

Research Article

Algorithm for Multiple Attribute Decision-Making with Interactive Archimedean Norm Operations Under Pythagorean Fuzzy Uncertainty

Lei Wang^{1,*}, Harish Garg^{2,*}

¹Department of Basic Teaching, Liaoning Technical University, Huludao, 125105, China

²School of Mathematics, Thapar Institute of Engineering and Technology, Deemed University, Patiala 147004, India

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ABSTRACT

Recently, a great attention is paid toward developing aggregation operators for Pythagorean fuzzy set (PFS). However, few of them have adopted the rules of Archimedean t-conorm and t-norm (ATT) to aggregate the numbers. Motivated by this, the keep interest of the present work is to define some Pythagorean fuzzy interaction aggregation operators with the aid of ATT. To do this, the objective of the work is divided into three folds. In the first fold, we define some interactive operations law for PFSs and propose their corresponding new interaction Pythagorean operators, namely Archimedean based Pythagorean fuzzy interactive weighted averaging (A-PFIWA) operator and Archimedean based Pythagorean fuzzy interactive weighted geometric (A-PFIWG) operator. In the second fold, we investigate their desirable properties and study the several special cases of the proposed ones with respect to the existing ones. Lastly, we design an algorithm to solve the multiple attribute decision-making issues with Pythagorean fuzzy uncertainties and explain their utilization with a numerical example. Further, several examples are discussed to demonstrate the validity and superiority of the proposed method. The novelties of proposed operators are that they not only can offer a more flexible method via selecting diverse forms of ATT, but also can reflect the interactive influence among membership degree and nonmembership degree for PFSs.

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1. INTRODUCTION

Multiple attribute decision-making (MADM) issue involving a series of approaches to choose optimal alternative or acquire their priority ordering in accordance with given attribute [1], which is a major branch of modern decision science. Considering the complicity of the realistic decision circumstances, decision makers (DMs) may realize that it is difficult to describe crisp values about attribute evaluation information. For this, the theory of intuitionistic fuzzy set (IFS) [2] was initiated by Atanassov in 1986, it suits for tackling the issue of insufficient evaluation information. IFS expresses attribute evaluation information by membership degree (MD), nonmembership degree (NMD) and hesitation degree, respectively, which is a useful extension of fuzzy sets [3]. Nowadays, plenty of researches [4–8] have been implemented on MADM techniques within intuitionistic fuzzy information.

More recently, the notion of Pythagorean fuzzy sets (PFS) was explored by Yager [9,10] as a useful assessment technique, which is successful extension of IFS. Similar to IFS, PFS is also represented by MD and NMD, the sum of MD and NMD are permitted to bigger than one, but the square sum of these two degrees is not exceed than one. Therefore, PFS possesses more superiority comparing with IFS to express the indeterminacy in both theory and application aspects [11–15]. In the field of MADM, some classical evaluation approaches are extend to PFS, including LINAMP [16], TODIM [17], VIKOR [18], ELECTRE [19] and so on. Zhang and Xu [20] developed detailed mathematical structure representation for PFS, they also proposed the notion of Pythagorean fuzzy numbers (PFNs), and corresponding fundamental operations on PFNs. Meanwhile, they developed ranking rules over PFNs by defined scores function. Additionally, they explored the distance between PFNs, and combined it and TOPSIS method to address Pythagorean fuzzy MADM issues. Apart from above mentioned evaluation methods, a large number of aggregation operators (AOs) [21–30] were proposed by some scholars to fuse PFNs employing the operations provided by Ref. [20]. For instance, in the light of the ordered weighted averaging (OWA) operator [31], Zhang [21] defined the Pythagorean fuzzy OWA operator and utilized it to manage group decision-making issue. Enlightened by the work [5], Ma and Xu [22] defined the Pythagorean fuzzy weighted averaging (PFWA) operator. Further, considering the information about MD and NMD should be equally handled, they initiated neutrality AOs, that is, symmetric PFWA, and also discussed the relationships over these AOs. Zeng

* Corresponding author. Email: ltuwl@126.com and harishg58iitr@gmail.com

et al. [23] discussed the Pythagorean fuzzy induced OWA weighted average operator and corresponding decision approach for settling group decision problems in Pythagorean fuzzy setting. In Ref. [24], several generalized Pythagorean fuzzy averaging AOs were presented utilizing the Einstein t-norm and t-conorm. Garg [25] proposed neutrality Pythagorean fuzzy geometric AOs for aggregating PFNs. Yang *et al.* [26] developed a series of Frank power AOs for interval-valued PFNs. Wu and Wei [27] created several Hamacher AOs to settle MADM issues with Pythagorean fuzzy information. Besides, in view of the relationship of attributes, some AOs [28–30] are initiated. However, Wei [32] pointed that the ranking orders are not reasonable by using above AOs [21–30] in some situations. For instance, let $\beta_1 = (\mu_1, \nu_1)$ and $\beta_2 = (\mu_2, \nu_2)$ be two PFNs, when $\mu_1 = 0$ and $\mu_2 \neq 0$, then employing the multiplication operation $\otimes$ in [20], we obtain $\mu_{\beta_1 \otimes \beta_2} = 0$. Which indicates that regardless of the value of MD μ_2 , the outcome $\mu_{\beta_1 \otimes \beta_2} = 0$ can be obtained always, it's an unreasonable situation. Similarly, the outcome $\nu_{\beta_1 \oplus \beta_2} = 0$ is always kept based on the addition operation $\oplus$ in [20], when $\nu_1 = 0$ and $\nu_2 \neq 0$. It's also an irrational situation. In order to overcome these shortcomings, enlightened by the idea [33] he defined the interactive operations on PFNs, and proposed various Pythagorean fuzzy AOs with the help of interactive operations, including Pythagorean fuzzy interactive weighted average, weighted geometric, ordered weighted average, ordered weighted geometric, hybrid average and hybrid geometric, labeled by PFIWA, PFIWG, PFIOWA, PFIOWG, PFIHA and PFIHG, respectively. After the work [33], certain interactive AOs [34–36] were presented within intuitionistic fuzzy circumstances. Based on the interactive operations [32], Gao *et al.* [37] proposed several interactive power AOs, which can reduce the impact of confused data on integration results. Taking the relationship over aggregated information into account, Wang and Li [38] explored the Pythagorean fuzzy interactive power Bonferroni means (BM) operator and weight form along with their properties and special cases. Zhu *et al.* [39] built interactive power partitioned BM operators for PFNs. Li *et al.* [40] created several new Pythagorean fuzzy interactive hybrid weighted AOs, the merit of these AOs is that possess the property of idempotency.

Meanwhile, alternative and appropriate operational laws play a crucial role in AOs. The Archimedean t-conorm and t-norm (ATT) [41,42] are formed using an additive function and a dual function. When the additive function was chosen with different expressions, ATT reduced to various t-conorm and t-norm, they are the Algebraic, Einstein, Hamacher and Frank t-conorms and t-norms and so on. ATT has been extended to diverse fuzzy circumstances, including IFSs [43,44], interval-valued hesitant fuzzy set [45], single-valued neutrosophic set [46] and dual hesitant fuzzy set [47]. Recently, based on the ATT, Yang *et al.* [48] proposed two Pythagorean fuzzy BM AOs for PFNs, and applied them to match investment selection problem.

From the above discussions, the existing operators [32], [27] for Pythagorean fuzzy information have the following two limitations:

1. Although the AOs presented in [32,37,38] can capture the interactions between MD and NMD for Pythagorean fuzzy information, the operational laws of these AOs only adopted algebra t-conorm and t-norm. It cannot offer a more flexible result.
2. AOs provided in [24,25,27] employing the Einstein and Hamacher operations, respectively, but they cannot acquire reasonable decision-making outcomes under some cases.

Clearly, the ATT [41,48] is a good alternative to avoid the first limitation, which contains many different types of the t-conorm and t-norm, and the algebra form is the special case of ATT. Interactive operations [32] can overcome the second limitation effectively via capturing the interactions between MD and NMD.

On the basis of above analysis, the main intentions of this study are:

1. To explore several novel interactive averaging and geometric AOs about PFNs to avoid the above two limitations with the aid of ATT.
2. To obtain some important properties and some specific cases of the defined newly AOs.
3. To build a new MADM approach by using the proposed interactive AOs in Pythagorean fuzzy context.
4. To demonstrate the availability and the merits of the constructed MADM approach.

An outline of this research is listed as follows: Section 2 concisely recalls the fundamental knowledge of PFSs, interactive operations on PFSs, and ATT. In Section 3, we define certain interactive operational rules on PFSs with the help of ATT. In Section 4, we propose Archimedean based PFIWA (A-PFIWA) and Archimedean based PFIWG (A-PFIWG) operators, and we discuss their desirable properties. In Section 5, we build a novel MADM technique utilizing the constructed A-PFIWA and A-PFIWG operators and in Section 6 we utilize a actual instance to show our technique and compare it with the previous research. Finally, Section 7 concludes the paper.

2. PRELIMINARIES

Some preconditions for PFS, interactive operational rules of PFS and ATT are briefly provided in this section.

2.1. Pythagorean Fuzzy Set

Definition 2.1. [9,10] A PFS P on universal set Z is defined as:

$$P = \{ \langle z, (\mu_P(z), \nu_P(z)) \rangle \mid z \in Z \}. \quad (1)$$

where $\mu_P(z)$, $\nu_P(z)$ stand for the MD and NMD for each $z \in Z$, respectively, which meets the conditions: $\mu_P(z), \nu_P(z) \in [0, 1]$ and $(\mu_P(z))^2 + (\nu_P(z))^2 \in [0, 1]$. Indeterminacy degree is $\pi_P(z) = \sqrt{1 - (\mu_P(z))^2 - (\nu_P(z))^2}$.

Zhang and Xu [20] defined $\beta = (\mu_\beta, \nu_\beta)$ as a PFN with $\mu_\beta, \nu_\beta \in [0, 1]$ and $\mu_\beta^2 + \nu_\beta^2 \in [0, 1]$. All PFNs are denoted by Ω . Wei [32] defined interactive operational laws for PFNs.

Definition 2.2. Suppose $\beta = (\mu_\beta, \nu_\beta)$, $\beta_1 = (\mu_{\beta_1}, \nu_{\beta_1})$ and $\beta_2 = (\mu_{\beta_2}, \nu_{\beta_2})$ are three PFNs, then the interactive operations are provided in the following expressions [32]:

$$\begin{aligned} (1) \quad \beta_1 \oplus \beta_2 &= \left(\sqrt{\mu_{\beta_1}^2 + \mu_{\beta_2}^2 - \mu_{\beta_1}^2 \mu_{\beta_2}^2}, \sqrt{\nu_{\beta_1}^2 + \nu_{\beta_2}^2 - \nu_{\beta_1}^2 \nu_{\beta_2}^2 - \mu_{\beta_1}^2 \nu_{\beta_2}^2 - \nu_{\beta_1}^2 \mu_{\beta_2}^2} \right); \\ (2) \quad \beta_1 \otimes \beta_2 &= \left(\sqrt{\mu_{\beta_1}^2 + \mu_{\beta_2}^2 - \mu_{\beta_1}^2 \mu_{\beta_2}^2 - \mu_{\beta_1}^2 \nu_{\beta_2}^2 - \nu_{\beta_1}^2 \mu_{\beta_2}^2}, \sqrt{\nu_{\beta_1}^2 + \nu_{\beta_2}^2 - \nu_{\beta_1}^2 \nu_{\beta_2}^2} \right); \\ (3) \quad k\beta &= \left(\sqrt{1 - (1 - \mu_\beta^2)^k}, \sqrt{(1 - \mu_\beta^2)^k - (1 - (\mu_\beta^2 + \nu_\beta^2))^k} \right), k > 0; \\ (4) \quad \beta^k &= \left(\sqrt{(1 - \nu_\beta^2)^k - (1 - (\mu_\beta^2 + \nu_\beta^2))^k}, \sqrt{1 - (1 - \nu_\beta^2)^k} \right), k > 0. \end{aligned}$$

Zhang and Xu [20] also have given the comparison rules for two PFNs.

Definition 2.3. [20] Let $\beta = (\mu_\beta, \nu_\beta)$ be a PFN, the score and accuracy of β are expressed by $s(\beta) = \mu_\beta^2 - \nu_\beta^2$ and $a(\beta) = \mu_\beta^2 + \nu_\beta^2$, respectively. Suppose β_1, β_2 are two PFNs, the following two expressions, as long as one holds

1. $s(\beta_1) > s(\beta_2)$;
2. $s(\beta_1) = s(\beta_2)$ and $a(\beta_1) > a(\beta_2)$.

then $\beta_1 \succ \beta_2$.

2.2. Archimedean t-conorm and t-norm

The notions of Archimedean t-norm and t-conorm were initiated by Klir and Yuan [41] and Nguyen and Walker [42], the formal definitions shown as follows:

Definition 2.4. [41,42] A t-norm is a mapping $T : [0, 1]^2 \rightarrow [0, 1]$, and meets the following properties:

1. $T(1, a) = a$, for all a ;
2. $T(a, b) = T(b, a)$, for all a and b ;
3. $T(a, T(b, c)) = T(T(a, b), c)$, for all a, b and c ;
4. If $a \leq b$ and $c \leq d$ then $T(a, c) \leq T(b, d)$.

Definition 2.5. [41,42] A t-conorm is a mapping $T^* : [0, 1]^2 \rightarrow [0, 1]$, and meets the following properties:

1. $T^*(0, a) = a$, for all a ;
2. $T^*(a, b) = T^*(b, a)$, for all a and b ;
3. $T^*(a, T^*(b, c)) = T^*(T^*(a, b), c)$, for all a, b and c ;
4. If $a \leq b$ and $c \leq d$ then $T^*(a, c) \leq T^*(b, d)$.

Ref. [49] indicated that a strict Archimedean t-norm is depicted through its following additive continuous function l :

$$T(a, b) = l^{-1}(l(a) + l(b)) \quad (2)$$

In which $l : [0, 1] \rightarrow [0, +\infty)$ is a strictly decreasing function and $l(1) = 0$.

Analogously, utilized to its dual t-conorm, the following expression can be derived:

$$T^*(a, b) = h^{-1}(h(a) + h(b)) \quad (3)$$

where $h(r) = l(1 - r)$.

3. ATT ON PFNs

Enlightened by the idea [50] and Equations (2) and (3), next, we shall develop new interactive operations rules for PFNs in the light of ATT.

Definition 3.1. Suppose $\beta = (\mu_\beta, \nu_\beta)$, $\beta_1 = (\mu_{\beta_1}, \nu_{\beta_1})$ and $\beta_2 = (\mu_{\beta_2}, \nu_{\beta_2})$ are three PFNs, then the novel operational rules for PFNs applying ATT are defined as follows:

$$\beta_1 \oplus \beta_2 = \left(\sqrt{T^*(\mu_{\beta_1}^2, \mu_{\beta_2}^2)}, \sqrt{T^*(\mu_{\beta_1}^2 + \nu_{\beta_1}^2, \mu_{\beta_2}^2 + \nu_{\beta_2}^2) - T^*(\mu_{\beta_1}^2, \mu_{\beta_2}^2)} \right) \quad (4)$$

$$\beta_1 \otimes \beta_2 = \left(\sqrt{T^*(\mu_{\beta_1}^2 + \nu_{\beta_1}^2, \mu_{\beta_2}^2 + \nu_{\beta_2}^2) - T^*(\nu_{\beta_1}^2, \nu_{\beta_2}^2)}, \sqrt{T^*(\nu_{\beta_1}^2, \nu_{\beta_2}^2)} \right) \quad (5)$$

Further, we can deduce

$$\beta_1 \oplus \beta_2 = \left(\sqrt{h^{-1}(h(\mu_{\beta_1}^2) + h(\mu_{\beta_2}^2))}, \sqrt{h^{-1}(h(\mu_{\beta_1}^2 + \nu_{\beta_1}^2) + h(\mu_{\beta_2}^2 + \nu_{\beta_2}^2)) - h^{-1}(h(\mu_{\beta_1}^2) + h(\mu_{\beta_2}^2))} \right) \quad (6)$$

$$\beta_1 \otimes \beta_2 = \left(\sqrt{h^{-1}(h(\mu_{\beta_1}^2 + \nu_{\beta_1}^2) + h(\mu_{\beta_2}^2 + \nu_{\beta_2}^2)) - h^{-1}(h(\nu_{\beta_1}^2) + h(\nu_{\beta_2}^2))}, \sqrt{h^{-1}(h(\nu_{\beta_1}^2) + h(\nu_{\beta_2}^2))} \right) \quad (7)$$

$$k\beta = \left(\sqrt{h^{-1}(kh(\mu_\beta^2))}, \sqrt{h^{-1}(kh(\mu_\beta^2 + \nu_\beta^2)) - h^{-1}(kh(\mu_\beta^2))} \right), k > 0 \quad (8)$$

$$\beta^k = \left(\sqrt{h^{-1}(kh(\mu_\beta^2 + \nu_\beta^2)) - h^{-1}(kh(\nu_\beta^2))}, \sqrt{h^{-1}(kh(\nu_\beta^2))} \right), k > 0 \quad (9)$$

Proof. Now, we shall verify the Equations (6) and (8) and the others are similar.

With the help of Equation (3), we have

$$0 \leq h^{-1}(h(\mu_{\beta_1}^2) + h(\mu_{\beta_2}^2)) \leq 1, 0 \leq h^{-1}(h(\mu_{\beta_1}^2 + \nu_{\beta_1}^2) + h(\mu_{\beta_2}^2 + \nu_{\beta_2}^2)) \leq 1.$$

Since $h(\cdot)$ is a strictly increasing function, so we can obtain

$$h^{-1}(h(\mu_{\beta_1}^2 + \nu_{\beta_1}^2) + h(\mu_{\beta_2}^2 + \nu_{\beta_2}^2)) \geq h^{-1}(h(\mu_{\beta_1}^2) + h(\mu_{\beta_2}^2)),$$

which further gives that

$$0 \leq h^{-1}(h(\mu_{\beta_1}^2 + \nu_{\beta_1}^2) + h(\mu_{\beta_2}^2 + \nu_{\beta_2}^2)) - h^{-1}(h(\mu_{\beta_1}^2) + h(\mu_{\beta_2}^2)) \leq 1.$$

Moreover,

$$\begin{aligned} 0 &\leq \left(\sqrt{h^{-1}(h(\mu_{\beta_1}^2) + h(\mu_{\beta_2}^2))} \right)^2 + \left(\sqrt{h^{-1}(h(\mu_{\beta_1}^2 + \nu_{\beta_1}^2) + h(\mu_{\beta_2}^2 + \nu_{\beta_2}^2)) - h^{-1}(h(\mu_{\beta_1}^2) + h(\mu_{\beta_2}^2))} \right)^2 \\ &= h^{-1}(h(\mu_{\beta_1}^2 + \nu_{\beta_1}^2) + h(\mu_{\beta_2}^2 + \nu_{\beta_2}^2)) \\ &\leq 1. \end{aligned}$$

Hence, Equation (6) holds.

Similarly, from Equation (3), we get

$$0 \leq h^{-1} \left(kh \left(\mu_{\beta}^2 \right) \right) \leq 1, \quad 0 \leq h^{-1} \left(kh \left(\mu_{\beta}^2 + \nu_{\beta}^2 \right) \right) \leq 1.$$

As $h(\cdot)$ is a strictly increasing function, then we have $h^{-1} \left(kh \left(\mu_{\beta}^2 + \nu_{\beta}^2 \right) \right) \geq h^{-1} \left(kh \left(\mu_{\beta}^2 \right) \right)$, that is $0 \leq h^{-1} \left(kh \left(\mu_{\beta}^2 + \nu_{\beta}^2 \right) \right) - h^{-1} \left(kh \left(\mu_{\beta}^2 \right) \right) \leq 1$. Additionally,

$$\begin{aligned} 0 &\leq \left(\sqrt{h^{-1} \left(kh \left(\mu_{\beta}^2 \right) \right)} \right)^2 + \left(\sqrt{h^{-1} \left(kh \left(\mu_{\beta}^2 + \nu_{\beta}^2 \right) \right)} - \sqrt{h^{-1} \left(kh \left(\mu_{\beta}^2 \right) \right)} \right)^2 \\ &= h^{-1} \left(kh \left(\mu_{\beta}^2 + \nu_{\beta}^2 \right) \right) \\ &\leq 1. \end{aligned}$$

So, Equation (8) holds.

If we set specific expressions to the function l , then the following several fundamental t-conorms and t-norms can be obtained:

Case 1: If $l(r) = -\log(r)$, then $h(r) = -\log(1-r)$, $l^{-1}(r) = e^{-r}$ and $h^{-1}(r) = 1 - e^{-r}$, based on Definition 3.1, we have

$$(1) \quad \beta_1 \oplus \beta_2 = \left(\sqrt{\mu_{\beta_1}^2 + \mu_{\beta_2}^2 - \mu_{\beta_1}^2 \mu_{\beta_2}^2}, \sqrt{\nu_{\beta_1}^2 + \nu_{\beta_2}^2 - \nu_{\beta_1}^2 \nu_{\beta_2}^2 - \mu_{\beta_1}^2 \nu_{\beta_2}^2 - \nu_{\beta_1}^2 \mu_{\beta_2}^2} \right) \quad (10)$$

$$(2) \quad \beta_1 \otimes \beta_2 = \left(\sqrt{\mu_{\beta_1}^2 + \mu_{\beta_2}^2 - \mu_{\beta_1}^2 \mu_{\beta_2}^2 - \mu_{\beta_1}^2 \nu_{\beta_2}^2 - \nu_{\beta_1}^2 \mu_{\beta_2}^2}, \sqrt{\nu_{\beta_1}^2 + \nu_{\beta_2}^2 - \nu_{\beta_1}^2 \nu_{\beta_2}^2} \right) \quad (11)$$

$$(3) \quad k\beta = \left(\sqrt{1 - \left(1 - \mu_{\beta}^2\right)^k}, \sqrt{\left(1 - \mu_{\beta}^2\right)^k - \left(1 - \left(\mu_{\beta}^2 + \nu_{\beta}^2\right)\right)^k} \right) \quad (12)$$

$$(4) \quad \beta^k = \left(\sqrt{\left(1 - \nu_{\beta}^2\right)^k - \left(1 - \left(\mu_{\beta}^2 + \nu_{\beta}^2\right)\right)^k}, \sqrt{1 - \left(1 - \nu_{\beta}^2\right)^k} \right) \quad (13)$$

which are Algebraic interactive operations on PFNs defined by Wei [32].

Case 2: If $l(r) = \log\left(\frac{2-r}{r}\right)$, then $h(r) = \log\left(\frac{1+r}{1-r}\right)$, and $h^{-1}(r) = \frac{e^r - 1}{e^r + 1}$, from Definition 3.1, we get

$$(1) \quad \beta_1 \oplus \beta_2 = \left(\sqrt{\frac{\prod_{j=1}^2 (1 + \mu_{\beta_j}^2) - \prod_{j=1}^2 (1 - \mu_{\beta_j}^2)}{\prod_{j=1}^2 (1 + \mu_{\beta_j}^2) + \prod_{j=1}^2 (1 - \mu_{\beta_j}^2)}}, \sqrt{\frac{\prod_{j=1}^2 (1 + \mu_{\beta_j}^2 + \nu_{\beta_j}^2) - \prod_{j=1}^2 (1 - \mu_{\beta_j}^2 - \nu_{\beta_j}^2)}{\prod_{j=1}^2 (1 + \mu_{\beta_j}^2 + \nu_{\beta_j}^2) + \prod_{j=1}^2 (1 - \mu_{\beta_j}^2 - \nu_{\beta_j}^2)}} - \frac{\prod_{j=1}^2 (1 + \mu_{\beta_j}^2) - \prod_{j=1}^2 (1 - \mu_{\beta_j}^2)}{\prod_{j=1}^2 (1 + \mu_{\beta_j}^2) + \prod_{j=1}^2 (1 - \mu_{\beta_j}^2)} \right) \quad (14)$$

$$(2) \quad \beta_1 \otimes \beta_2 = \left(\sqrt{\frac{\prod_{j=1}^2 (1 + \mu_{\beta_j}^2 + \nu_{\beta_j}^2) - \prod_{j=1}^2 (1 - \mu_{\beta_j}^2 - \nu_{\beta_j}^2)}{\prod_{j=1}^2 (1 + \mu_{\beta_j}^2 + \nu_{\beta_j}^2) + \prod_{j=1}^2 (1 - \mu_{\beta_j}^2 - \nu_{\beta_j}^2)}} - \frac{\prod_{j=1}^2 (1 + \nu_{\beta_j}^2) - \prod_{j=1}^2 (1 - \nu_{\beta_j}^2)}{\prod_{j=1}^2 (1 + \nu_{\beta_j}^2) + \prod_{j=1}^2 (1 - \nu_{\beta_j}^2)}, \sqrt{\frac{\prod_{j=1}^2 (1 + \nu_{\beta_j}^2) - \prod_{j=1}^2 (1 - \nu_{\beta_j}^2)}{\prod_{j=1}^2 (1 + \nu_{\beta_j}^2) + \prod_{j=1}^2 (1 - \nu_{\beta_j}^2)}} \right) \quad (15)$$

$$(3) \quad k\beta = \left(\sqrt{\frac{\left(1 + \mu_{\beta}^2\right)^k - \left(1 - \mu_{\beta}^2\right)^k}{\left(1 + \mu_{\beta}^2\right)^k + \left(1 - \mu_{\beta}^2\right)^k}}, \sqrt{\frac{\left(1 + \mu_{\beta}^2 + \nu_{\beta}^2\right)^k - \left(1 - \mu_{\beta}^2 - \nu_{\beta}^2\right)^k}{\left(1 + \mu_{\beta}^2 + \nu_{\beta}^2\right)^k + \left(1 - \mu_{\beta}^2 - \nu_{\beta}^2\right)^k}} - \frac{\left(1 + \mu_{\beta}^2\right)^k - \left(1 - \mu_{\beta}^2\right)^k}{\left(1 + \mu_{\beta}^2\right)^k + \left(1 - \mu_{\beta}^2\right)^k} \right) \quad (16)$$

$$(4) \beta^k = \left(\sqrt{\frac{(1 + \mu_\beta^2 + \nu_\beta^2)^k - (1 - \mu_\beta^2 - \nu_\beta^2)^k}{(1 + \mu_\beta^2 + \nu_\beta^2)^k + (1 - \mu_\beta^2 - \nu_\beta^2)^k}} - \frac{(1 + \nu_\beta^2)^k - (1 - \nu_\beta^2)^k}{(1 + \nu_\beta^2)^k + (1 - \nu_\beta^2)^k}, \sqrt{\frac{(1 + \nu_\beta^2)^k - (1 - \nu_\beta^2)^k}{(1 + \nu_\beta^2)^k + (1 - \nu_\beta^2)^k}} \right) \quad (17)$$

which are called Einstein interactive operations on PFNs.

- If $l(r) = \log\left(\frac{\rho+(1-\rho)r}{r}\right)$, $\rho > 0$, then $h(r) = \log\left(\frac{\rho+(1-\rho)(1-r)}{(1-r)}\right)$, and $h^{-1}(r) = \frac{e^r-1}{e^r+\rho-1}$, according to Definition 3.1, we get

$$(1) \beta_1 \oplus \beta_2 = \left(\sqrt{\frac{\prod_{j=1}^2 (\rho + (1-\rho)(1 - \mu_{\beta_j}^2)) - \prod_{j=1}^2 (1 - \mu_{\beta_j}^2)}{\prod_{j=1}^2 (\rho + (1-\rho)(1 - \mu_{\beta_j}^2)) + (\rho - 1) \prod_{j=1}^2 (1 - \mu_{\beta_j}^2)}}, \sqrt{\frac{\prod_{j=1}^2 (\rho + (1-\rho)(1 - \mu_{\beta_j}^2 - \nu_{\beta_j}^2)) - \prod_{j=1}^2 (1 - \mu_{\beta_j}^2 - \nu_{\beta_j}^2)}{\prod_{j=1}^2 (\rho + (1-\rho)(1 - \mu_{\beta_j}^2 - \nu_{\beta_j}^2)) + (\rho - 1) \prod_{j=1}^2 (1 - \mu_{\beta_j}^2 - \nu_{\beta_j}^2)}}, \right. \\ \left. \sqrt{\frac{\prod_{j=1}^2 (\rho + (1-\rho)(1 - \mu_{\beta_j}^2)) - \prod_{j=1}^2 (1 - \mu_{\beta_j}^2)}{\prod_{j=1}^2 (\rho + (1-\rho)(1 - \mu_{\beta_j}^2)) + (\rho - 1) \prod_{j=1}^2 (1 - \mu_{\beta_j}^2)}}} \right) \quad (18)$$

$$(2) \beta_1 \otimes \beta_2 = \left(\sqrt{\frac{\prod_{j=1}^2 (\rho + (1-\rho)(1 - \mu_{\beta_j}^2 - \nu_{\beta_j}^2)) - \prod_{j=1}^2 (1 - \mu_{\beta_j}^2 - \nu_{\beta_j}^2)}{\prod_{j=1}^2 (\rho + (1-\rho)(1 - \mu_{\beta_j}^2 - \nu_{\beta_j}^2)) + (\rho - 1) \prod_{j=1}^2 (1 - \mu_{\beta_j}^2 - \nu_{\beta_j}^2)}}, \sqrt{\frac{\prod_{j=1}^2 (\rho + (1-\rho)(1 - \nu_{\beta_j}^2)) - \prod_{j=1}^2 (1 - \nu_{\beta_j}^2)}{\prod_{j=1}^2 (\rho + (1-\rho)(1 - \nu_{\beta_j}^2)) + (\rho - 1) \prod_{j=1}^2 (1 - \nu_{\beta_j}^2)}}}, \right. \\ \left. \sqrt{\frac{\prod_{j=1}^2 (\rho + (1-\rho)(1 - \nu_{\beta_j}^2)) - \prod_{j=1}^2 (1 - \nu_{\beta_j}^2)}{\prod_{j=1}^2 (\rho + (1-\rho)(1 - \nu_{\beta_j}^2)) + (\rho - 1) \prod_{j=1}^2 (1 - \nu_{\beta_j}^2)}}} \right) \quad (19)$$

$$(3) k\beta = \left(\sqrt{\frac{(\rho + (1-\rho)(1 - \mu_\beta^2))^k - (1 - \mu_\beta^2)^k}{(\rho + (1-\rho)(1 - \mu_\beta^2))^k + (\rho - 1)(1 - \mu_\beta^2)^k}}, \sqrt{\frac{(\rho + (1-\rho)(1 - \mu_\beta^2 - \nu_\beta^2))^k - (1 - \mu_\beta^2 - \nu_\beta^2)^k}{(\rho + (1-\rho)(1 - \mu_\beta^2 - \nu_\beta^2))^k + (\rho - 1)(1 - \mu_\beta^2 - \nu_\beta^2)^k}}}, \right. \\ \left. \sqrt{\frac{(\rho + (1-\rho)(1 - \mu_\beta^2))^k - (1 - \mu_\beta^2)^k}{(\rho + (1-\rho)(1 - \mu_\beta^2))^k + (\rho - 1)(1 - \mu_\beta^2)^k}}}, \sqrt{\frac{(\rho + (1-\rho)(1 - \mu_\beta^2 - \nu_\beta^2))^k - (1 - \mu_\beta^2 - \nu_\beta^2)^k}{(\rho + (1-\rho)(1 - \mu_\beta^2 - \nu_\beta^2))^k + (\rho - 1)(1 - \mu_\beta^2 - \nu_\beta^2)^k}}}, \right) \quad (20)$$

$$(4) \beta^k = \left(\sqrt{\frac{\left(\rho + (1-\rho)(1-\mu_\beta^2 - \nu_\beta^2)\right)^k - (1-\mu_\beta^2 - \nu_\beta^2)^k}{\left(\rho + (1-\rho)(1-\mu_\beta^2 - \nu_\beta^2)\right)^k + (\rho-1)(1-\mu_\beta^2 - \nu_\beta^2)^k}}, \sqrt{\frac{\left(\rho + (1-\rho)(1-\nu_\beta^2)\right)^k - (1-\nu_\beta^2)^k}{\left(\rho + (1-\rho)(1-\nu_\beta^2)\right)^k + (\rho-1)(1-\nu_\beta^2)^k}} \right) \quad (21)$$

- If $l(r) = -\log\left(\frac{\varepsilon-1}{\varepsilon^r-1}\right)$, $\varepsilon > 1$, then $h(r) = -\log\left(\frac{\varepsilon-1}{\varepsilon^{1-r}-1}\right)$, and $h^{-1}(r) = 1 - \log_\varepsilon\left(\frac{\varepsilon-1+e^{-r}}{e^{-r}}\right)$, on the basis of Definition 3.1, we obtain:

$$(1) \beta_1 \oplus \beta_2 = \left(\sqrt{1 - \log_\varepsilon\left(1 + \frac{1}{\varepsilon-1} \prod_{j=1}^2 \left(\varepsilon^{1-\mu_{\beta_j}^2} - 1\right)\right)}, \sqrt{\log_\varepsilon\left(\frac{1 + \frac{1}{\varepsilon-1} \prod_{j=1}^2 \left(\varepsilon^{1-\mu_{\beta_j}^2} - 1\right)}{1 + \frac{1}{\varepsilon-1} \prod_{j=1}^2 \left(\varepsilon^{1-\mu_{\beta_j}^2 - \nu_{\beta_j}^2} - 1\right)}\right)} \right) \quad (22)$$

$$(2) \beta_1 \otimes \beta_2 = \left(\sqrt{\log_\varepsilon\left(\frac{1 + \frac{1}{\varepsilon-1} \prod_{j=1}^2 \left(\varepsilon^{1-\nu_{\beta_j}^2} - 1\right)}{1 + \frac{1}{\varepsilon-1} \prod_{j=1}^2 \left(\varepsilon^{1-\mu_{\beta_j}^2 - \nu_{\beta_j}^2} - 1\right)}\right)}, \sqrt{1 - \log_\varepsilon\left(1 + \frac{1}{\varepsilon-1} \prod_{j=1}^2 \left(\varepsilon^{1-\nu_{\beta_j}^2} - 1\right)\right)} \right) \quad (23)$$

$$(3) \lambda\beta = \left(\sqrt{1 - \log_\varepsilon\left(1 + \frac{\left(\varepsilon^{1-\mu_\beta^2} - 1\right)^\lambda}{(\varepsilon-1)^{\lambda-1}}\right)}, \sqrt{\log_\varepsilon\left(\frac{(\varepsilon-1)^{\lambda-1} + \left(\varepsilon^{1-\mu_\beta^2} - 1\right)^\lambda}{(\varepsilon-1)^{\lambda-1} + \left(\varepsilon^{1-\mu_\beta^2 - \nu_\beta^2} - 1\right)^\lambda}\right)} \right) \quad (24)$$

$$(4) \beta^\lambda = \left(\sqrt{\log_\varepsilon\left(\frac{(\varepsilon-1)^{\lambda-1} + \left(\varepsilon^{1-\nu_\beta^2} - 1\right)^\lambda}{(\varepsilon-1)^{\lambda-1} + \left(\varepsilon^{1-\mu_\beta^2 - \nu_\beta^2} - 1\right)^\lambda}\right)}, \sqrt{1 - \log_\varepsilon\left(1 + \frac{\left(\varepsilon^{1-\nu_\beta^2} - 1\right)^\lambda}{(\varepsilon-1)^{\lambda-1}}\right)} \right) \quad (25)$$

which are called Frank interactive operations on PFNs.

Furthermore, we can verify some operational laws straightforward as follows.

Theorem 3.1. Suppose $\beta = (\mu_\beta, \nu_\beta)$, $\beta_1 = (\mu_{\beta_1}, \nu_{\beta_1})$ and $\beta_2 = (\mu_{\beta_2}, \nu_{\beta_2})$ are three PFNs, and real numbers $k > 0$, $k_1 > 0$, $k_2 > 0$, then

1. $\beta_1 \oplus \beta_2 = \beta_2 \oplus \beta_1$.
2. $\beta_1 \otimes \beta_2 = \beta_2 \otimes \beta_1$.
3. $k(\beta_1 \oplus \beta_2) = k\beta_1 \oplus k\beta_2$.
4. $(\beta_1 \otimes \beta_2)^k = (\beta_1)^k \otimes (\beta_2)^k$.
5. $k_1\beta \oplus k_2\beta = (k_1 + k_2)\beta$.
6. $\beta^{k_1} \otimes \beta^{k_2} = \beta^{(k_1+k_2)}$.
7. $(\beta^{k_1})^{k_2} = \beta^{k_1 k_2}$.

Proof. We shall prove the parts (3) and (4), while others parts (1), (2), (5), (6), (7) can be proved analogously.

1. For PFNs $\beta_1 = (\mu_{\beta_1}, \nu_{\beta_1})$ and $\beta_2 = (\mu_{\beta_2}, \nu_{\beta_2})$ and a real $k > 0$, we have

$$\begin{aligned} k(\beta_1 \oplus \beta_2) &= k \left(\sqrt{h^{-1} \left(h \left(\mu_{\beta_1}^2 \right) + h \left(\mu_{\beta_2}^2 \right) \right)}, \sqrt{h^{-1} \left(h \left(\mu_{\beta_1}^2 + \nu_{\beta_1}^2 \right) + h \left(\mu_{\beta_2}^2 + \nu_{\beta_2}^2 \right) \right)} - h^{-1} \left(h \left(\mu_{\beta_1}^2 \right) + h \left(\mu_{\beta_2}^2 \right) \right) \right) \\ &= \left(\sqrt{h^{-1} \left(kh \left(h^{-1} \left(h \left(\mu_{\beta_1}^2 \right) + h \left(\mu_{\beta_2}^2 \right) \right) \right) \right)}, \sqrt{h^{-1} \left(kh \left(h^{-1} \left(h \left(\mu_{\beta_1}^2 + \nu_{\beta_1}^2 \right) + h \left(\mu_{\beta_2}^2 + \nu_{\beta_2}^2 \right) \right) \right) - h^{-1} \left(kh \left(h^{-1} \left(h \left(\mu_{\beta_1}^2 \right) + h \left(\mu_{\beta_2}^2 \right) \right) \right) \right) \right)} \right) \\ &= \left(\sqrt{h^{-1} \left(kh \left(\mu_{\beta_1}^2 \right) + kh \left(\mu_{\beta_2}^2 \right) \right)}, \sqrt{h^{-1} \left(kh \left(\mu_{\beta_1}^2 + \nu_{\beta_1}^2 \right) + kh \left(\mu_{\beta_2}^2 + \nu_{\beta_2}^2 \right) \right)} - h^{-1} \left(kh \left(\mu_{\beta_1}^2 \right) + kh \left(\mu_{\beta_2}^2 \right) \right) \right) \\ &= k\beta_1 \oplus k\beta_2 \end{aligned}$$

2. For PFNs β_1 and β_2 , we have

$$\begin{aligned} (\beta_1 \otimes \beta_2)^k &= \left(\sqrt{h^{-1} \left(h \left(\mu_{\beta_1}^2 + \nu_{\beta_1}^2 \right) + h \left(\mu_{\beta_2}^2 + \nu_{\beta_2}^2 \right) \right)} - h^{-1} \left(h \left(\nu_{\beta_1}^2 \right) + h \left(\nu_{\beta_2}^2 \right) \right) \right)^k, \sqrt{h^{-1} \left(h \left(\nu_{\beta_1}^2 \right) + h \left(\nu_{\beta_2}^2 \right) \right)} \\ &= \left(\sqrt{h^{-1} \left(kh \left(h^{-1} \left(h \left(\mu_{\beta_1}^2 + \nu_{\beta_1}^2 \right) + h \left(\mu_{\beta_2}^2 + \nu_{\beta_2}^2 \right) \right) \right) \right)} - h^{-1} \left(kh \left(h^{-1} \left(h \left(\nu_{\beta_1}^2 \right) + h \left(\nu_{\beta_2}^2 \right) \right) \right) \right) \right)^k, \\ &\quad \sqrt{h^{-1} \left(kh \left(h^{-1} \left(h \left(\nu_{\beta_1}^2 \right) + h \left(\nu_{\beta_2}^2 \right) \right) \right) \right)} \\ &= \left(\sqrt{h^{-1} \left(kh \left(\mu_{\beta_1}^2 + \nu_{\beta_1}^2 \right) + kh \left(\mu_{\beta_2}^2 + \nu_{\beta_2}^2 \right) \right)} - h^{-1} \left(kh \left(\nu_{\beta_1}^2 \right) + kh \left(\nu_{\beta_2}^2 \right) \right) \right)^k, \sqrt{h^{-1} \left(kh \left(\nu_{\beta_1}^2 \right) + kh \left(\nu_{\beta_2}^2 \right) \right)} \\ &= (\beta_1)^k \otimes (\beta_2)^k \end{aligned}$$

4. ARCHIMEDEAN BASED PYTHAGOREAN FUZZY INTERACTIVE AOs

The focus of this part is to establish some novel interactive AOs for PFNs based on the ATT.

4.1. Archimedean Based Pythagorean Fuzzy Interactive Averaging Operators

Definition 4.1. Suppose $\beta_j = (\mu_{\beta_j}, \nu_{\beta_j})$ ($j = 1, 2, 3, \dots, n$) is a group of PFNs, an A-PFIWA operator is a mapping $A - PFIWA : \Omega^n \rightarrow \Omega$, meeting

$$A - PFIWA (\beta_1, \beta_2, \dots, \beta_n) = \bigoplus_{j=1}^n \omega_j \beta_j \quad (26)$$

where, $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the weight vector of β_j ($j = 1, 2, 3, \dots, n$), satisfying $\omega_j \in [0, 1]$, $\sum_{j=1}^n \omega_j = 1$.

Theorem 4.1. Suppose $\beta_j = (\mu_{\beta_j}, \nu_{\beta_j})$ ($j = 1, 2, 3, \dots, n$) is a group of PFNs, then the aggregated outcome employing A-PFIWA operator is also a PFN, and

$$A - PFIWA (\beta_1, \beta_2, \dots, \beta_n) = \bigoplus_{j=1}^n \omega_j \beta_j = \left(\sqrt{h^{-1} \left(\sum_{j=1}^n \omega_j h \left(\mu_{\beta_j}^2 \right) \right)}, \sqrt{h^{-1} \left(\sum_{j=1}^n \omega_j h \left(\mu_{\beta_j}^2 + \nu_{\beta_j}^2 \right) \right)} - h^{-1} \left(\sum_{j=1}^n \omega_j h \left(\mu_{\beta_j}^2 \right) \right) \right) \quad (27)$$

Proof of Theorem 4.1.

We verify Equation (27) via utilizing mathematical induction on n .

1. When $n = 2$, and by the operations of the PFNs on β_1 and β_2 , we have

$$\omega_1 \beta_1 = \left(\sqrt{h^{-1} \left(\omega_1 h \left(\mu_{\beta_1}^2 \right) \right)}, \sqrt{h^{-1} \left(\omega_1 h \left(\mu_{\beta_1}^2 + \nu_{\beta_1}^2 \right) \right)} - h^{-1} \left(\omega_1 h \left(\mu_{\beta_1}^2 \right) \right) \right)$$

$$\omega_2\beta_2 = \left(\sqrt{h^{-1}(\omega_2 h(\mu_{\beta_2}^2))}, \sqrt{h^{-1}(\omega_2 h(\mu_{\beta_2}^2 + v_{\beta_2}^2)) - h^{-1}(\omega_2 h(\mu_{\beta_2}^2))} \right)$$

then

$$\begin{aligned} & \omega_1\beta_1 \oplus \omega_2\beta_2 \\ &= \left(\sqrt{h^{-1}\left(h\left(h^{-1}(\omega_1 h(\mu_{\beta_1}^2))\right) + h\left(h^{-1}(\omega_2 h(\mu_{\beta_2}^2))\right)\right)}, \sqrt{\left(h^{-1}\left(h\left(h^{-1}(\omega_1 h(\mu_{\beta_1}^2 + v_{\beta_1}^2))\right) + h\left(h^{-1}(\omega_2 h(\mu_{\beta_2}^2 + v_{\beta_2}^2))\right)\right)\right)} \right. \\ &\quad \left. - h^{-1}\left(h\left(h^{-1}(\omega_1 h(\mu_{\beta_1}^2))\right) + h\left(h^{-1}(\omega_2 h(\mu_{\beta_2}^2))\right)\right)\right) \right) \\ &= \left(\sqrt{h^{-1}(\omega_1 h(\mu_{\beta_1}^2) + \omega_2 h(\mu_{\beta_2}^2))}, \sqrt{\left(h^{-1}(\omega_1 h(\mu_{\beta_1}^2 + v_{\beta_1}^2) + \omega_2 h(\mu_{\beta_2}^2 + v_{\beta_2}^2))\right)} \right. \\ &\quad \left. - h^{-1}(\omega_1 h(\mu_{\beta_1}^2) + \omega_2 h(\mu_{\beta_2}^2))\right) \right) \\ &= \left(\sqrt{h^{-1}\left(\sum_{j=1}^2 \omega_j h(\mu_{\beta_j}^2)\right)}, \sqrt{h^{-1}\left(\sum_{j=1}^2 \omega_j h(\mu_{\beta_j}^2 + v_{\beta_j}^2)\right) - h^{-1}\left(\sum_{j=1}^2 \omega_j h(\mu_{\beta_j}^2)\right)} \right) \end{aligned}$$

2. Assume Equation (27) is true for $n = k$, that is,

$$A - PFIWA(\beta_1, \beta_2, \dots, \beta_k) = \bigoplus_{j=1}^k \omega_j \beta_j = \left(\sqrt{h^{-1}\left(\sum_{j=1}^k \omega_j h(\mu_{\beta_j}^2)\right)}, \sqrt{h^{-1}\left(\sum_{j=1}^k \omega_j h(\mu_{\beta_j}^2 + v_{\beta_j}^2)\right) - h^{-1}\left(\sum_{j=1}^k \omega_j h(\mu_{\beta_j}^2)\right)} \right)$$

Then when $n = k + 1$, by Definition 3.1, we obtain

$$\begin{aligned} & A - PFIWA(\beta_1, \beta_2, \dots, \beta_k, \beta_{k+1}) = A - PFIWA(\beta_1, \beta_2, \dots, \beta_k) \oplus \omega_{k+1}\beta_{k+1} \\ &= \left(\sqrt{h^{-1}\left(\sum_{j=1}^k \omega_j h(\mu_{\beta_j}^2)\right)}, \sqrt{h^{-1}\left(\sum_{j=1}^k \omega_j h(\mu_{\beta_j}^2 + v_{\beta_j}^2)\right) - h^{-1}\left(\sum_{j=1}^k \omega_j h(\mu_{\beta_j}^2)\right)} \right) \\ &\quad \oplus \left(\sqrt{h^{-1}(\omega_{k+1} h(\mu_{\beta_{k+1}}^2))}, \sqrt{h^{-1}(\omega_{k+1} h(\mu_{\beta_{k+1}}^2 + v_{\beta_{k+1}}^2)) - h^{-1}(\omega_{k+1} h(\mu_{\beta_{k+1}}^2))} \right) \\ &= \left(\sqrt{h^{-1}\left(h\left(h^{-1}\left(\sum_{j=1}^k \omega_j h(\mu_{\beta_j}^2)\right)\right) + h\left(h^{-1}(\omega_{k+1} h(\mu_{\beta_{k+1}}^2))\right)\right)}, \right. \\ &\quad \left. \sqrt{h^{-1}\left(h\left(h^{-1}\left(\sum_{j=1}^k \omega_j h(\mu_{\beta_j}^2 + v_{\beta_j}^2)\right)\right) + h\left(h^{-1}(\omega_{k+1} h(\mu_{\beta_{k+1}}^2 + v_{\beta_{k+1}}^2))\right)\right) - h^{-1}\left(h\left(h^{-1}\left(\sum_{j=1}^k \omega_j h(\mu_{\beta_j}^2)\right)\right) + h\left(h^{-1}(\omega_{k+1} h(\mu_{\beta_{k+1}}^2))\right)\right)} \right) \\ &= \left(\sqrt{h^{-1}\left(\sum_{j=1}^k \omega_j h(\mu_{\beta_j}^2) + \omega_{k+1} h(\mu_{\beta_{k+1}}^2)\right)}, \sqrt{\left(h^{-1}\left(\sum_{j=1}^k \omega_j h(\mu_{\beta_j}^2 + v_{\beta_j}^2)\right) + \omega_{k+1} h(\mu_{\beta_{k+1}}^2 + v_{\beta_{k+1}}^2)\right)} \right. \\ &\quad \left. - h^{-1}\left(\sum_{j=1}^k \omega_j h(\mu_{\beta_j}^2) + \omega_{k+1} h(\mu_{\beta_{k+1}}^2)\right)} \right) \\ &= \left(\sqrt{h^{-1}\left(\sum_{j=1}^{k+1} \omega_j h(\mu_{\beta_j}^2)\right)}, \sqrt{h^{-1}\left(\sum_{j=1}^{k+1} \omega_j h(\mu_{\beta_j}^2 + v_{\beta_j}^2)\right) - h^{-1}\left(\sum_{j=1}^{k+1} \omega_j h(\mu_{\beta_j}^2)\right)} \right) \end{aligned}$$

That is, Equation (27) is true for $n = k + 1$.

Hence, according to the result of (1) and (2), Equation (27) is true for all n .

Furthermore, function $h(r)$ is strictly increasing, then

$$\begin{aligned} 0 &\leq \left(\sqrt{h^{-1} \left(\sum_{j=1}^n \omega_j h(\mu_{\beta_j}^2) \right)} \right)^2 + \left(\sqrt{h^{-1} \left(\sum_{j=1}^n \omega_j h(\mu_{\beta_j}^2 + \nu_{\beta_j}^2) \right)} - h^{-1} \left(\sum_{j=1}^n \omega_j h(\mu_{\beta_j}^2) \right) \right)^2 \\ &= h^{-1} \left(\sum_{j=1}^n \omega_j h(\mu_{\beta_j}^2 + \nu_{\beta_j}^2) \right) \leq h^{-1} \left(\sum_{j=1}^n \omega_j h(1) \right) = h^{-1} h(1) \left(\sum_{j=1}^n \omega_j \right) = 1 \end{aligned}$$

Therefore, the proof is completed.

When we adopt diverse forms of the function $l(r)$, then some especial AOs can be determined as follows:

Case 1: If $l(r) = -\log(r)$, according to Theorem 4.1, then A-PFIWA operator becomes the PFIWA operator introduced by Wei [32].

$$PFIWA(\beta_1, \beta_2, \dots, \beta_n) = \left(\sqrt{1 - \prod_{j=1}^n (1 - \mu_{\beta_j}^2)^{\omega_j}}, \sqrt{\prod_{j=1}^n (1 - \mu_{\beta_j}^2)^{\omega_j} - \prod_{j=1}^n (1 - \mu_{\beta_j}^2 - \nu_{\beta_j}^2)^{\omega_j}} \right) \quad (28)$$

Case 2: If $l(r) = \log\left(\frac{2-r}{r}\right)$, from Theorem 4.1, A-PFIWA operator becomes

$$\begin{aligned} EPFIWA(\beta_1, \beta_2, \dots, \beta_n) \\ = \left(\sqrt{\frac{\prod_{j=1}^n (1 + \mu_{\beta_j}^2)^{\omega_j} - \prod_{j=1}^n (1 - \mu_{\beta_j}^2)^{\omega_j}}{\prod_{j=1}^n (1 + \mu_{\beta_j}^2)^{\omega_j} + \prod_{j=1}^n (1 - \mu_{\beta_j}^2)^{\omega_j}}}, \right. \\ \left. \sqrt{\frac{\prod_{j=1}^n (1 + \mu_{\beta_j}^2 + \nu_{\beta_j}^2)^{\omega_j} - \prod_{j=1}^n (1 - \mu_{\beta_j}^2 - \nu_{\beta_j}^2)^{\omega_j}}{\prod_{j=1}^n (1 + \mu_{\beta_j}^2 + \nu_{\beta_j}^2)^{\omega_j} + \prod_{j=1}^n (1 - \mu_{\beta_j}^2 - \nu_{\beta_j}^2)^{\omega_j}} - \frac{\prod_{j=1}^n (1 + \mu_{\beta_j}^2)^{\omega_j} - \prod_{j=1}^n (1 - \mu_{\beta_j}^2)^{\omega_j}}{\prod_{j=1}^n (1 + \mu_{\beta_j}^2)^{\omega_j} + \prod_{j=1}^n (1 - \mu_{\beta_j}^2)^{\omega_j}}} \right) \end{aligned} \quad (29)$$

which is called the Einstein PFIWA (EPFIWA) operator.

Case 3: If $l(r) = \log\left(\frac{\rho+(1-\rho)r}{r}\right)$, $\rho > 0$, then in the light of Theorem 4.1, A-PFIWA operator becomes

$$\begin{aligned} HPFIWA(\beta_1, \beta_2, \dots, \beta_n) \\ = \left(\sqrt{\frac{\prod_{j=1}^n (\rho + (1-\rho)(1 - \mu_{\beta_j}^2))^{\omega_j} - \prod_{j=1}^n (1 - \mu_{\beta_j}^2)^{\omega_j}}{\prod_{j=1}^n (\rho + (1-\rho)(1 - \mu_{\beta_j}^2))^{\omega_j} + (\rho - 1) \prod_{j=1}^n (1 - \mu_{\beta_j}^2)^{\omega_j}}}, \right. \\ \left. \sqrt{\frac{\prod_{j=1}^n (\rho + (1-\rho)(1 - \mu_{\beta_j}^2 - \nu_{\beta_j}^2))^{\omega_j} - \prod_{j=1}^n (1 - \mu_{\beta_j}^2 - \nu_{\beta_j}^2)^{\omega_j}}{\prod_{j=1}^n (\rho + (1-\rho)(1 - \mu_{\beta_j}^2 - \nu_{\beta_j}^2))^{\omega_j} + (\rho - 1) \prod_{j=1}^n (1 - \mu_{\beta_j}^2 - \nu_{\beta_j}^2)^{\omega_j}} - \frac{\prod_{j=1}^n (\rho + (1-\rho)(1 - \mu_{\beta_j}^2))^{\omega_j} - \prod_{j=1}^n (1 - \mu_{\beta_j}^2)^{\omega_j}}{\prod_{j=1}^n (\rho + (1-\rho)(1 - \mu_{\beta_j}^2))^{\omega_j} + (\rho - 1) \prod_{j=1}^n (1 - \mu_{\beta_j}^2)^{\omega_j}}} \right) \end{aligned} \quad (30)$$

which is called the Hamacher PFIWA (HPFIWA) operator.

Particularly, when $\rho = 1$, then Equation (27) can become the PFIWA operator [32], when $\rho = 2$, then Equation (27) can become the EPFIWA operator.

Case 4: If $l(r) = -\log\left(\frac{\varepsilon-1}{\varepsilon^r-1}\right)$, $\varepsilon > 1$, then A-PFIWA operator becomes

$$FPFIWA(\beta_1, \beta_2, \dots, \beta_n) = \left(\sqrt[1 - \log_\varepsilon\left(1 + \prod_{j=1}^n \left(\varepsilon^{1-\mu_{\beta_j}^2} - 1\right)^{\omega_j}\right)}{\log_\varepsilon\left(\frac{1 + \prod_{j=1}^n \left(\varepsilon^{1-\mu_{\beta_j}^2} - 1\right)^{\omega_j}}{1 + \prod_{j=1}^n \left(\varepsilon^{1-\mu_{\beta_j}^2 - \nu_{\beta_j}^2} - 1\right)^{\omega_j}}\right)} \right) \quad (31)$$

which is called the Frank PFIWA (FPFIWA) operator.

The following useful properties of the explored A-PFIWA operator are discussed carefully.

Property 1 (Idempotency). If all $\beta_j = (\mu_{\beta_j}, \nu_{\beta_j})$ ($j = 1, 2, 3, \dots, n$) are equal to $\beta = (\mu_\beta, \nu_\beta)$, then

$$A - PFIWA(\beta_1, \beta_2, \dots, \beta_n) = \beta \quad (32)$$

Proof. Since $\beta_j = \beta = (\mu, \nu)$ ($j = 1, 2, 3, \dots, n$), based on Theorem 3.1, then we have

$$\begin{aligned} A - PFIWA(\beta_1, \beta_2, \dots, \beta_n) &= \left(\sqrt[h^{-1}\left(\sum_{j=1}^n \omega_j h(\mu_{\beta_j}^2)\right) }{ h^{-1}\left(\sum_{j=1}^n \omega_j h(\mu_{\beta_j}^2 + \nu_{\beta_j}^2)\right) - h^{-1}\left(\sum_{j=1}^n \omega_j h(\mu_{\beta_j}^2)\right) } \right) \\ &= \left(\sqrt[h^{-1}\left(h(\mu_{\beta_j}^2)\sum_{j=1}^n \omega_j\right) }{ h^{-1}\left(h(\mu_{\beta_j}^2 + \nu_{\beta_j}^2)\sum_{j=1}^n \omega_j\right) - h^{-1}\left(h(\mu_{\beta_j}^2)\sum_{j=1}^n \omega_j\right) } \right) \\ &= \left(\sqrt[h^{-1}\left(h(\mu_{\beta_j}^2)\right) }{ h^{-1}\left(h(\mu_{\beta_j}^2 + \nu_{\beta_j}^2)\right) - h^{-1}\left(h(\mu_{\beta_j}^2)\right) } \right) \\ &= (\mu_\beta, \nu_\beta) = \beta \end{aligned}$$

Hence, the Property 1 holds.

Property 2 (Monotonicity). Let $\beta_j = (\mu_{\beta_j}, \nu_{\beta_j})$ and $\alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j})$ ($j = 1, 2, 3, \dots, n$) are two groups of PFNs, if $\mu_{\beta_j} \geq \mu_{\alpha_j}$ and $\mu_{\beta_j}^2 + \nu_{\beta_j}^2 \leq \mu_{\alpha_j}^2 + \nu_{\alpha_j}^2$ then

$$s(A - PFIWA(\beta_1, \beta_2, \dots, \beta_n)) \geq s(A - PFIWA(\alpha_1, \alpha_2, \dots, \alpha_n)) \quad (33)$$

Proof. From Equation (3), we known that $h(r)$ is strictly monotonically increasing, as $\mu_{\beta_j} \geq \mu_{\alpha_j}$, $\mu_{\beta_j}^2 + \nu_{\beta_j}^2 \leq \mu_{\alpha_j}^2 + \nu_{\alpha_j}^2$. Then, we obtain

$$h^{-1}\left(\sum_{j=1}^n \omega_j h(\mu_{\beta_j}^2)\right) \geq h^{-1}\left(\sum_{j=1}^n \omega_j h(\mu_{\alpha_j}^2)\right),$$

$$h^{-1}\left(\sum_{j=1}^n \omega_j h(\mu_{\beta_j}^2 + \nu_{\beta_j}^2)\right) \leq h^{-1}\left(\sum_{j=1}^n \omega_j h(\mu_{\alpha_j}^2 + \nu_{\alpha_j}^2)\right).$$

Further, $h^{-1}\left(\sum_{j=1}^n \omega_j h(\mu_{\beta_j}^2 + \nu_{\beta_j}^2)\right) - h^{-1}\left(\sum_{j=1}^n \omega_j h(\mu_{\beta_j}^2)\right) \leq h^{-1}\left(\sum_{j=1}^n \omega_j h(\mu_{\alpha_j}^2 + \nu_{\alpha_j}^2)\right) - h^{-1}\left(\sum_{j=1}^n \omega_j h(\mu_{\alpha_j}^2)\right)$. On the basis of Definition 2.3 and Equation (27), we have

$$s(A - PFIWA(\beta_1, \beta_2, \dots, \beta_n)) \geq s(A - PFIWA(\alpha_1, \alpha_2, \dots, \alpha_n)).$$

Property 3 (Boundedness). Let $\beta_j = (\mu_{\beta_j}, \nu_{\beta_j})$ be a group of PFNs, then

$$s(\beta_-) \leq s(A - PFIWA(\beta_1, \beta_2, \dots, \beta_n)) \leq s(\beta_+) \quad (34)$$

where $\beta_- = \left(\min\{\mu_{\beta_j}\}, \sqrt{\max\{\mu_{\beta_j}^2 + \nu_{\beta_j}^2\} - \min\{\mu_{\beta_j}^2\}} \right)$, $\beta_+ = \left(\max\{\mu_{\beta_j}\}, \nu_{\beta_+} \right)$

and

$$\nu_{\beta_+} = \begin{cases} \sqrt{\min\{\mu_{\beta_j}^2 + \nu_{\beta_j}^2\} - \max\{\mu_{\beta_j}^2\}}, & \min\{\mu_{\beta_j}^2 + \nu_{\beta_j}^2\} - \max\{\mu_{\beta_j}^2\} > 0 \\ 0, & \min\{\mu_{\beta_j}^2 + \nu_{\beta_j}^2\} - \max\{\mu_{\beta_j}^2\} \leq 0 \end{cases}$$

Proof. Since, $\mu_{\beta_j}^2 \geq \min\{\mu_{\beta_j}^2\}$, then

$$h^{-1} \left(\sum_{j=1}^n \omega_j h \left(\mu_{\beta_j}^2 \right) \right) \geq h^{-1} \left(\sum_{j=1}^n \omega_j h \left(\min\{\mu_{\beta_j}^2\} \right) \right) = \min\{\mu_{\beta_j}^2\},$$

and

$$\sqrt{h^{-1} \left(\sum_{j=1}^n \omega_j h \left(\mu_{\beta_j}^2 \right) \right)} \geq \min\{\mu_{\beta_j}\}.$$

Because $\mu_{\beta_j}^2 + \nu_{\beta_j}^2 \leq \max\{\mu_{\beta_j}^2 + \nu_{\beta_j}^2\}$, then

$$h^{-1} \left(\sum_{j=1}^n \omega_j h \left(\mu_{\beta_j}^2 + \nu_{\beta_j}^2 \right) \right) \leq h^{-1} \left(\sum_{j=1}^n \omega_j h \left(\max\{\mu_{\beta_j}^2 + \nu_{\beta_j}^2\} \right) \right) = \max\{\mu_{\beta_j}^2 + \nu_{\beta_j}^2\}$$

and

$$\sqrt{h^{-1} \left(\sum_{j=1}^n \omega_j h \left(\mu_{\beta_j}^2 + \nu_{\beta_j}^2 \right) \right) - h^{-1} \left(\sum_{j=1}^n \omega_j h \left(\mu_{\beta_j}^2 \right) \right)} \leq \sqrt{\max\{\mu_{\beta_j}^2 + \nu_{\beta_j}^2\} - \min\{\mu_{\beta_j}^2\}}$$

Thus, from Definition 2.3 and Equation (27), we have $s(\beta_-) \leq s(A - PFIWA(\beta_1, \beta_2, \dots, \beta_n))$. Similarity,

$$h^{-1} \left(\sum_{j=1}^n \omega_j h \left(\mu_{\beta_j}^2 \right) \right) \leq \max\{\mu_{\beta_j}^2\}$$

, then

$$\sqrt{h^{-1} \left(\sum_{j=1}^n \omega_j h \left(\mu_{\beta_j}^2 \right) \right)} \leq \max\{\mu_{\beta_j}\}$$

Further,

$$h^{-1} \left(\sum_{j=1}^n \omega_j h \left(\mu_{\beta_j}^2 + \nu_{\beta_j}^2 \right) \right) - h^{-1} \left(\sum_{j=1}^n \omega_j h \left(\mu_{\beta_j}^2 \right) \right) \geq \min\{\mu_{\beta_j}^2 + \nu_{\beta_j}^2\} - \max\{\mu_{\beta_j}^2\}$$

and

$$h^{-1} \left(\sum_{j=1}^n \omega_j h \left(\mu_{\beta_j}^2 + \nu_{\beta_j}^2 \right) \right) - h^{-1} \left(\sum_{j=1}^n \omega_j h \left(\mu_{\beta_j}^2 \right) \right) \geq 0$$

Thus,

$$h^{-1} \left(\sum_{j=1}^n \omega_j h \left(\mu_{\beta_j}^2 + \nu_{\beta_j}^2 \right) \right) - h^{-1} \left(\sum_{j=1}^n \omega_j h \left(\mu_{\beta_j}^2 \right) \right) \geq \max\{0, \min\{\mu_{\beta_j}^2 + \nu_{\beta_j}^2\} - \max\{\mu_{\beta_j}^2\}\}$$

If $\min\{\mu_{\beta_j}^2 + \nu_{\beta_j}^2\} - \max\{\mu_{\beta_j}^2\} > 0$, then

$$\sqrt{h^{-1} \left(\sum_{j=1}^n \omega_j h \left(\mu_{\beta_j}^2 + \nu_{\beta_j}^2 \right) \right) - h^{-1} \left(\sum_{j=1}^n \omega_j h \left(\mu_{\beta_j}^2 \right) \right)} \geq \sqrt{\min\{\mu_{\beta_j}^2 + \nu_{\beta_j}^2\} - \max\{\mu_{\beta_j}^2\}}$$

If $\min\{\mu_{\beta_j}^2 + \nu_{\beta_j}^2\} - \max\{\mu_{\beta_j}^2\} \leq 0$, then

$$\sqrt{h^{-1}\left(\sum_{j=1}^n \omega_j h(\mu_{\beta_j}^2 + \nu_{\beta_j}^2)\right)} - h^{-1}\left(\sum_{j=1}^n \omega_j h(\mu_{\beta_j}^2)\right) \geq 0$$

According to Definition 2.3 and Equation (27), we get $s(A - PFIWA(\beta_1, \beta_2, \dots, \beta_n)) \leq s(\beta_+)$. Therefore, the Property 3 holds.

Property 4 (Shift-invariance). Let $\beta_j = (\mu_{\beta_j}, \nu_{\beta_j})$ be a group of PFNs and $\alpha = (\mu_{\alpha}, \nu_{\alpha})$ be a PFN, then

$$A - PFIWA(\beta_1 \oplus \alpha, \beta_2 \oplus \alpha, \dots, \beta_n \oplus \alpha) = A - PFIWA(\beta_1, \beta_2, \dots, \beta_n) \oplus \alpha \quad (35)$$

Proof. Since,

$$\beta_j \oplus \alpha = \left(\sqrt{h^{-1}\left(h(\mu_{\beta_j}^2) + h(\mu_{\alpha}^2)\right)}, \sqrt{\frac{h^{-1}\left(h(\mu_{\beta_j}^2 + \nu_{\beta_j}^2) + h(\mu_{\alpha}^2 + \nu_{\alpha}^2)\right)}{-h^{-1}\left(h(\mu_{\beta_j}^2) + h(\mu_{\alpha}^2)\right)}} \right) \text{ then}$$

$$A - PFIWA(\beta_1 \oplus \alpha, \beta_2 \oplus \alpha, \dots, \beta_n \oplus \alpha)$$

$$\begin{aligned} &= \left(\sqrt{h^{-1}\left(\sum_{j=1}^n \omega_j h\left(h^{-1}\left(h(\mu_{\beta_j}^2) + h(\mu_{\alpha}^2)\right)\right)\right)}, \sqrt{h^{-1}\left(\sum_{j=1}^n \omega_j h\left(h^{-1}\left(h(\mu_{\beta_j}^2 + \nu_{\beta_j}^2) + h(\mu_{\alpha}^2 + \nu_{\alpha}^2)\right)\right) - h^{-1}\left(\sum_{j=1}^n \omega_j h\left(h^{-1}\left(h(\mu_{\beta_j}^2) + h(\mu_{\alpha}^2)\right)\right)\right)} \right) \\ &= \left(\sqrt{h^{-1}\left(\sum_{j=1}^n \omega_j \left(h(\mu_{\beta_j}^2) + h(\mu_{\alpha}^2)\right)\right)}, \sqrt{h^{-1}\left(\sum_{j=1}^n \omega_j \left(h(\mu_{\beta_j}^2 + \nu_{\beta_j}^2) + h(\mu_{\alpha}^2 + \nu_{\alpha}^2)\right)\right) - h^{-1}\left(\sum_{j=1}^n \omega_j \left(h(\mu_{\beta_j}^2) + h(\mu_{\alpha}^2)\right)\right)} \right) \\ &= \left(\sqrt{h^{-1}\left(\sum_{j=1}^n \omega_j h(\mu_{\beta_j}^2) + h(\mu_{\alpha}^2)\right)}, \sqrt{h^{-1}\left(\sum_{j=1}^n \omega_j h(\mu_{\beta_j}^2 + \nu_{\beta_j}^2) + h(\mu_{\alpha}^2 + \nu_{\alpha}^2)\right) - h^{-1}\left(\sum_{j=1}^n \omega_j h(\mu_{\beta_j}^2) + h(\mu_{\alpha}^2)\right)} \right) \end{aligned}$$

and

$$\begin{aligned} A - PFIWA(\beta_1, \beta_2, \dots, \beta_n) \oplus \alpha &= \left(\sqrt{h^{-1}\left(\sum_{j=1}^n \omega_j h(\mu_{\beta_j}^2)\right)}, \sqrt{h^{-1}\left(\sum_{j=1}^n \omega_j h(\mu_{\beta_j}^2 + \nu_{\beta_j}^2)\right) - h^{-1}\left(\sum_{j=1}^n \omega_j h(\mu_{\beta_j}^2)\right)} \right) \oplus (\mu_{\alpha}, \nu_{\alpha}) \\ &= \left(\sqrt{h^{-1}\left(h\left(h^{-1}\left(\sum_{j=1}^n \omega_j h(\mu_{\beta_j}^2)\right)\right) + h(\mu_{\alpha}^2)\right)}, \right. \\ &\quad \left. \sqrt{\frac{h^{-1}\left(h\left(h^{-1}\left(\sum_{j=1}^n \omega_j h(\mu_{\beta_j}^2 + \nu_{\beta_j}^2)\right)\right) + h(\mu_{\alpha}^2 + \nu_{\alpha}^2)\right)}{-h^{-1}\left(h\left(h^{-1}\left(\sum_{j=1}^n \omega_j h(\mu_{\beta_j}^2)\right)\right) + h(\mu_{\alpha}^2)\right)}} \right) \\ &= \left(\sqrt{h^{-1}\left(\sum_{j=1}^n \omega_j h(\mu_{\beta_j}^2) + h(\mu_{\alpha}^2)\right)}, \sqrt{\frac{h^{-1}\left(\sum_{j=1}^n \omega_j h(\mu_{\beta_j}^2 + \nu_{\beta_j}^2) + h(\mu_{\alpha}^2 + \nu_{\alpha}^2)\right)}{-h^{-1}\left(\sum_{j=1}^n \omega_j h(\mu_{\beta_j}^2) + h(\mu_{\alpha}^2)\right)}} \right) \end{aligned}$$

Thus, the Property 4 is correct.

Property 5. Let $\beta_j = (\mu_{\beta_j}, \nu_{\beta_j})$ be a group of PFNs, if $k > 0$ then

$$A - PFIWA(k\beta_1, k\beta_2, \dots, k\beta_n) = k(A - PFIWA(\beta_1, \beta_2, \dots, \beta_n)) \quad (36)$$

Proof. Because

$$k\beta_j = \left(\sqrt{h^{-1}\left(kh(\mu_{\beta_j}^2)\right)}, \sqrt{h^{-1}\left(kh(\mu_{\beta_j}^2 + \nu_{\beta_j}^2)\right) - h^{-1}\left(kh(\mu_{\beta_j}^2)\right)} \right),$$

so

$$\begin{aligned}
 A - PFIWA(k\beta_1, k\beta_2, \dots, k\beta_n) &= \left(\sqrt{h^{-1} \left(\sum_{j=1}^n \omega_j h \left(h^{-1} \left(kh \left(\mu_{\beta_j}^2 \right) \right) \right) \right)} \right. \\
 &\quad \left. \sqrt{h^{-1} \left(\sum_{j=1}^n \omega_j h \left(h^{-1} \left(kh \left(\mu_{\beta_j}^2 + \nu_{\beta_j}^2 \right) \right) \right) \right)} - h^{-1} \left(\sum_{j=1}^n \omega_j h \left(h^{-1} \left(kh \left(\mu_{\beta_j}^2 \right) \right) \right) \right) \right) \\
 &= \left(\sqrt{h^{-1} \left(k \sum_{j=1}^n \omega_j h \left(\mu_{\beta_j}^2 \right) \right)} \right. \\
 &\quad \left. \sqrt{h^{-1} \left(k \sum_{j=1}^n \omega_j h \left(\mu_{\beta_j}^2 + \nu_{\beta_j}^2 \right) \right)} - h^{-1} \left(k \sum_{j=1}^n \omega_j h \left(\mu_{\beta_j}^2 \right) \right) \right)
 \end{aligned}$$

and

$$\begin{aligned}
 k(A - PFIWA(\beta_1, \beta_2, \dots, \beta_n)) &= k \left(\sqrt{h^{-1} \left(\sum_{j=1}^n \omega_j h \left(\mu_{\beta_j}^2 \right) \right)} \right. \\
 &\quad \left. \sqrt{h^{-1} \left(\sum_{j=1}^n \omega_j h \left(\mu_{\beta_j}^2 + \nu_{\beta_j}^2 \right) \right)} - h^{-1} \left(\sum_{j=1}^n \omega_j h \left(\mu_{\beta_j}^2 \right) \right) \right) \\
 &= \left(\sqrt{h^{-1} \left(kh \left(h^{-1} \left(\sum_{j=1}^n \omega_j h \left(\mu_{\beta_j}^2 \right) \right) \right) \right)} \right. \\
 &\quad \left. \sqrt{h^{-1} \left(kh \left(h^{-1} \left(\sum_{j=1}^n \omega_j h \left(\mu_{\beta_j}^2 + \nu_{\beta_j}^2 \right) \right) \right) \right)} - h^{-1} \left(kh \left(h^{-1} \left(\sum_{j=1}^n \omega_j h \left(\mu_{\beta_j}^2 \right) \right) \right) \right) \right) \\
 &= \left(\sqrt{h^{-1} \left(k \sum_{j=1}^n \omega_j h \left(\mu_{\beta_j}^2 \right) \right)} \right. \\
 &\quad \left. \sqrt{h^{-1} \left(k \sum_{j=1}^n \omega_j h \left(\mu_{\beta_j}^2 + \nu_{\beta_j}^2 \right) \right)} - h^{-1} \left(k \sum_{j=1}^n \omega_j h \left(\mu_{\beta_j}^2 \right) \right) \right)
 \end{aligned}$$

In the light of Properties 4 and 5, we can obtain the following properties:

Property 6. Suppose $\alpha = (\mu_\alpha, \nu_\alpha)$ is a PFN and $\beta_j = (\mu_{\beta_j}, \nu_{\beta_j})$ is a group of PFNs, if $k > 0$, then

$$A - PFIWA(k\beta_1 \oplus \alpha, k\beta_2 \oplus \alpha, \dots, k\beta_n \oplus \alpha) = kA - PFIWA(\beta_1, \beta_2, \dots, \beta_n) \oplus \alpha \quad (37)$$

Proof. It is straightforward, so omitted here.

Property 7. Let $\beta_j = (\mu_{\beta_j}, \nu_{\beta_j})$ and $\alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j})$ ($j = 1, 2, 3, \dots, n$) be two groups of PFNs, then

$$A - PFIWA(\beta_1 \oplus \alpha_1, \beta_2 \oplus \alpha_2, \dots, \beta_n \oplus \alpha_n) = A - PFIWA(\beta_1, \beta_2, \dots, \beta_n) \oplus A - PFIWA(\alpha_1, \alpha_2, \dots, \alpha_n) \quad (38)$$

Proof. For PFNs β_j and α_j , we have

$$\beta_j \oplus \alpha_j = \left(\sqrt{h^{-1} \left(h \left(\mu_{\beta_j}^2 \right) + h \left(\mu_{\alpha_j}^2 \right) \right)} \right. \\
 \left. \sqrt{h^{-1} \left(h \left(\mu_{\beta_j}^2 + \nu_{\beta_j}^2 \right) + h \left(\mu_{\alpha_j}^2 + \nu_{\alpha_j}^2 \right) \right)} - h^{-1} \left(h \left(\mu_{\beta_j}^2 \right) + h \left(\mu_{\alpha_j}^2 \right) \right) \right), \text{ then}$$

$$\begin{aligned}
 &A - PFIWA(\beta_1 \oplus \alpha_1, \beta_2 \oplus \alpha_2, \dots, \beta_n \oplus \alpha_n) \\
 &= \left(\sqrt{h^{-1} \left(\sum_{j=1}^n \omega_j h \left(h^{-1} \left(h \left(\mu_{\beta_j}^2 \right) + h \left(\mu_{\alpha_j}^2 \right) \right) \right) \right)} \right. \\
 &\quad \left. \sqrt{h^{-1} \left(\sum_{j=1}^n \omega_j h \left(h^{-1} \left(h \left(\mu_{\beta_j}^2 + \nu_{\beta_j}^2 \right) + h \left(\mu_{\alpha_j}^2 + \nu_{\alpha_j}^2 \right) \right) \right)} - h^{-1} \left(\sum_{j=1}^n \omega_j h \left(h^{-1} \left(h \left(\mu_{\beta_j}^2 \right) + h \left(\mu_{\alpha_j}^2 \right) \right) \right) \right) \right) \\
 &= \left(\sqrt{h^{-1} \left(\sum_{j=1}^n \omega_j \left(h \left(\mu_{\beta_j}^2 \right) + h \left(\mu_{\alpha_j}^2 \right) \right) \right)} \right. \\
 &\quad \left. \sqrt{h^{-1} \left(\sum_{j=1}^n \omega_j \left(h \left(\mu_{\beta_j}^2 + \nu_{\beta_j}^2 \right) + h \left(\mu_{\alpha_j}^2 + \nu_{\alpha_j}^2 \right) \right) \right)} - h^{-1} \left(\sum_{j=1}^n \omega_j \left(h \left(\mu_{\beta_j}^2 \right) + h \left(\mu_{\alpha_j}^2 \right) \right) \right) \right)
 \end{aligned}$$

and

$$\begin{aligned}
 A - PFIWA(\beta_1, \beta_2, \dots, \beta_n) \oplus A - PFIWA(\alpha_1, \alpha_2, \dots, \alpha_n) &= \left(\sqrt{h^{-1} \left(\sum_{j=1}^n \omega_j h(\mu_{\beta_j}^2) \right)}, \sqrt{h^{-1} \left(\sum_{j=1}^n \omega_j h(\mu_{\beta_j}^2 + \nu_{\beta_j}^2) \right) - h^{-1} \left(\sum_{j=1}^n \omega_j h(\mu_{\beta_j}^2) \right)} \right) \\
 &\oplus \left(\sqrt{h^{-1} \left(\sum_{j=1}^n \omega_j h(\mu_{\alpha_j}^2) \right)}, \sqrt{h^{-1} \left(\sum_{j=1}^n \omega_j h(\mu_{\alpha_j}^2 + \nu_{\alpha_j}^2) \right) - h^{-1} \left(\sum_{j=1}^n \omega_j h(\mu_{\alpha_j}^2) \right)} \right) \\
 &= \left(\sqrt{h^{-1} \left(\sum_{j=1}^n \omega_j h(\mu_{\beta_j}^2) + h(\mu_{\alpha_j}^2) \right)}, \right. \\
 &\quad \left. \sqrt{h^{-1} \left(\sum_{j=1}^n \omega_j h(\mu_{\beta_j}^2 + \nu_{\beta_j}^2) \right) - h^{-1} \left(\sum_{j=1}^n \omega_j h(\mu_{\beta_j}^2) + h(\mu_{\alpha_j}^2) \right)} \right)
 \end{aligned}$$

Then, the result of Property 7 holds.

4.2. Archimedean Based Pythagorean Fuzzy Interactive Geometric Operators

Definition 4.2. Suppose $\beta_j = (\mu_{\beta_j}, \nu_{\beta_j})$ ($j = 1, 2, 3, \dots, n$) is a group of PFNs, an A-PFIWG operator is a mapping $A - PFIWG : \Omega^n \rightarrow \Omega$, meeting

$$A - PFIWG(\beta_1, \beta_2, \dots, \beta_n) = \bigotimes_{j=1}^n \beta_j^{\omega_j} \quad (39)$$

where, $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the weight vector of β_j ($j = 1, 2, 3, \dots, n$), satisfying $\omega_j \in [0, 1]$, $\sum_{j=1}^n \omega_j = 1$.

Similar to Theorem 4.1, we can derive the following result:

Theorem 4.2. Suppose $\beta_j = (\mu_{\beta_j}, \nu_{\beta_j})$ ($j = 1, 2, 3, \dots, n$) is a group of PFNs, then the outcome employing A-PFIWG operator is again a PFN, and

$$A - PFIWG(\beta_1, \beta_2, \dots, \beta_n) = \bigotimes_{j=1}^n \beta_j^{\omega_j} = \left(\sqrt{h^{-1} \left(\sum_{j=1}^n \omega_j h(\mu_{\beta_j}^2 + \nu_{\beta_j}^2) \right) - h^{-1} \left(\sum_{j=1}^n \omega_j h(\nu_{\beta_j}^2) \right)}, \sqrt{h^{-1} \left(\sum_{j=1}^n \omega_j h(\nu_{\beta_j}^2) \right)} \right) \quad (40)$$

Proof. The way of proof is like to Theorem 4.1, so omitted here.

Similar to the A-PFIWA operator, if we utilize different forms of function $l(r)$, then we can acquire the following certain particular AOs:

Case 1: Let $l(r) = -\log(r)$, then utilizing Theorem 4.2, A-PFIWG operator can become the PFIWG operator introduced by Wei [32].

$$PFIWG(\beta_1, \beta_2, \dots, \beta_n) = \left(\sqrt{\prod_{j=1}^n (1 - \nu_{\beta_j}^2)^{\omega_j} - \prod_{j=1}^n (1 - \mu_{\beta_j}^2 - \nu_{\beta_j}^2)^{\omega_j}}, \sqrt{1 - \prod_{j=1}^n (1 - \nu_{\beta_j}^2)^{\omega_j}} \right) \quad (41)$$

Case 2: Let $l(r) = \log\left(\frac{2-r}{r}\right)$, then by Theorem 4.2, A-PFIWA operator becomes

$$\begin{aligned}
 EPFIWG(\beta_1, \beta_2, \dots, \beta_n) &= \left(\sqrt{\frac{\prod_{j=1}^n (1 + \mu_{\beta_j}^2 + \nu_{\beta_j}^2)^{\omega_j} - \prod_{j=1}^n (1 - \mu_{\beta_j}^2 - \nu_{\beta_j}^2)^{\omega_j}}{\prod_{j=1}^n (1 + \mu_{\beta_j}^2 + \nu_{\beta_j}^2)^{\omega_j} + \prod_{j=1}^n (1 - \mu_{\beta_j}^2 - \nu_{\beta_j}^2)^{\omega_j}} - \frac{\prod_{j=1}^n (1 + \nu_{\beta_j}^2)^{\omega_j} - \prod_{j=1}^n (1 - \nu_{\beta_j}^2)^{\omega_j}}{\prod_{j=1}^n (1 + \nu_{\beta_j}^2)^{\omega_j} + \prod_{j=1}^n (1 - \nu_{\beta_j}^2)^{\omega_j}}}, \sqrt{\frac{\prod_{j=1}^n (1 + \nu_{\beta_j}^2)^{\omega_j} - \prod_{j=1}^n (1 - \nu_{\beta_j}^2)^{\omega_j}}{\prod_{j=1}^n (1 + \nu_{\beta_j}^2)^{\omega_j} + \prod_{j=1}^n (1 - \nu_{\beta_j}^2)^{\omega_j}}} \right) \quad (42)
 \end{aligned}$$

which is called the Einstein PFIWG (EPFIWG) operator.

Case 3: Let $l(r) = \log\left(\frac{\rho+(1-\rho)r}{r}\right)$, $\rho > 0$, then with the aid of Theorem 4.2, A-PFIWG operator becomes

$$HPFIWG(\beta_1, \beta_2, \dots, \beta_n) = \left(\sqrt[n]{\frac{\prod_{j=1}^n \left(\rho + (1-\rho) \left(1 - \mu_{\beta_j}^2 - \nu_{\beta_j}^2 \right) \right)^{\omega_j} - \prod_{j=1}^n \left(1 - \mu_{\beta_j}^2 - \nu_{\beta_j}^2 \right)^{\omega_j}}{\prod_{j=1}^n \left(\rho + (1-\rho) \left(1 - \mu_{\beta_j}^2 - \nu_{\beta_j}^2 \right) \right)^{\omega_j} + (\rho-1) \prod_{j=1}^n \left(1 - \mu_{\beta_j}^2 - \nu_{\beta_j}^2 \right)^{\omega_j}}}, \sqrt[n]{\frac{\prod_{j=1}^n \left(\rho + (1-\rho) \left(1 - \nu_{\beta_j}^2 \right) \right)^{\omega_j} - \prod_{j=1}^n \left(1 - \nu_{\beta_j}^2 \right)^{\omega_j}}{\prod_{j=1}^n \left(\rho + (1-\rho) \left(1 - \nu_{\beta_j}^2 \right) \right)^{\omega_j} + (\rho-1) \prod_{j=1}^n \left(1 - \nu_{\beta_j}^2 \right)^{\omega_j}}} \right) \quad (43)$$

which is called the Hammer PFIWG (HPFIWG) operator.

Specially, when $\rho = 1$, then Equation (40) can become the PFIWG operator [32], if $\rho = 2$, then Equation (40) can become the EPFIWG operator.

Case 4: Let $l(r) = -\log\left(\frac{\varepsilon-1}{\varepsilon r-1}\right)$, $\varepsilon > 1$, then in the light of Theorem 4.2, A-PFIWG operator becomes

$$FPFIWG(\beta_1, \beta_2, \dots, \beta_n) = \left(\sqrt[n]{\log_{\varepsilon} \left(\frac{1 + \prod_{j=1}^n \left(\varepsilon^{1-\nu_{\beta_j}^2} - 1 \right)^{\omega_j}}{1 + \prod_{j=1}^n \left(\varepsilon^{1-\mu_{\beta_j}^2 - \nu_{\beta_j}^2} - 1 \right)^{\omega_j}} \right)}, \sqrt[n]{1 - \log_{\varepsilon} \left(1 + \prod_{j=1}^n \left(\varepsilon^{1-\nu_{\beta_j}^2} - 1 \right)^{\omega_j} \right)} \right) \quad (44)$$

which is called the Frank PFIWG (FPFIWG) operator.

Like properties of A-PFIWA operator, we can prove the following outcomes of A-PFIWG easily.

Property 8. Let $\beta_j = (\mu_{\beta_j}, \nu_{\beta_j})$ and $\alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j})$ ($j = 1, 2, 3, \dots, n$) are two groups of PFNs, if $\mu_{\beta_j}^2 + \nu_{\beta_j}^2 \geq \mu_{\alpha_j}^2 + \nu_{\alpha_j}^2$ and $\nu_{\beta_j} \leq \nu_{\alpha_j}$ then

$$s(A - PFIWG(\beta_1, \beta_2, \dots, \beta_n)) \geq s(A - PFIWG(\alpha_1, \alpha_2, \dots, \alpha_n)) \quad (45)$$

Property 9. Let $\beta_j = (\mu_{\beta_j}, \nu_{\beta_j})$ be a group of PFNs, then

$$s(\beta_-) \leq s(A - PFIWG(\beta_1, \beta_2, \dots, \beta_n)) \leq s(\beta_+) \quad (46)$$

where $\beta_- = (\mu_{\beta_-}, \max\{\nu_{\beta_j}\})$ with

$$\mu_{\beta_-} = \begin{cases} \sqrt{\min\{\mu_{\beta_j}^2 + \nu_{\beta_j}^2\} - \max\{\nu_{\beta_j}^2\}}, & \min\{\mu_{\beta_j}^2 + \nu_{\beta_j}^2\} - \max\{\nu_{\beta_j}^2\} > 0 \\ 0, & \min\{\mu_{\beta_j}^2 + \nu_{\beta_j}^2\} - \max\{\nu_{\beta_j}^2\} \leq 0 \end{cases}$$

and $\beta_+ = (\sqrt{\max\{\mu_{\beta_j}^2 + \nu_{\beta_j}^2\} - \min\{\nu_{\beta_j}^2\}}, \min\{\nu_{\beta_j}\})$.

In addition, the remainder properties of A-PFIWG operator are the same as A-PFIWA operator.

5. A NOVEL APPROACH FOR MADM UTILIZING THE CREATED AOs

In a MADM problem with PFNs, consider a collection of alternatives denoted by $A = \{A_1, A_2, \dots, A_m\}$, and attributes $C = \{C_1, C_2, \dots, C_n\}$ whose relevant weight is described by $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$, where $\omega_j \in [0, 1]$ and $\sum_{j=1}^n \omega_j = 1$. The DMs provide their evaluation values in PFNs form, $d_{ij} = (\mu_{d_{ij}}, \nu_{d_{ij}})$ ($i = 1, 2, \dots, m; j = 1, 2, \dots, n$) for the candidate alternative $A_i \in A$ ($i = 1, 2, \dots, m$) concerning the attribute $C_j \in C$ ($j = 1, 2, \dots, n$), thus the decision assessment matrix $D = (d_{ij})_{m \times n}$ can be established, such that $\mu_{d_{ij}}, \nu_{d_{ij}} \in [0, 1]$, and

Table 1 | The Pythagorean fuzzy decision matrix D

	C_1	C_2	C_3	C_4	C_5
A_1	(0.6,0.3)	(0.7,0.1)	(0.9,0.2)	(0.4,0.5)	(0.8,0.2)
A_2	(0.4,0.7)	(0.5,0.7)	(0.8,0.4)	(0.8,0.1)	(0.5,0.6)
A_3	(0.8,0.5)	(0.3,0.5)	(0.8,0.4)	(0.7,0.5)	(0.8,0.0)
A_4	(0.4,0.2)	(0.5,0.6)	(0.6,0.7)	(0.9,0.4)	(0.7,0.6)

$0 \leq (\mu_{d_{ij}})^2 + (\nu_{d_{ij}})^2 \leq 1$. Then, with the aid of the proposed A-PFIWA and A-PFIWG operators, the ranking of the whole alternatives are obtained through the steps below:

Step 1. Convert the preference matrix $D = (d_{ij})_{m \times n}$ into the normalized preference matrix $Y = (y_{ij})_{m \times n}$, in which

$$y_{ij} = \begin{cases} d_{ij}, & \text{for benefit attribute } C_j \\ d_{ij}^c, & \text{for cost attribute } C_j \end{cases}.$$

where $d_{ij}^c = (\nu_{d_{ij}}, \mu_{d_{ij}})$, $(i = 1, 2, \dots, m; j = 1, 2, \dots, n)$ [20,22].

Step 2. Aggregate the PFNs y_{ij} via utilizing the A-PFIWA operator as follows:

$$y_i = A - PFIWA(y_{i1}, y_{i2}, \dots, y_{in})$$

or

$$y_i = A - PFIWG(y_{i1}, y_{i2}, \dots, y_{in}).$$

Step 3. Determine the score value $s(y_i)$ ($i = 1, 2, \dots, m$) for every collective value y_i ($i = 1, 2, \dots, m$).

Step 4. Generate the ranking of all the candidate alternatives A_i ($i = 1, 2, \dots, m$) according to $s(y_i)$ ($i = 1, 2, \dots, m$), and then acquire the optimal alternative(s).

6. PRACTICAL APPLICATION EXAMPLE

In this part, we consider an actual MADM issue involving the assessment of online payment service providers (revised from Refs. [30,38]) to show the application and the working process of the presented approach.

6.1. Description

GCB Bank Ltd., with over 60 years of efforts has been made to encourage and support business growth in Ghana. In order to expand its business scope and for the convenience of its customers, GCB Bank Ltd. prepare to purchase a new online payment system. Therefore, how to choose the appropriate online payment service providers will possess vital impact on the development of the GCB Bank Ltd itself, meanwhile, it is an important task for the e-banking director. Four underlying service providers $\{A_1, A_2, A_3, A_4\}$ were regarded as possible providers for chosen after careful research. The e-banking director considers the following five attributes, which are technical ability and innovation (C_1), truly competitive rates (C_2), performance history (C_3), technical competence (C_4) and logistics (C_5). $\omega = (0.3, 0.1, 0.2, 0.1, 0.3)^T$ is the corresponding weight. The feature of provider A_i ($i = 1, 2, 3, 4$) in terms of attribute C_j are described via the following preference matrix $D = (d_{ij})_{4 \times 5}$, which is provided in Table 1.

6.2. Decision Process

The presented approach to address the MADM issue is provided as follows:

Step 1. In viewing of each alternative is benefit form, so the normalization does not need.

Step 2. On the basis of the HPFIWA operator (we set $\rho = 3$), the collective value y_i ($i = 1, 2, \dots, 4$) is obtained as follows:

$$y_1 = (0.7449, 0.2695), y_2 = (0.5392, 0.5761),$$

Table 2 | The aggregation results of different case of A-PFIWA operators.

Aggregation Operators	Score Value $s(y_i)$	Ranking
PFIWA	$s(y_1) = 0.5110, s(y_2) = 0.0686, s(y_3) = 0.4127, s(y_4) = 0.0648$	$A_1 > A_3 > A_2 > A_4$
EPFIWA	$s(y_1) = 0.4924, s(y_2) = 0.0375, s(y_3) = 0.4055, s(y_4) = 0.0504$	$A_1 > A_3 > A_4 > A_2$
HPFIWA	$s(y_1) = 0.4822, s(y_2) = 0.0200, s(y_3) = 0.3998, s(y_4) = 0.0429$	$A_1 > A_3 > A_4 > A_2$
FPFIWA	$s(y_1) = 0.5021, s(y_2) = 0.0546, s(y_3) = 0.4102, s(y_4) = 0.0587$	$A_1 > A_3 > A_4 > A_2$

$$y_3 = (0.7617, 0.4248), y_4 = (0.6248, 0.5895).$$

Step 3. The score value of y_i ($i = 1, 2, \dots, 4$) is determined as follows:

$$s(y_1) = 0.4822; s(y_2) = 0.0200,$$

$$s(y_3) = 0.3998; s(y_4) = 0.0429$$

Step 4. Because $s(y_1) > s(y_3) > s(y_4) > s(y_2)$, then we can get the priority result

$$A_1 > A_3 > A_4 > A_2$$

Hence, the optimal alternative is A_1 .

Analogously, we deal with the above Example based on the HPFIWG operator:

Step 1. is the same.

Step 2. Based on the HPFIWG operator (we set $\rho = 3$), the comprehensive value y_i ($i = 1, 2, \dots, 4$) is acquired as follows:

$$y_1 = (0.7453, 0.2683), y_2 = (0.5869, 0.5825),$$

$$y_3 = (0.7786, 0.3930), y_4 = (0.6838, 0.5199).$$

Step 3. The score value of y_i ($i = 1, 2, \dots, 4$) is obtained as follows:

$$s(y_1) = 0.4836; s(y_2) = 0.0051,$$

$$s(y_3) = 0.4517; s(y_4) = 0.1973$$

Step 4. Since $s(y_1) > s(y_3) > s(y_4) > s(y_2)$, then we can determine the priority result

$$A_1 > A_3 > A_4 > A_2$$

Hence, the optimal choice is A_1 .

We can see that the ranking orders are the same by using the HPFIWA operator and HPFIWG operator.

6.3. Discussion and Comparison Analysis

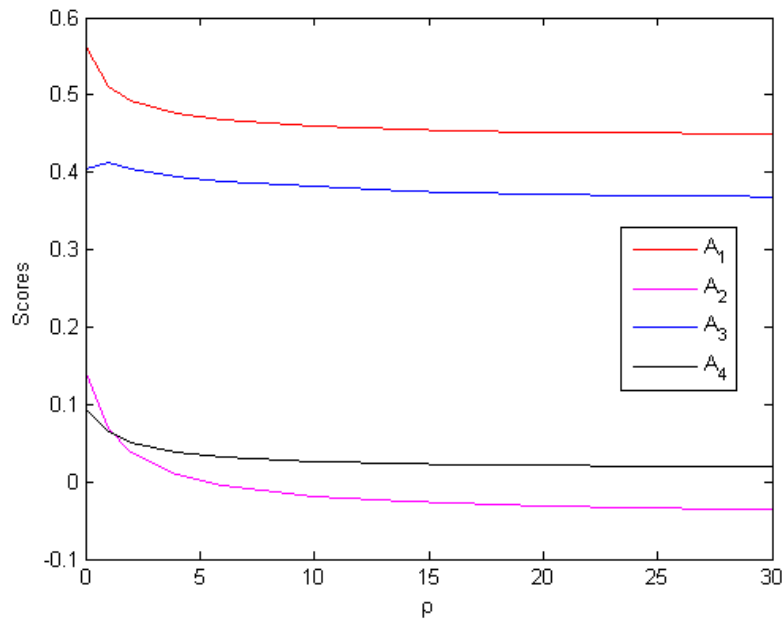
6.3.1. The influence of diverse specific AOs on decision results

In what follows, we will observe the effect on the decision results by utilizing diverse specific AOs. The proposed AOs: PFIWA, EPFIWA, HPFIWA, FPFIWA, PFIWG, EPFIWG, HPFIWG and FPFIWG operators are used to settle above example. In HPFIWA and HPFIWG operators, we take parameter value ρ as 3, and in FPFIWA and FPFIWG operators, we set parameter value $\varepsilon = 2$. The final score values and ranking outcomes are listed in Tables 2 and 3.

As displayed in Tables 2 and 3, the score values of the same alternative are different by utilizing diverse specific algebraic and geometric AOs. The ranking results are slightly different via using PFIWA and PFIWG operators, the reason is that PFIWA operator highlights the function of whole PFNs, while PFIWG operator reflects the function of individual PFN, the ones are the same by employing other AOs. However, the optimal choice is completely consistent by all AOs, that is, A_1 , which shows that diverse specific AOs have no significant impact on the ranking outcomes, and the decision-making approach is relatively stable.

Table 3 | The aggregation results of different case of A-PFIWG operators.

Aggregation Operators	Score Value $s(y_i)$	Ranking
PFIWG	$s(y_1) = 0.4894, s(y_2) = -0.0246, s(y_3) = 0.4420, s(y_4) = 0.1873$	$A_1 > A_3 > A_4 > A_2$
EPFIWG	$s(y_1) = 0.4845, s(y_2) = -0.0073, s(y_3) = 0.4461, s(y_4) = 0.1912$	$A_1 > A_3 > A_4 > A_2$
HPFIWG	$s(y_1) = 0.4836, s(y_2) = 0.0051, s(y_3) = 0.4517, s(y_4) = 0.1973$	$A_1 > A_3 > A_4 > A_2$
FPFIWG	$s(y_1) = 0.4860, s(y_2) = -0.0178, s(y_3) = 0.4422, s(y_4) = 0.1863$	$A_1 > A_3 > A_4 > A_2$

**Figure 1** | Scores for alternatives determined using the Hamacher Pythagorean fuzzy interactive weighted averaging (HPFIWA) operator.

6.3.2. The effect of diverse parameter values on MADM results

Next, we analyze the influence of parameter value changes on decision results. As we discussed before, if parameter value $\rho = 1$, then the HPFIWA and HPFIWG operators become the PFIWA and PFIWG operators, respectively; if $\rho = 2$, then the HPFIWA and HPFIWG operators become the EPFIWA and EPFIWG operators, respectively. Therefore, we choose HPFIWA, HPFIWG, FPFIWA and FPFIWG these four operators for analysis.

First, we set the diverse value ρ from 0 to 30 in HPFIWA and HPFIWG operators to settle above example. The variation trend of score results with the change of parameter ρ is shown in Figures 1 and 2, respectively.

From Figure 1, we can obtain that the scores determined by the HPFIWA operator of alternative A_3 first increases and then decreases as the value ρ increase, the reason is that the score function of alternative A_3 is a monotone increasing function about parameter ρ in interval $[0, 0.4097]$ and a monotone decreasing function about parameter ρ in interval $(0.4097, 30]$, whereas the scores of other alternatives determined by the HPFIWA operator decrease as the value ρ increase. Since the parameter ρ is smaller, the score values of all alternatives are larger, then for an optimistic DM, smaller ρ is advised. Further analysis, we can get the following results:

1. When $\rho \in (0, 1.2275]$, the ranking is $A_1 > A_3 > A_2 > A_4$.
2. When $\rho \in (1.2275, 30]$, the ranking is $A_1 > A_3 > A_4 > A_2$.

As shown in Figure 2, the scores determined employing the HPFIWG operator of alternative A_2 increase as the value ρ increase, whereas the scores of other alternatives derived from the HPFIWG operator first decrease and then increase when the value ρ increase. Because the parameter ρ is bigger, the score values of all alternatives are larger, then as an optimistic DM, bigger ρ is advised. Further analysis, we can get the following results:

1. When $\rho \in (0, 11.8182]$, the ranking is $A_1 > A_3 > A_2 > A_4$.

2. When $\rho \in (11.8182, 30]$, the ranking is $A_3 > A_1 > A_4 > A_2$.

Furthermore, if the HPFIWA and HPFIWG operators are replaced respectively with FPFIWA and FPFIWG operators to resolve above example, then the score outcomes with the change of parameter ε are described in Figures 3 and 4, respectively.

In the light of Figure 3, we can get that the scores acquired via the FPFIWA operator of alternative A_4 first decreases and then increases as the value ε increases, whereas the scores of other alternatives determined utilizing the FPFIWA operator decrease when the value ε increases.

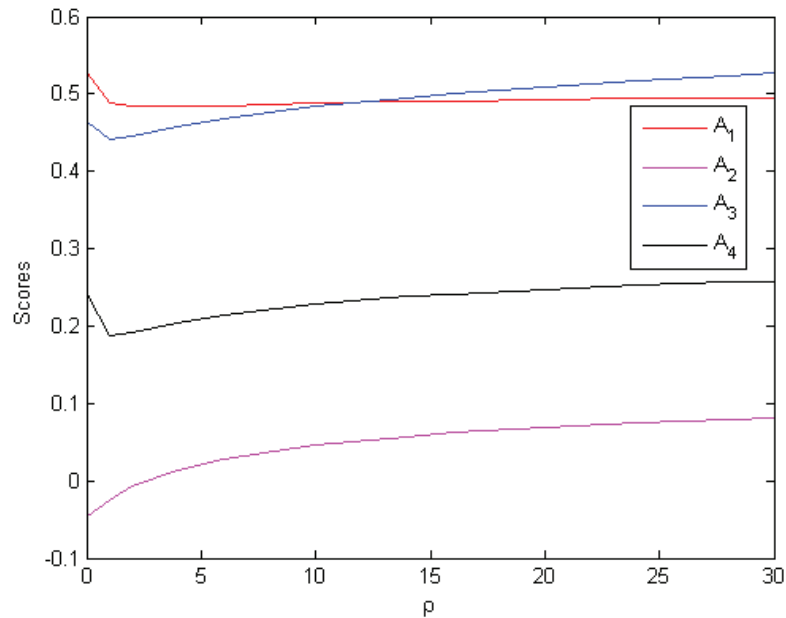


Figure 2 | Scores for alternatives determined using the Hamacher Pythagorean fuzzy interactive weighted geometric (HPFIWG) operator.

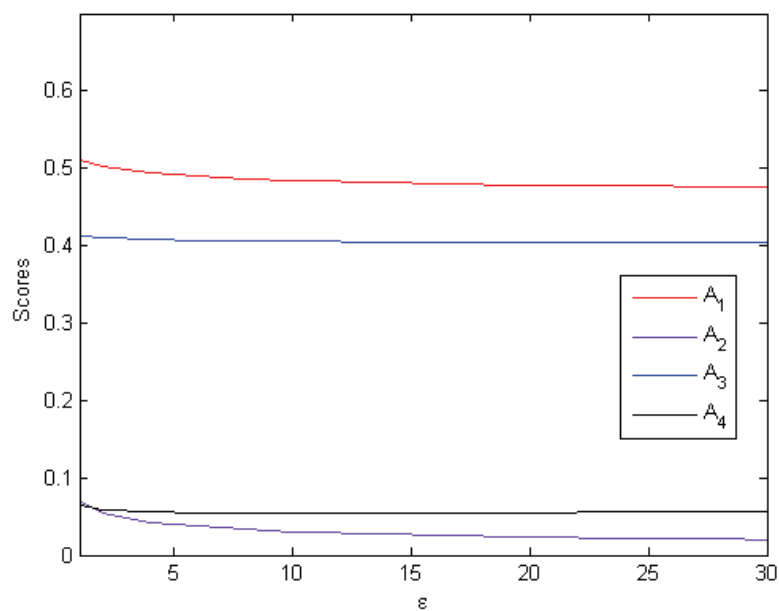


Figure 3 | Scores for alternatives determined using the Frank Pythagorean fuzzy interactive weighted averaging (FPFIWA) operator.

Because the parameter ε is smaller, the score values of all alternatives are larger, then as an optimistic DM, smaller ε is advised. Further research, the following results can be obtained:

1. When $\varepsilon \in (0, 1.4185]$, the ranking is $A_1 > A_3 > A_2 > A_4$.
2. When $\varepsilon \in (1.4185, 30]$, the ranking is $A_1 > A_3 > A_4 > A_2$.

Based on Figure 4, we notice that the scores generated via the FPFIWG operator of alternatives A_1 and A_4 decrease when the value ε increases, whereas the changing trend of scores for alternative A_2 is just the opposite of alternatives A_1 and A_4 . The scores acquired using the FPFIWG operator of alternative A_3 first increases and then decreases when the value ε increases. And regardless of the scores of the four alternatives change, the ranking results are always the same, that is $A_1 > A_3 > A_4 > A_2$.

6.3.3. Comparative analysis with the relevant approaches

In order to show the effectiveness and merits of proposed approach (As mentioned above, we select HPFIWA and HPFIWG operators to stand for the proposed AOs), we will compare with the following methods: Ma and Xu's method [22] based on PFWA operator, Garg's method [24] based on PFEWA operator, Wu and Wei's method [27] based on PFHWA operator, Wei's method [32] based on PFIWA operator, Gao *et al.*'s method [37] based on PFIPIWA operator and Wang and Li's method [38] based on WPFIPBM operator. The evaluation values in Table 1 are employed, and the comparison results are provided in Table 4.

This Table reveals that the ranking outcomes utilizing the presented method are same with the methods defined by [32,37,38], which indicates that our novel proposed approach is valid. Moreover, the ranking outcomes are different from the methods given by [22,24,27]. Further discussions are listed as follows:

1. Ma and Xu's method [22] is on the base of PFWA operator, the PFWA operator adopted the algebra t-conorm and t-norm, it is a particular form of ATT. Although the calculation process is relatively easy, it cannot capture the interaction over MD and NMD of PFNs. Hence the ranking output by Ma and Xu's method [22] is different from our defined method.
2. Garg's method [24] and Wu and Wei's method [27] are based on the PFEWA and PFHWA operators, respectively. The PFEWA operator adopted the Einstein t-conorm and t-norm, and the PFHWA operator adopted the Hamacher t-conorm and t-norm. Whereas, the presented method used the ATT, and Einstein t-conorm and t-norm, Hamacher t-conorm and t-norm are all special cases of ATT. On the other hand, like Ma and Xu's method [22], the methods [24,27] also do not reflect the interactive influence among MD and NMD, so the ranking outputs by these two methods [24,27] are different from our constructed method. It reveals the superiority of our built method.

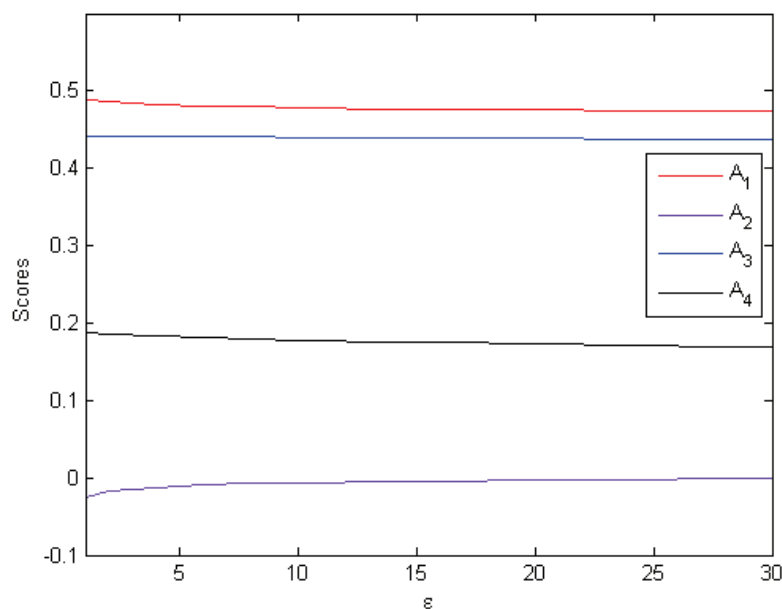


Figure 4 | Scores for alternatives determined using the Frank Pythagorean fuzzy interactive weighted geometric (FPFIWG) operator.

Table 4 | The aggregating results by different approaches.

Approaches	Score Value $s(y_i)$	Ranking
Ma and Xu's method [22]		
based on PFWAoperator	$s(y_1) = 0.5223, s(y_2) = 0.1354, s(y_3) = 0.5910, s(y_4) = 0.2337$	$A_3 > A_1 > A_4 > A_2$
Garg's method [24]		
based on PFEWA operator	$s(y_1) = 0.5080, s(y_2) = 0.1059, s(y_3) = 0.5844, s(y_4) = 0.2075$	$A_3 > A_1 > A_4 > A_2$
Wu and Wei's method [27]		
based on PFHWA operator	$s(y_1) = 0.5008, s(y_2) = 0.0916, s(y_3) = 0.5802, s(y_4) = 0.1948$	$A_3 > A_1 > A_4 > A_2$
Wei's method [32]		
based on PFIWA operator	$s(y_1) = 0.5110, s(y_2) = 0.0686, s(y_3) = 0.4127, s(y_4) = 0.0648$	$A_1 > A_3 > A_2 > A_4$
Gao <i>et al.</i> 's method [37]		
based on PFIPWAoperator	$s(y_1) = 0.5060, s(y_2) = 0.0534, s(y_3) = 0.4207, s(y_4) = 0.0626$	$A_1 > A_3 > A_4 > A_2$
Wang and Li's method [38] based on		
WPFIPBM operator ($p = q = 1$)	$s(y_1) = 0.4981, s(y_2) = 0.0719, s(y_3) = 0.4780, s(y_4) = 0.1887$	$A_1 > A_3 > A_4 > A_2$
Proposed method based on		
HPFIWA operator ($\rho = 3$)	$s(y_1) = 0.4822, s(y_2) = 0.0200, s(y_3) = 0.3998, s(y_4) = 0.0429$	$A_1 > A_3 > A_4 > A_2$
Proposed method based on		
HPFIWG operator ($\rho = 3$)	$s(y_1) = 0.4836, s(y_2) = 0.0051, s(y_3) = 0.4517, s(y_4) = 0.1973$	$A_1 > A_3 > A_4 > A_2$

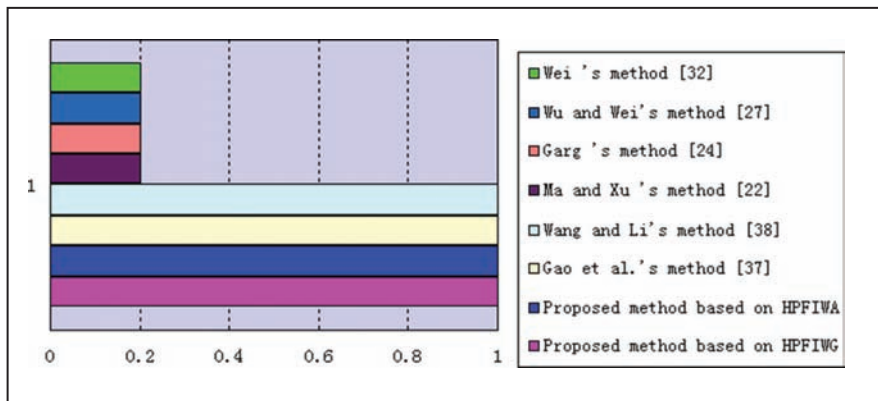


Figure 5 | Spearman correlation: presented methods versus others.

- Wei's method [32] is based on PFIWA operator, which is the special case of proposed HPFIWA operator, when the function $l(r)$ takes $-\log(r)$.
- Gao *et al.*'s method [37] is on the basis of PFIPWA operator, which can reduce the impact of confused data during the aggregation process. Although the PFIPWA operator can capture the interactive influence over MD and NMD, but it utilizes algebra t-conorm and t-norm.
- Wang and Li's method [38] is based on WPFIPBM operator, to settle above MADM issue, we take the valued $p = q = 1$, and then decision result coincides with the presented HPFIWA operator. It displays that the effectiveness of our defined AOs. But the superiority of our built method is that it uses ATT, so it is more flexible.

As shown in Figure 5, the spearman's coefficient of proposed method (HPFIWA, HPFIWG), Gao *et al.*'s method [37] and Wang and Li's method [38] are all 1, which reveals the validity of the explored method. Furthermore, the spearman's coefficient of Ma and Xu's method [22], Garg's method [24], Wu and Wei's method [27] and Wei's method [32] are all 0.2, which indicates the superiority of the constructed method.

In a word, the superiorities of our explored method are summarized as follows: (1) It can offer a more flexible method via selecting diverse forms of ATT; (2) It can reflect the interactive influence among MD and NMD for PFNs in the decision-making process.

7. CONCLUSION

ATT can derive many famous t-conorms and t-norms, PFS is a remarkable technique to depict the fuzziness and indeterminacy in MADM issues. In this study, we have constructed several novel interactive operational rules for PFNs in the light of ATT, based on which, some novel interactive AOs are explored, they are A-PFIWA operator and A-PFIWG operator. In addition, we discussed their properties, such as their idempotency, monotonicity boundedness, shift-invariance and so on. When the parameter varies in the presented AOs, we acquire

certain specific Pythagorean fuzzy interactive AOs. Further, we have provided a decision-making approach for MADM issue on the basis of these AOs. Finally, an actual issue on assessment of online payment service providers is settled to show the availability of presented approach, meanwhile, a detailed investigation are also provided with respect to the change trend of decision results when the parameter varies in introduced AOs. Furthermore, the superiority of built approach is given by comparing to relevant approaches. The shortcomings of the operators proposed in this paper are that: (1) They do not consider the interrelationship between the input PFNs; (2) The aggregated result is relatively complicated by using the proposed operators.

In the succeeding research, we shall extend the constructed approach to other applications [51–58], or study the parameter determination of Bonferroni operators [59] and also extend to proportional interval type-2 hesitant fuzzy set [60].

CONFLICTS OF INTEREST

The authors declare no conflict of interests regarding the publication for the paper.

AUTHORS' CONTRIBUTIONS

L.W. (LeiWang) conceived this work, L.W. and H.G. (Harish Garg) compiled the computing program by Matlab and analyzed the data, L.W. and H.G. wrote the paper. Finally, all the authors have read and approved the final manuscript. All authors have read and agreed to the published version of the manuscript.

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