, 1.9019)	(.9972, 2.2699)	(.6274, 1.3527)
	R_2		(.7650, 2.3676)	(.7010, 1.9810)	(1.0017, 2.2927)	(.4704, 1.4206)
	R_3		(.7562, 2.2291)	(.8193, 1.8892)	(.9983, 2.2566)	(.4084, 1.3584)
$T = 1$	R_1	(24, 12)	(.9397, 2.6172)	(.9236, 2.005)	(.9914, 2.2393)	(.7703, 1.5099)
$n = 30$	R_2		(.9701, 2.8885)	(.9108, 2.1780)	(1.0738, 2.3458)	(.6580, 1.5680)
	R_3		(.9279, 2.4779)	(1.0181, 2.0732)	(.9868, 2.2468)	(.6903, 1.5560)
	R_1	(28, 12)	(.7395, 2.1049)	(.7718, 1.6812)	(.9914, 2.2393)	(.6553, 1.4315)
	R_2		(.7436, 2.1699)	(.5731, 1.7665)	(.9906, 2.2547)	(.6577, 1.4760)
	R_3		(.7385, 2.0540)	(.8062, 1.7203)	(.9924, 2.2247)	(.6855, 1.4748)
$T = .9$	R_1	(32, 16)	(1.0002, 2.4490)	(.8962, 1.8438)	(1.1108, 2.3608)	(.7789, 1.6099)
$n = 40$	R_2		(.9412, 2.7433)	(.1307, 1.2828)	(1.1205, 2.2163)	(.6769, 1.5792)
	R_3		(.9926, 2.3262)	(.8434, 1.8434)	(.8856, 2.1356)	(.6478, 1.4578)
	R_1	(36, 16)	(.8564, 2.1526)	(.8331, 1.7041)	(1.0656, 2.1389)	(.8282, 1.4961)
	R_2		(.8697, 2.2672)	(.7225, 1.7225)	(1.1205, 2.2163)	(.6769, 1.5792)
	R_3		(.8438, 2.0504)	(.8360, 1.8347)	(.8856, 2.1356)	(.6478, 1.4578)
$T = 1$	R_1	(32, 16)	(1.0272, 2.4420)	(.9564, 1.7657)	(1.1267, 2.1733)	(.8402, 1.4860)
$n = 40$	R_2		(1.0441, 2.6153)	(.9800, 2.1277)	(1.1241, 2.2104)	(.7897, 1.5510)
	R_3		(1.0113, 2.3059)	(1.0161, 1.8946)	(1.1267, 2.1385)	(.9484, 1.6682)
	R_1	(36, 16)	(.8475, 2.0420)	(.7461, 1.4314)	(1.0645, 2.1216)	(.8135, 1.4988)
	R_2		(.8484, 2.0984)	(.8712, 1.8350)	(1.0617, 2.1388)	(.7665, 1.5244)
	R_3		(.8546, 2.0141)	(.8851, 1.7370)	(1.1143, 2.1146)	(.8303, 1.5211)

ACI, approximate confidence interval; HPD, highest posterior density.

The average estimates, mean squared errors (MSEs) and average confidence intervals based on 10000 replications have been reported in Tables 1–3. For more compression, the MSEs of the proposed estimators and lengths of intervals are presented in Figures 2 and 3 for some values of generalized progressive censoring schemes.

It can be observed that for fixed n, T and k as m increases, the average estimates and the MSEs of the parameters decreases. Also, the average lengths of approximate and HPD intervals tend to decrease with increasing the effective sample size m (see also, Figures 2 and 3).

For fixed m, k and T as sample size n increases the MSEs of all the estimators decreases (see also, Figure 2). Similar trend is observed for fixed n, k and m as T increases (see also, Figure 2).

Tabulated values also show that the importance sampling estimates are better choice among all rivals and for all values of n, m, k and T. In particular, the MCMC Bayes estimates of α and β under LI loss function perform better than other respective estimators for all values of n, m, k and T.

Thus, we recommend Bayesian point and interval estimations of the parameters using importance sampling algorithm.

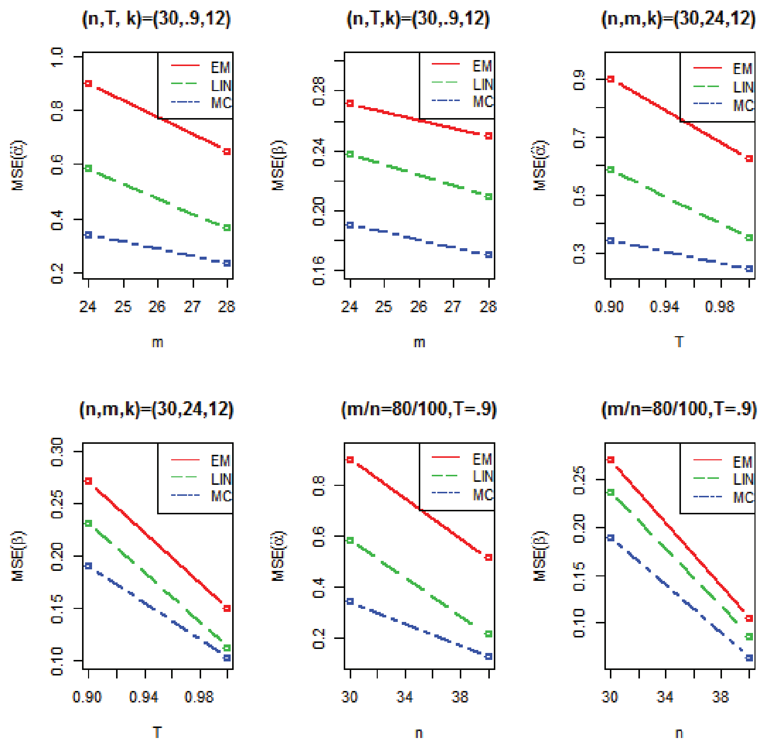

Figure 2 The mean squared errors (MSEs) of the proposed estimators (*expectation-maximization (EM), Lindley, Markov Chain Monte Carlo (MCMC)*) for different choices of n,T and m.

5. REAL DATA ANALYSIS

In order to illustrate all the inferential results established for the BXII distribution, we analyze one data set from Wingo [20]. We first make an inference whether the BXII distribution fits the given data set. For this purpose, we compute the Kolmogorov-Smirnov (K-S) distances between the empirical distribution and the fitted distribution functions based on MLEs, it is 0.1019, and the associated p-value is 0.9719. The p value suggests that BXII distribution can be considered as an adequate model for the given data set. We take ($R_1 = 2$ and $R_j = 0$ for $j \neq 1$) and consider different GPHCS Cases by taking selected choices of m, k, and T values which are listed below:

Case 1: $m = 18, k = 16, T = 0.8,$

Case 2: $m = 18, k = 8, T = 1,$

Case 3: $m = 18, k = 8, T = 2.$

The maximum likelihood estimates and the approximate Bayesian estimates using Lindley's approximation and MCMC algorithm are presented in Table 4. The upper and lower bounds for the %95 approximate and HPD confidence intervals of α and β are presented in Tables 5 and 6 respectively. Because we have no prior information about the unknown parameters, we assume the non-informative gamma priors of the unknown parameters which are defined as $a_1 = a_2 = b_1 = b_2 = 0$. In general, the Bayes estimates are smaller than the MLEs.

6. CONCLUSIONS

In this paper, different point and interval estimation problems have taken into consideration under classical and Bayesian framework when lifetime data following BXII distribution are observed under GPHCS. It is observed that the MLEs cannot be derived in the closed form and

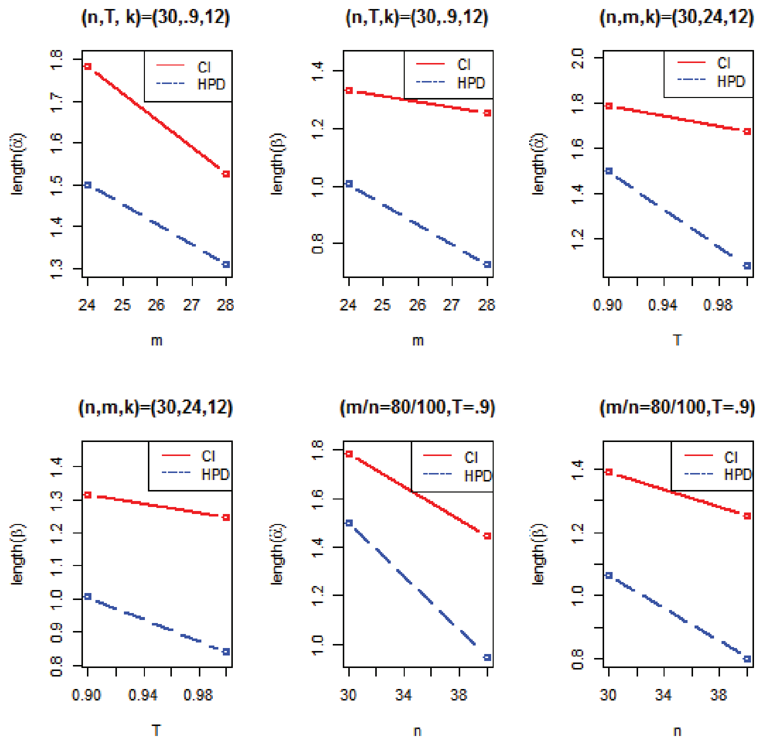

Figure 3 The length of approximate confidence intervals (ACIs) and highest posterior densities (HPDs) for different choices of n, T and m.

instead the traditional Newton-Raphson algorithm, we suggest the EM algorithm to compute them. By applying two different approximation approaches like Lindley's and the importance sampling algorithm, the Bayesian estimates of the parameters under different symmetric and asymmetric loss functions have been obtained. Based on the EM framework and MCMC technique, the confidence intervals have also been constructed. We compared performance of different proposed estimators using Monte Carlo simulations and observed that the Bayes estimates based on importance sampling algorithm outperforms other proposed estimators. An illustrative example is also provided in support of the proposed methods.

Table 4 The MLEs and Bayesian estimates of α and β for $(n, m) = (20, 18)$.

				Lindely		MCMC	
	k	T	$\hat{\alpha}_{EM}$	$\hat{\alpha}_{SE}$	$\hat{\alpha}_{LI}$	$\hat{\alpha}_{SE}$	$\hat{\alpha}_{LI}$
Case 1: $(T < X_k < X_m)$	16	.8	4.48017	4.3678	3.6591	3.8213	3.4523
Case 2: $(X_k < T < X_m)$	8	1	2.8606	2.8167	2.3670	2.6084	2.4212
Case 3: $(X_k < X_m < T)$	8	2	2.9191	2.8879	2.5179	2.8658	2.6769
	k	T	$\hat{\beta}_{EM}$	$\hat{\beta}_{SE}$	$\hat{\beta}_{LI}$	$\hat{\beta}_{SE}$	$\hat{\beta}_{LI}$
Case 1: $(T < X_k < X_m)$	16	.8	6.8144	7.3354	6.0906	5.4495	5.2748
Case 2: $(X_k < T < X_m)$	8	1	5.8888	6.2755	5.2147	4.9963	4.7647
Case 3: $(X_k < X_m < T)$	8	2	5.9872	6.1919	5.4047	5.7884	5.3493

MCMC, Markov Chain Monte Carlo.

Table 5 The ACIs and HPD intervals of α for $(n, m) = (20, 18)$.

	k	T	%95 ACIs	HPD Credible Intervals
Case 1: $(T < X_k < X_m)$	16	.8	(1.9531, 7.0070)	(1.4170, 4.3683)
Case 2: $(X_k < T < X_m)$	8	1	(1.2622, 5.8302)	(1.0755, 3.1945)
Case 3: $(X_k < X_m < T)$	8	2	(1.5281, 4.3101)	(1.4502, 3.8189)

ACI, approximate confidence interval; HPD, highest posterior density.

Table 6 The ACIs and HPD intervals of β for $(n, m) = (20, 18)$.

	k	T	%95 ACIs	HPD Credible Intervals
Case 1: $(T < X_k < X_m)$	16	.8	(4.4475, 9.1813)	(3.3368, 6.2481)
Case 2: $(X_k < T < X_m)$	8	1	(3.6823, 8.0954)	(3.8160, 5.9253)
Case 3: $(X_k < X_m < T)$	8	2	(3.8680, 6.8192)	(3.7736, 6.0916)

ACI, approximate confidence interval; HPD, highest posterior density.

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