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Research Article

Advanced Soft Relation and Soft Mapping

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ABSTRACT

The research data in this manuscript is drawn from four main sections: First, the union and the intersection of an arbitrary family of soft sets are introduced and further results for various soft set operations are obtained. Second, by using the soft relation introduced by Babitha and Sunil, its some important results are obtained. Third, by using the soft mapping by them, its advanced properties are proved and studied. Finally, taken together, these results suggest that there is an association between soft mappings and soft equivalence relations. This organization dispels an overly rigorous or formal view of soft relations and soft mappings and offers some strong pedagogical value in that the discrete discussions can sometimes serve to motivate the more abstract continuous discussions.

1. INTRODUCTION

In order to solve complicated problems in economics, engineering, environmental science, medical science and social science, methods in classical mathematics may not be successfully used because of various uncertainties arising in these problems. Awareness of uncertainly is not recent, having possibly first been described in seventeenth-century France when the two great French mathematicians, Blaise Pascal and Pierre de Fermat, corresponded over two problems from games of chance. While mathematical theories such as probability theory, fuzzy set theory [1], rough set theory [2], vague set theory [3] and interval-valued set theory [4,5] are well known and often useful approaches to describing uncertainly, each of these theories has its inherent difficulties as pointed out in [6].

In 1999, the Russian researcher Molodtsov introduced the concept of soft set. He proposed it as a completely generic mathematical tool for modelling uncertainties. After that time, works on soft set theory are progressing rapidly. Today, soft set theory is a well-established branch of mathematics that finds applications in every area of scholarly activity from music to physics, and in daily experience from weather prediction to predicting the risks of new medical treatments. We can summarize some of the main works: Maji *et al.* [7] defined several operations on soft sets and made a theoretical study on the theory of soft sets. Ali *et al.* [8] introduced some new operations on soft sets and improved the notion of complement of a soft set and proved that DeMorgan's laws in soft set theory. Qin *et al.* [9] studied soft sets in the sense of lattice theory. Majumdar and Samanta [10] investigated soft mappings. Feng *et al.* [11] compared soft sets with rough sets. Also Feng *et al.* [12] initiated the study of soft semirings. Moreover, Feng *et al.* [13] applied soft relations to semigroups. Kanwal and Shabir [14] studied rough approximation of a fuzzy set in semigroups via soft relations. Acar *et al.* [15] applied the concept of soft set to ring theory. Also Aktas and cCavgman [16] applied it to group theory. Furthermore, cCavgman and Karataacs [17] and Shabir and Naz [18] investigated soft topology. Wardowski [19] dealt with fixed point problems of a soft mapping. Feng and Pedrycz [20] investigated decompositions of fuzzy soft sets. Feng and Li [21] introduced soft product operations. Kanwal *et al.* [22] studied generalized approximation of substructures in quantales using soft relations.

In this paper, we study in the following directions: First, we introduce the union and the intersection of an arbitrary family of soft sets and obtain further results for various soft set operations. Second, by using the soft relation introduced by Babitha and Sunil [23], we obtain its some results. Third, by using the soft mapping by them, we prove its various properties. Finally, we investigate some relations between soft mappings and soft equivalence relations. The findings of this study have a number of important implications for future practice.

2. PRELIMINARIES

In this section, we introduce some concepts of soft sets. We refer to [6–8] for details.

Throughout this paper, let U be an initial universe set, let E be the set of all possible parameters under consideration with respect to

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U and let $P(U)$ be the power set of U. Usually, parameters are attribute, characteristic or properties in U.

More information on soft sets would help us to establish a greater degree of accuracy throughout the paper.

Definition 2.1. [6] Let $A \subset E$. Then a pair (F, A) is called a *soft set over* U if $F : A \to P(U)$ is a mapping.

In other words, a soft set over U is a parameterized family of subsets of U. For each $e \in A$, $F(e)$ may be considered as the set e-approximate elements of (F, A).

Definition 2.2. [7] Let (F, A) and (G, B) be two soft over a common universe U. Then we say that (F, A) is a *soft subset* of (G, B) [or (G, B) is a *soft super set* of (F, A)], denoted by $(F, A)\tilde{\subset}(G, B)$ if

i. $A \subset B$

ii. $F(e) = G(e), \forall e \in A$.

In particular, if $(F, A)\tilde{\subset}(G, B)$ and $(G, B)\tilde{\subset}(F, A)$, then (F, A) and (G, B) are said to be *equal* and denoted by $(F, A) = (G, B)$.

It is clear that if $A \subset B \subset E$, then $(F, A)\tilde{\subset}(F, B)$.

Definition 2.3. [7] Let (F, A) and (G, B) be two soft sets over a common universe U. Then the *union* of (F, A) and (G, B), denoted by $(F, A)\tilde{\cup}(G, B)$, is the soft set (H, C) defined as follows:

i. $C = A \cup B$,

ii.

$$H(e) = \begin{cases} F(e) & \text{if e} \in \text{A} - \text{B}, \\ G(e) & \text{if e} \in \text{B} - \text{A}, \\ F(e) \cup G(e) & \text{if e} \in \text{A} \cap \text{B}, \forall \text{e} \in \text{C}. \end{cases}$$

Definition 2.4. [8] Let (F, A) and (G, B) be two soft sets over a common universe U.

(a) The *extended intersection* of (F, A) and (G, B), denoted by $(F, A) \sqcap_{\varepsilon} (G, B)$, is the soft set (H, C) defined as follows:
   i. $C = A \cup B$,
   ii.

$$H(e) = \begin{cases} F(e) & \text{if e} \in \text{A} - \text{B}, \\ G(e) & \text{if e} \in \text{B} - \text{A}, \\ F(e) \cup G(e) & \text{if e} \in \text{A} \cap \text{B}, \forall \text{e} \in \text{C}. \end{cases}$$

(b) Let $A \cap B \neq \emptyset$. Then the *restricted intersection* (or *bi-intersection* (See [12]) of (F, A) and (G, B), denoted by $(F, A) \Cap (G, B)$ (or $(F, A)\tilde{\sqcap}(G, B)$), is the soft set (H, C) defined as follows:
   i. $C = A \cap B$,
   ii. $H(e) = F(e) \cap G(e), \forall e \in C$.

(c) Let $A \cap B \neq \emptyset$. Then the *restricted union* of (F, A) and (G, B), denoted by $(F, A) \cup_R (G, B)$ is the soft set (H, C) defined as follows:
   i. $C = A \cap B$,
   ii. $H(e) = F(e) \cup G(e), \forall e \in C$.

Definition 2.5. [8] Let (F, A) be a soft set over U.

i. (F, A) is called a *relative null soft set* (with respect to A), denoted by $\emptyset_A$, if $F(e) = \emptyset, \forall e \in A$.

ii. (F, A) is called a *relative whole soft set* (with respect to A), denoted by U_A, if $F(e) = U, \forall e \in A$.

Definition 2.6. [8] Let (F, A) be a soft set over U. Then the *relative complement* of (F, A), denoted by $(F, A)^r$, is the soft set (F^r, A) defined as follows:

$F^r : A \to P(U)$ is a mapping given by $F^r(e) = U - F(e), \forall e \in A$.

Result 2.7.

([8], Theorem 4.1) Let (F, A) and (G, B) be two soft sets over a common universe U such that $A \cap B \neq \emptyset$. Then

1. $((F, A) \cup_R (G, B))^r = (F, A)^r \tilde{\sqcap} (G, B)^r$,
2. $((F, A)\tilde{\sqcap}(G, B))^r = (F, A)^r \cup_R (G, B)^r$.

Result 2.8.

([9], Theorems 11–14) The operations $\tilde{\cup}, \sqcap_{\varepsilon}, \cup_R$ and $\tilde{\sqcap}$ are idempotent associative and commutative, respectively.

Result 2.9.

([9], Theorem 15) The absorption laws with respect to operations $\tilde{\cup}$ and $\tilde{\sqcap}$ hold. That is, let (F, A) and (G, B) be two soft sets over a common universe U. Then

1. $((F, A)\tilde{\cup}(G, B))\tilde{\sqcap}(F, A) = (F, A)$,
2. $((F, A)\tilde{\sqcap}(G, B))\tilde{\cup}(F, A) = (F, A)$.

Result 2.10.

([9], Theorem 16) Let $S(U, E) = \{(F, A) : A \subset E \text{ and } F : A \to P(U)\}$.

1. $(S(U, E), \tilde{\cup}, \tilde{\sqcap})$ is a distributive lattice.
2. Let $\leq_1$ be the order relation in $(S(U, E), \tilde{\cup}, \tilde{\sqcap})$ and let (F, A), $(G, B) \in S(U, E)$. Then $(F, A) \leq_1 (G, B)$ if and only if $A \subset B$ and $F(e) \subset G(e), \forall e \in A$.

It is clear that $(S(U, E), \tilde{\cup}, \tilde{\sqcap})$ is a bounded lattice, U_E and $\emptyset_{\emptyset}$ are upper bound and lower bound respectively.

Result 2.11.

([9], Corollary 17) Let $A \subset E$ be fixed and let $S_A = \{(F, A) : F : A \to P(U)\}$. Then S_A is a sublattice of $(S(U, E), \tilde{\cup}, \tilde{\sqcap})$. In particular, U_A and $\emptyset_A$ are the greatest element and the least element in $(S_A, \tilde{\cup}, \tilde{\sqcap})$, respectively.

Result 2.12.

([9], Theorem 18) Let (F, A) and (G, B) be two soft sets over a common universe U. Then

1. $((F, A) \cup_R (G, B)) \tilde{\sqcap}_\varepsilon (F, A) = (F, A)$,
2. $((F, A) \tilde{\sqcap}_\varepsilon (G, B)) \cup_R (F, A) = (F, A)$.

Result 2.13.

([9], Theorem 19) Let $(F, A), (G, B), (H, C) \in S(U, E)$. Then $(F, A) \cup_R ((G, B) \sqcap_\varepsilon (H, C)) = ((F, A) \cup_R (G, B)) \sqcap_\varepsilon ((F, A) \cup_R (H, C))$.

Result 2.14.

([9], Theorem 20).

1. $(S(U, E), \cup_R, \sqcap_\varepsilon)$ is a distributive lattice.
2. Let $\leq_2$ be the order relation in $(S(U, E), \cup_R, \sqcap_\varepsilon)$ and let (F, A), $(G, B) \in S(U, E)$. Then $(F, A) \leq_2 (G, B)$ if and only if $B \subset A$ and $F(e) \subset G(e), \forall e \in B$.

Result 2.15.

([9], Corollary 21) S_A is a sublattice of $(S(U, E), \cup_R, \sqcap_\varepsilon)$.

It is obvious that the lattice structure $(S(U, E), \tilde{\cup}, \tilde{\sqcap})$ is different from that of $(S(U, E), \cup_R, \sqcap_\varepsilon)$.

3. FURTHER RESULTS OF A SOFT SET

In this section, we take a closer look at the structure of soft sets.

Definition 3.1. [12] Let $(F_j, A_j)_{j\in J}$ be a nonempty family of soft sets over a common universe U, where J is an index class.

(a) The *union* of $(F_j, A_j)_{j\in J}$, denoted by $\tilde{\cup}_{j\in J}(F_j, A_j)$, is the soft set (H, B) defined as follows:
   i. $B = \cup_{j\in J} A_j$,
   ii. $H(x) = \cup_{j\in J(x)} A_j(x), \forall x \in B$, where $J(x) = \{j \in J : x \in A_j\}$.

(b) Let $\cap_{j\in J} A_j \neq \emptyset$. Then the *bi-intersection* (or *restricted intersection*) of $(F_j, A_j)_{j\in J}$, denoted by $\tilde{\sqcap}_{j\in J}(F_j, A_j)$, is the soft set (H, B) defined as follows:
   i. $B = \cap_{j\in J} A_j$,
   ii. $H(x) = \cap_{j\in J} F_j(x), \forall x \in B$.

Proposition 3.2. *Let $(F, A) \in S(U, E)$ and let $(F_\alpha, A_\alpha)_{\alpha\in\Gamma} \subset S(U, E)$ such that $\cap_{\alpha\in\Gamma} A_\alpha \neq \emptyset$.*

1. $(F, A)\tilde{\sqcap}(\tilde{\cup}_{\alpha\in\Gamma}(F_\alpha, A_\alpha)) = \tilde{\cup}_{\alpha\in\Gamma}((F, A)\tilde{\sqcap}(F_\alpha, A_\alpha))$.
2. $(F, A)\tilde{\cup}(\tilde{\sqcap}_{\alpha\in\Gamma}(F_\alpha, A_\alpha)) = \tilde{\sqcap}_{\alpha\in\Gamma}((F, A)\tilde{\cup}(F_\alpha, A_\alpha))$.

Proof. 1. Let

$$(F, A)\tilde{\sqcap}(\tilde{\cup}_{\alpha\in\Gamma}(F_\alpha, A_\alpha)) = (K, A \cap (\cup_{\alpha\in\Gamma} A_\alpha))$$
$$\cup_{\alpha\in\Gamma}((F, A)\tilde{\sqcap}(F_\alpha, A_\alpha)) = (L, \cup_{\alpha\in\Gamma}(A \cap A_\alpha)).$$

Let $e \in A \cap (\cup_{\alpha\in\Gamma} A_\alpha)$ and let $\Gamma(e) = \{\alpha \in \Gamma : e \in A_\alpha\}$. Then $e \in \cup_{\alpha\in\Gamma}(A \cap A_\alpha)$. Thus

$$\begin{aligned} K(e) &= A(e) \cap (\cup_{\alpha\in\Gamma(e)} A_\alpha(e)) \\ &= \cup_{\alpha\in\Gamma(e)}(A(e) \cap A_\alpha(e)) \\ &= \cup_{\alpha\in\Gamma(e)} A\tilde{\sqcap}A_\alpha \\ &= \cup_{\alpha\in\Gamma(e)}[(F, A)\tilde{\sqcap}(F_\alpha, A_\alpha)](e) \\ &= [\tilde{\cup}_{\alpha\in\Gamma}((F, A)\tilde{\sqcap}(F_\alpha, A_\alpha))](e) \\ &= L(e). \end{aligned}$$

So the equality holds.

2. By the similar arguments, we can prove that the equality holds.

The following is the immediate result of Definitions 2.6 and 3.1

Proposition 3.3. *Let $(F_\alpha, A_\alpha)_{\alpha\in\Gamma}$ be a nonempty family of soft sets over a common universe U such that $\cap_{\alpha\in\Gamma} A_\alpha \neq \emptyset$. Then*

1. $(\tilde{\cup}_{\alpha\in\Gamma}(F_\alpha, A_\alpha))^r = \tilde{\sqcap}_{\alpha\in\Gamma}(F_\alpha, A_\alpha)^r$,
2. $(\tilde{\sqcap}_{\alpha\in\Gamma}(F_\alpha, A_\alpha))^r = \tilde{\cup}_{\alpha\in\Gamma}(F_\alpha, A_\alpha)^r$.

Definition 3.4. Let $(F_j, A_j)_{j\in J}$ be a nonempty family of soft sets over a common universe U such that $\cap_{j\in J} A_j \neq \emptyset$.

(a) The *restricted union* of $(F_j, A_j)_{j\in J}$, denoted by $\cup_{R,j\in J}(F_j, A_j)$, is the soft set (H, B) defined as follows:
   i. $B = \cap_{j\in J} A_j$,
   ii. $H(e) = \cup_{j\in} F_j(e), \forall e \in B$.

(b) The *extended intersection* of $(F_j, A_j)_{j\in J}$, denoted by $\sqcap_{\varepsilon,j\in J}(F_j, A_j)$, is the soft set (H, C) defined as follows:
   i. $C = \cup_{j\in J} A_j$,
   ii.

$$H(e) = \begin{cases} F_j(e) & \text{if } e \in A_j - \cup_{i\neq j\in J} A_j, \\ \cup_{j\in J} F_j(e) & \text{if } e \in \cap_{j\in J} A_j, \ \forall e \in \cup_{j\in J} A_i. \end{cases}$$

Proposition 3.5. *Let $(F_\alpha, A_\alpha)_{\alpha\in\Gamma} \subset S(U, E)$ such that $\cap_{\alpha\in\Gamma} A_\alpha \neq \emptyset$.*

1. $(\cup_{R,\alpha\in\Gamma}(F_\alpha, A_\alpha))^r = \sqcap_{\varepsilon,\alpha\in\Gamma}(F_\alpha, A_\alpha)^r$.
2. $(\sqcap_{\varepsilon,\alpha\in\Gamma}(F_\alpha, A_\alpha))^r = \cup_{R,\alpha\in\Gamma}(F_\alpha, A_\alpha)^r$.

Proof. 1. Let $\cup_{R,\alpha\in\Gamma}(F_\alpha, A_\alpha) = (H, C)$, where $H(e) = \cup_{\alpha\in\Gamma} F_\alpha(e)$ for each $e \in C = \cap_{\alpha\in\Gamma} A_\alpha$.
Then $(\cup_{R,\alpha\in\Gamma}(F_\alpha, A_\alpha))^r = (H, C)^r$. Thus by Definition 2.6,

$$\begin{aligned} H^r(e) &= U - [\cup_{\alpha\in\Gamma} F_\alpha(e)] \\ &= \cap_{\alpha\in\Gamma}[U - F_\alpha(e)] \\ &= \cap_{\alpha\in\Gamma} F_\alpha^r(e) \end{aligned}$$

for each $e \in C$.
Now let $\tilde{\sqcap}_{\alpha\in\Gamma}(F_\alpha, A_\alpha)^r = \tilde{\sqcap}_{\alpha\in\Gamma}(F^r_\alpha, A_\alpha) = (K, C)$ and let $e \in C$. Then $K(e) = \cap_{\alpha\in\Gamma} F^r_\alpha(e) = H^r(e)$. So the equality holds.

2. By the similar arguments, we can show that the quality holds.

Proposition 3.6. *Let* $(F, A) \in S(U, E)$ *and let* $(F_\alpha, A_\alpha)_{\alpha\in\Gamma} \subset S(U, E)$.

1. $(F, A) \cup_R (\sqcap_{\varepsilon,\alpha\in\Gamma}(F_\alpha, A_\alpha)) = \sqcap_{\varepsilon,\alpha\in\Gamma}[(F, A) \cup_R (F_\alpha, A_\alpha)]$.
2. $(F, A) \sqcap_\varepsilon (\cup_{R,\alpha\in\Gamma}(F_\alpha, A_\alpha)) = \cup_{R,\alpha\in\Gamma}[(F, A) \sqcap_\varepsilon (F_\alpha, A_\alpha)]$.

Proof. 1. Let $(F, A) \cup_R (\sqcap_{\varepsilon,\alpha\in\Gamma}(F_\alpha, A_\alpha)) = (H, A \cap (\cup_{\alpha\in\Gamma} A_\alpha))$ and let $\sqcap_{\varepsilon,\alpha\in\Gamma}[(F, A) \cup_R (F_\alpha, A_\alpha)] = (K, \cup_{\alpha\in\Gamma}(A \cap A_\alpha))$. Let $e \in A \cap (\cup_{\alpha\in\Gamma} A_\alpha)$. Then $e \in A$ and $e \in \cup_{\alpha\in\Gamma} A_\alpha$,

i. If $e \in A, e \in A_\alpha$ and $e \notin \cup_{\alpha\neq\beta\in\Gamma} A_\beta$, then $H(e) = F(e) \cup F_\alpha(e) = K(e)$.

ii. If $e \in A$ and $e \in \cap_{\alpha\in\Gamma} A_\alpha$, then

$$\begin{aligned} H(e) &= F(e) \cup (\cap_{\alpha\in\Gamma} F_\alpha(e)) \\ &= \cap_{\alpha\in\Gamma}(F(e) \cup F_\alpha(e)) \\ &= H(e). \end{aligned}$$

Thus, in either cases, the equality holds.

2. By the similar arguments, we can show that the equality holds.

The following is the immediate result of Results 2.8, 2.12, 2.13, 2.14 and Proposition 3.6.

Theorem 3.7. $(S(U, E), \cup_R, \sqcap_\varepsilon)$ *is a complete distributive lattice.*

The following is the immediate result of Result 2.15 and Theorem 3.7.

Corollary 3.8. S_A is a complete sublattice of $(S(U, E), \cup_R, \sqcap_\varepsilon)$.

Note that the lattice structure $(S(U, E), \tilde{\cup}, \tilde{\sqcap})$ is different from that of $(S(U, E), \cup_R, \sqcap_\varepsilon)$.

4. FURTHER RESULTS OF A SOFT RELATION

In this section, we take a closer look at the structure of soft relations.

Definition 4.1. [23] Let $(F, A), (G, B) \in S(U, E)$. Then the *Cartesian product* of (F, A) and (G, B), denoted by $(F, A) \times (G, B)$, is soft set $(F \times G, A \times B)$ over $U \times U$ defined as follows:

$F \times G : A \times B \to P(U \times U)$ is a set-valued mapping given by $(F \times G)(a, b) = F(a) \times G(b), \forall (a, b) \in A \times B$, i.e., $(F \times G)(a, b) = \{(h_i, h_j) \in U \times U : h_i \in F(a)$ and $h_j \in G(b)\}$.

The Cartesian product of three or more nonempty soft sets can be defined by generating Definition 4.1. The Cartesian product $(F_1, A_1) \times (F_2, A_2) \times \cdots \times (F_n, A_n)$ of the nonempty soft sets $(F_1, A_1), (F_2, A_2), \cdots, (F_n, A_n)$ is the soft set of all ordered n-tuple $(h_1, h_2, \cdots, h_n)$, where $h_i \in F_i(a_i), \forall a_i \in A_i$.

Definition 4.2. [23] Let $(F, A), (G, B) \in S(U.E)$. Then R is called a *soft relation from* (F, A) *to* (G, B) if $R \tilde{\subset} (F, A) \times (G, B)$, equivalently, there exists $S \subset A \times B$ such that $R = (F \times G|_S, S)$, where $(F \times G|_S)(a, b) = F(a) \times F(b), \forall (a, b) \in S$.

In this case, we will write $R = \{F(a) \times G(b) : (a, b) \in S\}$ and $F(a) RG(b)$ iff $F(a) \times G(b) \in R$.

In particular, any soft subset of $(F, A) \times (F, A)$ is called a *soft set relation* on [or *in*] (F, A).

Definition 4.3. [23] Let R be a soft relation from (F, A) to $(G.B)$.

i. The *domain* of R, denoted by dom R, is the soft set (D, A_1) defined as follows:

$$A_1 = \{a \in A : \exists\, b \in B \text{ s.t. } F(a) \times G(b) \in R\} \text{ and}$$
$$D(a) = F(a), \forall a \in A,$$

ii. The *range* of R, denoted by ran R, is the soft set (RG, B_1) defined as follows:

$$B_1 = \{b \in B : \exists\, ba \in A \text{ s.t. } F(a) \times G(b) \in R\} \text{ and}$$
$$RG(b) = G(b), \forall b \in B_1.$$

Thus we can easily see that

$$\text{dom}R = \{F(a) \in (F, A) : \exists\, G(b) \in (G, B) \text{ s.t. } F(a) \times G(b) \in R\}$$

and

$$\text{ran}R = \{G(b) \in (G, B) : \exists\, F(a) \in (F, A) \text{ s.t. } F(a) \times G(b) \in R\}.$$

Definition 4.4. [23] Let $(F, A), (G, B), (H, C) \in S(U, E)$, $R \tilde{\subset} (F, A) \times (G, B)$ and $S \tilde{\subset} (G, B) \times (H, C)$. Then the *composition* of R and S, denoted by $S \circ R$, is a soft relation from (F, A) to (H, C) defined as follows: For each $F(a) \times H(c) \in (F, A) \times (H, C)$, $F(a) \times H(c) \in S \circ R$ iff $\exists\, G(b) \in (G, B)$ s.t. $F(a) \times G(b) \in R$ and $G(b) \times H(c) \in S$.

Definition 4.5. [23] Let $R \tilde{\subset} (F, A) \times (G, B)$. Then the *inverse* of R, denoted by R^{-1}, is a soft relation from (G, B) and (F, A) defined as follows: For each $F(a) \times G(b) \in (F, A) \times (G, B)$,

$$F(a) \times G(b) \in R \text{ iff } G(b) \times F(a) \in R^{-1}.$$

Result 4.6.

([23], Theorem 4.17) Let $R \tilde{\subset} (F, A) \times (G, B)$ and $S \tilde{\subset} (G, B) \times (H, C)$. Then $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.

Proposition 4.7. *Let* $R \tilde{\subset} (F, A) \times (G, B)$ *and* $S \tilde{\subset} (G, B) \times (H, C)$. *Then*

1. *dom* $R =$ *ran* R^{-1}, *ran* $R =$ *dom* R^{-1},
2. *dom* $(S \circ R) \subset$ *dom* R, *ran* $(S \circ R) \subset$ *ran*S.

Proof. (1) From Definitions 4.1 and 4.5, the proofs are clear.

(2) Let $F(a) \in$ dom $(S \circ R)$. Then by Definition 4.3,

$$\exists H(c) \in (H, C) \text{ s.t. } F(a) \times H(c) \in S \circ R.$$

Thus by Definition 4.4, $\exists G(b) \in (G, B)$ s.t. $F(a) \times G(b) \in R$ and $G(b) \times H(c) \in S$.

So $F(a) \in$ dom R. Hence dom $(S \circ R) \subset$ dom R. Similarly, we can show that ran $(S \circ R) \subset$ ran S.

The following is the immediate result of Proposition 4.7 and Definition 4.3.

Corollary 4.8. Let $R\tilde{\subset}(F,A)\times(G,B)$ and $S\tilde{\subset}(G,B)\times(H,C)$. If $\mathrm{ran}R\subset\mathrm{dom}S$, then $\mathrm{dom}(S\circ R)=\mathrm{dom}R$.

The following is the immediate result of Definitions 4.2, 4.4 and 4.5.

Proposition 4.9. *Let $R, R_1, R_2, R_3, S_1, S_2$ be soft relations. Then we have the following results:*

1. $R_1\circ(R_2,\circ R_3)=(R_1,R_2)\circ R_3$.
2. *If $R_1\subset S_1$ and $R_2\subset S_2$, then $R_1\circ R_2\subset S_1\circ S_2$.*
   In particular, if $S_1\subset S_2$, then $R\circ S_1\subset R\circ S_2$.
3. $R_1\circ(R_2\cup R_3)=(R_1\circ R_2)\cup(R_1\circ R_3)$. $R_1\circ(R_2\cap R_3)=(R_1\circ R_2)\cap(R_1\circ R_3)$.
4. *If $R_1\subset R_2$, then $R_2^{-1}\subset R_1^{-1}$.*
5. $(R^{-1})^{-1}=R$.
6. $(R_1\cup R_2)^{-1}=R_1^{-1}\cup R_2^{-1}, (R_1\cap R_2)^{-1}=R_1^{-1}\cap R_2^{-1}$.

The following is the immediate result of Proposition 4.9 (1).

Corollary 4.10. Let $R\tilde{\subset}(F,A)\times(G,B)$ and $S\tilde{\subset}(G,B)\times(H,C)$. If $R\circ S=S\circ R$, then $(R\circ S)\circ(R\circ S)=(S\circ S)\circ(R\circ R)$.

Definition 4.11. [23] Let R be a soft relation on (F,A). Then R is said to be

i. *reflexive*, if $F(a)\times F(a)\in R, \forall a\in A$.
ii. *symmetric*, if $F(a)\times F(b)\in R\Rightarrow F(b)\times F(a)\in R, \forall(a,b)\in A\times A$.
iii. *transitive*, if $F(a)\times F(b)\in R, F(b)\times F(a)\in R\Rightarrow F(a)\times F(c)\in R, \forall a,b,c\in A$.
iv. an equivalence relation, if it is reflexive, symmetric and transitive.
   We will denote the set of all soft equivalence relations on (F,A) as $\mathrm{SRel}_E((F,A))$.

Definition 4.12. [23] Let R be a soft relation on (F,A). Then R is called the *soft identity relation* on (F,A), if $F(a)\times F(a)\in R, \forall a\in A$.

It is clear that if R is a soft reflexive relation on (F,A) if and only if $I_{(F,A)}\subset R$.

Proposition 4.13. *Let $R, S\tilde{\subset}(F,A)\times(F,A)$. If S is reflexive, then $R\subset S\circ R$ and $R\subset R\circ S$.*

Proof. Let $F(a)\times F(b)\in R$. Since S is reflexive, $F(a)\times F(a)\in S$ and $F(b)\times F(b)\in S$. Thus $F(a)\times F(b)\in S\circ R$ and $F(a)\times F(b)\in R\circ S$. So $R\subset S\circ R$ and $R\subset R\circ S$. ☒

Theorem 4.14. *Let $R, S\tilde{\subset}(F,A)\times(F,A)$, let R be reflexive and let S be reflexive and transitive. Then $R\subset S$ if and only if $R\circ S=S$.*

Proof. ($\Rightarrow$): Suppose $R\subset S$. Let $F(a)\times F(c)\in R\circ S$. Then

$$\exists\, F(b)\in(F,A) \text{ s.t. } F(a)\times F(b)\in S \text{ and } F(a)\times F(c)\in R.$$

Since $R\subset S, F(b)\times F(c)\in S$. Since S is transitive, $F(a)\times F(c)\in S$. Thus $R\circ S\subset S$.

On the other hand, since R is reflexive, by Proposition 3.10, $S\subset R\circ S$, So $R\circ S=S$.

($\Leftarrow$): Suppose $R\circ S=S$. Let $F(a)\times F(b)\in R$. Since S is reflexive, $F(a)\times F(a)\in S$. Then $F(a)\times F(b)\in R\circ S$. Thus $F(a)\times F(b)\in S$. So $R\subset S$. This completes the proof.

Theorem 4.15. *Let $R\tilde{\subset}(F,A)\times(F,A)$.*

1. *R is symmetric if and only if $R=R^{-1}$.*
2. *R is transitive if and only if $R\circ R\subset R$.*

Proof. 1. ($\Rightarrow$): Suppose R is symmetric and let $F(a)\times F(b)\in(F,A)\times(F,A)$. Then

$$\begin{aligned} F(a)\times F(b)\in R &\Leftrightarrow F(b)\times F(a)\in R \\ &\qquad\text{[By the hypothesis]} \\ &\Leftrightarrow F(a)\times F(b)\in R^{-1} \\ &\qquad\text{[By Definition 4.5]} \end{aligned}$$

Thus $R=R^{-1}$.($\Leftarrow$): Suppose $R=R^{-1}$ and let $F(a)\times F(b)\in(F,A)\times(F,A)$. Then

$$\begin{aligned} F(a)\times F(b)\in R &\Rightarrow F(a)\times F(b)\in R^{-1} \\ &\qquad\text{[By the hypothesis]} \\ &\Rightarrow F(b)\times F(a)\in R. \end{aligned}$$

2. ($\Rightarrow$): Suppose R is transitive and let $F(a)\times F(b)\in R\circ R$. Then

$$\exists F(c)\in(F,A)\text{s.t.}F(a)\times F(c)\in R \text{ and } F(c)\times F(b)\in R.$$

Thus, by the hypothesis, $F(a)\times F(b)\in R$. Thus $R\circ R\subset R$.
($\Leftarrow$): Suppose $R\circ R\subset R$. Let $F(a)\times F(b)\in R$ and $F(b)\times F(c)\in R$. Then clearly $F(a)\times F(c)\in R\circ R$. Thus, by the hypothesis, $F(a)\times F(c)\in R$. So R is transitive.

Theorem 4.16. *Let $R, S\in SRel_E((F,A))$. Then $R\circ S\in SRel_E((F,A))$ if and only if $R\circ S=S\circ R$.*

Proof. ($\Rightarrow$): Suppose $R\circ S\in \mathrm{SRel}_E((F,A))$. Let $F(a)\times F(c)\in(F,A)\times(F,A)$. Then

$$\begin{aligned} &F(a)\times F(c)\in R\circ S \\ &\Leftrightarrow \exists\, F(b)\in(F,A) \text{ s.t. } F(a)\times F(b)\in S \text{ and } F(b)\times F(c)\in R \\ &\qquad\text{[By Definition 4.4]} \\ &\Leftrightarrow \exists\, F(b)\in(F,A) \text{ s.t. } F(c)\times F(b)\in R \text{ and } F(b)\times F(a)\in S \\ &\qquad\text{[Since } R \text{ and } S \text{ are symmetric]} \\ &\Leftrightarrow F(c)\times F(a)\in S\circ R \text{ [By Definition 4.4]} \\ &\Leftrightarrow F(a)\times F(c)\in S\circ R \text{ [By the hypothesis]} \end{aligned}$$

Thus $R\circ S=S\circ R$.

($\Leftarrow$): Suppose $R\circ S=S\circ R$. Let $F(a)\in(F,A)$. Since R and S are reflexive, $F(a)\times F(a)\in R$ and $F(a)\times F(a)\in S$. Then $F(a)\times F(b)\in S\circ R$. Thus, by the hypothesis, $F(a)\times F(a)\in R\circ S$. So $R\circ S$ is reflexive.

Now, suppose $F(a)\times F(b)\in R\circ S$. Then

$$\exists\, F(c)\in(F,A) \text{ s.t. } F(a)\times F(c)\in S \text{ and } F(c)\times F(b)\in R.$$

Since R and S are symmetric,

$$F(b) \times F(c) \in R \text{ and } F(c) \times F(a) \in S.$$

Thus $F(b) \times F(a) \in S \circ R$. So $F(b) \times F(a) \in R \circ S$. Hence $R \circ S$ is symmetric.

Finally,

$$\begin{aligned}(R \circ S) \circ (R \circ S) &= R \circ (S \circ R) \circ S \\ &= R \circ (R \circ S) \circ S \\ &= (R \circ R) \circ (S \circ S) \\ &\subset R \circ S\end{aligned}$$

Thus $R \circ S$ is transitive. Therefore $R \circ S \in \mathrm{SRel}_E((F, A))$.

Definition 4.17. [23] Let $R \in \mathrm{SRel}_E((F, A))$ and let $F(a) \in (F, A)$. Then the set

$$\{F(b) \in (F, A) : F(b) \times F(a) \in R\}$$

is called the *equivalence class determined by* $F(a)$ and denoted by $[F(a)]$, $F(a)/R$, $R_{F(a)}$ or $(F, A)_{F(a)}$.

The set $\{[F(a)] : F(a) \in (F, A)\}$ is called the *quotient soft set of* (F, A) *under* R and denoted by $(F, A)/R$.

Result 4.18.

([23], Lemma 4.5) Let $R \in \mathrm{SRel}_E((F, A))$. Then for any $F(a), F(b) \in (F, A)$,

$$F(a) \times F(b) \in R \text{ if and only if } [F(a)] = [F(b)].$$

Definition 4.19. [23] Let (F, A) be a soft set over U and let $P = \{(F_i, A_i) : i \in I\}$ be a collection of nonempty soft subsets of (F, A). Then P is called a *partition* of (F, A) if

i. $(F, A) = \tilde{\cup}_{i \in I}(F_i, A_i)$.

ii. $A_i \cap A_j = \emptyset$, whenever $i \neq j, \forall i, j \in I$.

In this case, each member of P is called a *block* of (F, A).

Definition 4.20. Let $R, S \in \mathrm{SRel}_E((F, A))$ and let $R \subset S$. Then the *image of* S *under* R, denoted by S/R, is an ordinary relation on $(F, A)/R$ defined as follows:

$$S/R = \{(F(a)/R, F(b)/R) : F(a) \times F(b) \in S\}.$$

Proposition 4.21. *Let* $R, S \in \mathrm{SRel}_E((F, A))$ *and let* $R \subset S$. *Then* S/R *is an ordinary equivalence relation on* $(F, A)/R$.

Proof. i. Since S is reflexive, $F(a) \times F(a) \in S, \forall F(a) \in (F, A)$. Then, by the definition of S/R, $(F(a)/R, F(a)/R) \in S/R$, $\forall F(a)/R \in (F, A)/R$. Thus S/R is reflexive.

ii. Suppose $(F(a)/R, F(b)/R) \in S/R$. Then $F(a) \times F(b) \in S$. Since S is symmetric, $F(b) \times F(a) \in S$. Thus $(F(b)/R, F(a)/R) \in S/R$. So S/R is symmetric.

iii. Suppose $(F(a)/R, F(b)/R) \in S/R$ and $(F(b)/R, F(c)/R) \in S/R$. Then $F(a) \times F(b) \in S$ and $F(b) \times F(c) \in S$. Since S is transitive, $F(a) \times F(c) \in S$. Thus $(F(a)/R, F(c)/R) \in S/R$. So S/R is transitive. Hence, by (i), (ii) and (iii), S/R is an ordinary equivalence relation on $(F, A)/R$.

Example 4.22. Let $A = \{a, b, c, d\}$, let $U = \{u_1, u_2, u_3, u_4, u_5\}$ and let (F, A) be the soft set over U given by

$$F(a) = \{u_1, u_2\},\ F(b) = \{u_3\},\ F(c) = \{u_3, u_4\} \text{ and } F(d) = \{u_4, u_5\}.$$

Consider two soft relations on (F, A) defined as follows: $R = \{F(a) \times F(a), F(b) \times F(b), F(c) \times F(c), F(d) \times F(d), F(a) \times F(b), F(b) \times F(a)\}$ and $S = \{F(a) \times F(a), F(b) \times F(b), F(c) \times F(c), F(d) \times F(d), F(a) \times F(b), F(b) \times F(a), F(c) \times F(d), F(d) \times F(c)\}$

Then clearly $R \subset S$. Thus by Definition 4.17,

$$(F, A)/R = \{F(a)/R, F(c)/R, F(d)/R\}.$$

So, by Definition 4.20,

$$\begin{aligned}S/R = \{&(F(a)/R, F(a)/R),\ (F(c)/R, F(c)/R),\ (F(d)/R, F(d)/R), \\ &(F(c)/R, F(d)/R),\ (F(d)/R, F(c)/R)\}.\end{aligned}$$

Furthermore, S/R is an ordinary equivalence relation on $(F, A)/R$. □

Proposition 4.23. *Let* $R, S, T \in \mathrm{SRel}_E((F, A))$ *be soft equivalence relation on* (F, A) *and let* $R \subset S \subset T$. *Then*

1. $S/R \subset T/R$,
2. $R \subset S \circ T$,
3. *If* $S \circ T \in \mathrm{SRel}_E((F, A))$, *then* $S/R \circ T/R = (S \circ T)/R$,
4. $S/R \circ T/R$ *is an ordinary equivalence relation on* $(F, A)/R$.

Proof. 1. Let $(F(a)/R, F(b)/R) \in S/R$. Then $F(a) \times F(b) \in S$. Since $S \subset T$, $F(a) \times F(b) \in T$. Thus $(F(a)/R, F(b)/R) \in T/R$. So $S/R \subset T/R$.

2. Let $F(a) \times F(b) \in R$. Since $R \subset S \subset T$, $F(a) \times F(b) \in T$. Since S is reflexive, $F(b) \times F(b) \in S$. Thus $F(a) \times F(b) \in S \circ T$. So $R \subset S \circ T$.

3. Suppose $S \circ T \in \mathrm{SRel}_E((F, A))$. Let $(F(a)/R, F(c)/R) \in S/R \circ T/R$. Then $\exists F(b) \in (F, A)$ s.t. $(F(a)/R, F(b)/R) \in T/R$ and $(F(b)/R, F(c)/R) \in S/R$. Thus $F(a) \times F(b) \in T$ and $F(b) \times F(c) \in S$. So $F(a) \times F(c) \in S \circ T$. By (2) and the hypothesis, $(F(a)/R, F(c)/R) \in S \circ T/R$.

   Hence $S/R \circ T/R \subset (S \circ T)/R$.

   Now let $(F(a)/R, F(c)/R) \in S \circ T/R$. Then $F(a) \times F(c) \in S \circ T$. Thus $\exists F(b) \in (F, A)$ s.t. $F(a) \times F(b) \in T$ and $F(b) \times F(c) \in S$.

   So $(F(a)/R, F(b)/R) \in T/R$ and $(F(b)/R, F(c)/R) \in S/R$.

   Hence $(F(a)/R, F(c)/R) \in S/R \circ T/R$, i.e., $S \circ T/R \subset S/R \circ T/R$.

   Therefore $S/R \circ T/R = (S \circ T)/R$.

4. The proof is obvious.

The results in this section indicate further results in soft relations. Then the next section moves on to discuss the results of soft mapping.

5. FURTHER RESULTS OF A SOFT MAPPING

This section is to develop results of soft mapping that is compatible with previous sections.

Definition 5.1. [23] Let (F, A) and (G, B) be two nonempty soft sets over a common universe U and let $f\tilde{\subset}(F, A)\times(G, B)$. Then f is called a *soft mapping from* (F, A) *to* (G, B), denoted by $f : (F, A) \to (G, B)$, if it satisfies the following conditions:

i. $\forall F(a) \in (F, A), \exists G(b) \in (G, B)$ s.t. $F(a) \times G(b) \in f$,

ii. $F(a) \times G(b) \in f$ and $F(a) \times G(c) \in f \Rightarrow G(b) = G(c)$.

In this case, if $F(a) \times G(b) \in f$, then we write $f(F(a)) = G(b)$.

Theorem 5.2. *Let (F, A) and (G, B) be two nonempty soft sets over U and let $f\tilde{\subset}(F, A) \times (G, B)$. Then $f : (F, A) \to (G, B)$ if and only if*

1. *$F(a) \times F(b) \in f$ and $F(a) \times G(c) \in f \Rightarrow G(b) = G(c)$.*
2. *$\mathrm{dom} f = (F, A)$.*
3. *$\mathrm{ran} f\tilde{\subset} (G, B)$.*

Proof. ($\Rightarrow$): Suppose $f : (F, A) \to (G, B)$ is a soft mapping.

1. From Definition 5.1 (ii), it is obvious.
2. Let $\mathrm{dom} f = (D, A_1)$. Then by Definition 4.3,

$$A_1 = \{a \in A : \exists b \in B \text{ s.t. } F(a) \times G(b) \in f\}$$

and

$$D(a) = F(a), \forall a \in A_1.$$

Thus by Definition 2.2, $(D, A_1)\tilde{\subset} (F, A)$, i.e., $\mathrm{dom} f\tilde{\subset} (F, A)$. Now let $a \in A$. Then clearly, $F(a) \in (F, A)$. By Definition 5.1 (i),

$$\exists\, G(b) \in (G, B) \text{ s.t. } F(a) \times G(b) \in f.$$

Thus $\exists b \in B$ s.t. $F(a) \times G(b) \in f$. So $A \subset A_1$ and $D(a) = F(a), \forall a \in A$. Hence $(F, A)\tilde{\subset}(D, A_1) = \mathrm{dom} f$. Therefore $\mathrm{dom} f = (F, A)$.

3. Let $\mathrm{ran} f = (RG, B_1)$. Then by Definition 4.3,

$$B_1 = \{b \in B : \exists a \in A \text{ s.t. } F(a) \times G(b) \in f\}$$

and

$$RG(b) = G(b), \forall b \in B_1.$$

Thus by Definition 2.2, $(RG, B_1)\tilde{\subset} (G, B)$. So $\mathrm{ran} f\tilde{\subset} (G.B)$.

($\Leftarrow$): Suppose the necessary conditions hold.

i. Let $F(a) \times G(b) \in f$. Then by Definition 4.3,

$$F(a) \in \mathrm{dom} f \text{ and } G(b) \in \mathrm{ran} f.$$

Thus by the hypotheses (2) and (3), $F(a) \in (F, A)$ and $G(b) \in (G, B)$.

So, $F(a) \times G(b) \in (F, A) \times (G, B)$. Hence $f\tilde{\subset}(F, A) \times (G, B)$.

ii. Let $F(a) \in (F, A)$. Since $\mathrm{dom} f = (F, A)$, $F(a) \in \mathrm{dom} f$. Then

$$\exists b \in B \text{ s.t. } F(a) \times G(b) \in f.$$

But $G(b) \in \mathrm{ran} f$. Since $\mathrm{ran} f \subset (G, B)$, $G(b) \in (G, B)$. Thus Definition 5.1 (i) holds.

Hence by Definition 5.1, $f : (F, A) \to (G, B)$ is a soft mapping.

The following is the immediate result of Theorem 5.2.

Corollary 5.3. Let $f : (F, A) \to (G, B)$ be a soft mapping. If $\mathrm{ran} f\tilde{\subset}(H, C)$, then $f : (F, A) \to (H, C)$ is a soft mapping.

The following is the immediate result of Definitions 2.2 and 5.1.

Theorem 5.4. *Let $f : (F, A) \to (G, B)$ and $g : (F, A) \to (G, B)$ be two soft mappings. Then $f = g$ if and only if $f(F(a)) = g(F(a))$, $\forall a \in A$.*

Definition 5.5. [23] Let $f : (F, A) \to (G, B)$ be a soft mapping. Then f is said to be

i. *injective* (or *one-one*), if $F(a) \neq F(b) \Rightarrow f(F(a)) \neq f(F(b))$.

ii. *surjective* (or *onto*), if $\mathrm{ran} f = (G, B)$.

iii. *bijective*, if it is injective and surjective.

From Definitions 4.3 and 5.5, it is obvious that $f : (F, A) \to (G, B)$ is a soft set surjective mapping if and only if $\forall b \in B, \exists a \in A$ s.t. $F(a) \times G(b) \in f$, i.e., $f(F(a)) = G(b)$.

Also from Definition 4.11, Theorem 5.2 and Definition 5.5, it is clear that the soft set identity relation $I_{(F,A)}$ on (F, A) is a soft set mapping $I_{(F,A)} : (F, A) \to (F, A)$. Furthermore, $I_{(F,A)}$ is bijective. In this case, $I_{(F,A)}$ is called the *soft identity mapping* and simply denoted by I or 1.

Proposition 5.6. *Let $f : (F, A) \to (G, B)$ and $g : (G, B) \to (H, C)$ be two soft mappings. Then $g \circ f : (F, A) \to (H, C)$ is a soft set mapping.*

In this case, we write $(g \circ f)(F(a)) = g(f(F(a)), \forall a \in A$.

Proof. i. Since $f : (F, A) \to (G, B)$ and $g : (G, B) \to (H, C)$ are soft mappings, $\mathrm{dom} f = A$, $\mathrm{ran} f \subset B$ and $\mathrm{dom} g = B$, $\mathrm{ran} g \subset C$. Then $\mathrm{ran} f \subset \mathrm{dom} g$. Thus, by Corollary 4.8, $\mathrm{dom}\,(g \circ f) = \mathrm{dom} f = A$.
On the other hand, by Proposition 4.7,

$$\mathrm{ran}\,(g \circ f) \subset \mathrm{ran}\, g \subset C.$$

So $\mathrm{dom}\,(g \circ f) = A$ and $\mathrm{ran}\,(g \circ f) \subset C$.

ii. Suppose $F(a) \times H(c) \in g \circ f$ and $F(a) \times H(d) \in g \circ f$. Then $\exists b_1 \in B$ s.t. $F(a) \times G(b_1) \in f$, $G(b_1) \times H(c) \in g$ and $\exists b_2 \in B$ s.t. $F(a) \times G(b_2) \in f$, $G(b_2) \times H(d) \in g$.

Since f is a soft mapping, $G(b_1) = G(b_2)$. Thus $b_1 = b_2$.

So $G(b_1) \times H(c) \in g$ and $G(b_1) \times H(d) \in g$. Since g is a soft mapping, $H(c) = H(d)$. Hence by Theorem 5.2, $g \circ f : (F, A) \to (H, C)$ is a soft mapping.

The following is the immediate result of Proposition 5.6.

Corollary 5.7. Let $f : (F, A) \to (G, B)$ be a soft mapping. Then $I_(G, B) \circ f = f$ and $f \circ 1_{(F,A)} = f$.

Definition 5.8. A soft set mapping $f : (F, A) \to (G, B)$ is said to be *invertible* if $f^{-1} : (G, B) \to (F, A)$ is a soft mapping.

From Definition 4.5 and 5.1, it is obvious that if $f : (F, A) \to (G, B)$ is invertible, then $f(F(a)) = G(b)$ if and only if $F(a) = f^{-1}(G(b))$.

Lemma 5.9. ([23], Theorem 5.11) Let $f : (F, A) \to (G, B)$ be a soft mapping. If f is bijective, then $f^{-1} : (G, B) \to (F, A)$ is bijective.

Proof. i. By Theorem 5.2, $\mathrm{dom}\, f = (F, A)$. By Definition 5.5 (ii), $\mathrm{ran}\, f = (G, B)$. Then by Proposition 4.7, $\mathrm{dom}\, f^{-1} = \mathrm{ran}\, f = (G, B)$ and $\mathrm{ran}\, f^{-1} = \mathrm{dom}\, f = (F, A)$.

ii. Suppose $G(b) \times F(a_1) \in f^{-1}$ and $G(b) \times F(a_2) \in f^{-1}$. Then $F(a_1) \times G(b) \in f$ and $F(a_2) \times G(b) \in f$.
Since f is injective, $F(a_1) = F(a_2)$. Thus by Theorem 5.2, $f^{-1} : (G, B) \to (F, A)$ is a soft mapping.

iii. Suppose $G(b_1) \times F(a) \in f^{-1}$ and $G(b_2) \times F(a) \in f^{-1}$. Then $F(a) \times G(b_1) \in f$ and $F(a) \times G(b_2) \in f$.
Since f is a soft mapping, $G(b_1) = G(b_2)$. Thus f^{-1} is injective.

iv. Since $\mathrm{ran}\, f^{-1} = (F, A)$, f^{-1} is surjective.
Hence $f^{-1} : (G, B) \to (F, A)$ is bijective.

Lemma 5.10. Let $f : (F, A) \to (G, B)$ be a soft mapping. If f is invertible, then f is bijective.

Proof. Suppose f is invertible. Then by Definition 5.8, $f^{-1} : (G, B) \to (F, A)$ is a soft mapping. Thus, by Theorem 5.2, $\mathrm{dom}\, f^{-1} = (G, B)$. So by Proposition 4.7 (1), $\mathrm{ran}\, f = (G, B)$. Hence f is surjective.

Now suppose $F(a_1) \times G(b) \in f$ and $F(a_2) \times G(b) \in f$. Then $G(b) \times F(a_1) \in f^{-1}$ and $G(b) \times F(a_2) \in f^{-1}$.

Since f^{-1} is a soft mapping, $F(a_1) = F(a_2)$., Thus f is injective.

Therefore f is bijective.

The following is the immediate result of Lemmas 5.9 and 5.10.

Theorem 5.11. *Let $f : (F, A) \to (G, B)$ be a soft mapping. Then f is invertible if and only if f is bijective.*

Lemma 5.12. Let $f : (F, A) \to (G, B)$ be a soft mapping. If f is invertible, then $f^{-1} \circ f = I_{(F,A)}$ and $f \circ f^{-1} = I_{(G,B)}$.

Proof. Suppose f is invertible. Let $G(b) = f(F(a))$. Then

$$\begin{aligned}(f^{-1} \circ f)(F(a)) &= f^{-1}(f(F(a))) \text{ [By Proposition 5.6]}\\ &= f^{-1}(G(b))\\ &= F(a)\\ &= I_{(F,A)}(F(a)).\end{aligned}$$

Thus $f^{-1} \circ f = I_{(F,A)}$.

Similarly, we can prove that $f \circ f^{-1} = I_{(G,B)}$. ☒

Lemma 5.13. Let $f : (F, A) \to (G, B)$ and $g : (G, B) \to (F, A)$ be soft mappings. If $g \circ f = I_{(F,A)}$ and $f \circ g = I_{(G,B)}$, then f is bijective and $g = f^{-1}$.

In this case, f^{-1} is called the *soft inverse mapping* of f.

Proof. i. For any $a_1, a_2 \in A$, suppose $f(F(a_1)) = f(F(a_2))$. Then

$$\begin{aligned}g(f(F(a_1))) &= g(f(F(a_2))) \text{ [Since } g \text{ is a soft mapping]}\\ &\Rightarrow (g \circ f)(F(a_1)) = (g \circ f)(F(a_2))\\ &\qquad \text{[By Proposition 5.6]}\\ &\Rightarrow F(a_1) = F(a_2). \text{ [Since } g \circ f = I_{(F,A)}]\end{aligned}$$

Thus f is injective.

ii. Let $b \in B$. Then

$$\begin{aligned}G(b) &= I_{(G,B)}(G(b))\\ &= (f \circ g)(G(b)) \text{ [Since} f \circ g = I_{(G,B)} \text{]}\\ &= f(g(G(b))). \text{ [By Proposition 5.6]}\end{aligned}$$

Since g is a soft mapping, $g(G(b) \in (F, A)$. Let $F(a) = g(G(b))$. Then clearly $F(a) \in (F, A)$ and $G(b) = f(F(a))$. Then f is surjective.

iii. Let $G(b) \times F(a) \in g$. Since g is a soft mapping,

$$g(G(b)) = F(a).$$

Then

$$\begin{aligned}f(F(a) &= f(g(G(b))) \text{ [Since } f \text{ is a soft mapping]}\\ &= (f \circ g)(G(b)) \text{ [By Proposition 5.6]}\\ &= I_{(G,B)}(G(b)) \text{ [Since} f \circ g = I_{(G,B)} \text{]}\\ &= G(b).\end{aligned}$$

Thus $F(a) \times G(b) \in f$, i.e., $G(b) \times F(a) \in f^{-1}$.
So $g \subset f^{-1}$.
Now let $G(b) \times F(a) \in f^{-1}$. Then $F(a) \times G(b) \in f$.
Since f is a soft mapping, $f(F(a)) = G(b)$.
Thus $g(G(b)) = g(f(F(a))) = (g \circ f)(F(a)) = I_{(F,A)}(F(a)) = F(a)$.
So $G(b) \times F(a) \in g$, i.e., $f^{-1} \subset g$. Hence $g = f^{-1}$.
This completes the proof.

These examples of proofs in the Lemma 5.13 illustrate the fact that constructing proofs in an axiomatized theory is a very laborious and tedious process. Many small technical lemmas need to be established from the axioms, which renders these proofs very lengthy and often unintuitive.

The following is the immediate result of Lemmas 5.12 and 5.13.

Theorem 5.14. *Let $f : (F, A) \to (G, B)$ be a soft mapping. Then f is invertible if and only if $\exists$ a soft mapping $g : (G, B) \to (F, A)$ s.t. $g \circ f = I_{(F,A)}$ and $f \circ g = I_{(G,B)}$. In fact $g = f^{-1}$.*

6. THE RELATION BETWEEN SOFT EQUIVALENCE RELATIONS AND MAPPINGS

The purpose of Section 6 is to explore the relationship between soft equivalence relations and mappings.

Definition 6.1. Let $f : (F, A) \to (G, B)$ be a soft mapping and let $R \in \mathrm{SRel}_E((G, B))$. Then the *preimage of* R *under* f, denoted by $f^{-1}(R)$, is a soft set relation on (F, A) defined as follows:

$$f^{-1}(R) = \{F(a) \times F(b) \in (F, A) \times (F, A) : f(F(a)) \times f(F(b)) \in R\}$$

The following is the immediate result of Definition 6.1.

Proposition 6.2. *$f^{-1}(R)$ is a soft equivalence relation on (F, A).*

Proposition 6.3. *Let $f : (F, A) \to (G, B)$ be a soft mapping. We define a soft set relation R on (F, A) as follows:*

$$R = \{F(a) \times F(b) \in (F, A) \times (F, A) : f(F(a)) = f(F(b))\}.$$

Then $R \in SRel_E((G, B))$.

In this case, R will be called the soft equivalence relation induced by f and will be denoted by $R = R_F$.

Proof.
i. For each $a \in A$, $f(F(a)) = f(F(a))$. Then by the definition of R, $F(a) \times F(a) \in R$. Thus R is reflexive.
ii. Suppose $F(a) \times F(b) \in R$. Then $f(F(a)) = f(F(b))$. Thus $f(F(b)) = f(F(a))$. So $F(b) \times F(a) \in R$. Hence R is symmetric.
iii. Suppose $F(a) \times F(b) \in R$ and $F(b) \times F(c) \in R$. Then $f(F(a)) = f(F(b))$ and $f(F(b)) = f(F(c))$. Thus $f(F(a)) = f(F(c))$.
So $F(a) \times F(c) \in R$. Hence R is transitive.
Therefore by (i), (ii) and (iii), $R \in \mathrm{SRel}_E((G, B))$ is an equivalence relation.

The following is the immediate result of Definitions 5.1 and 5.5.

Proposition 6.4. *Let $R \in SRel_E((F, A)))$ and let $f \tilde{\subset} (F, A) \times (F, A)/R$ such that $f(F(a)) = [F(a)]$ for each $a \in A$. Then $f : (F, A) \to (F, A)/R$ is a soft mapping.*

In this case, f will be called the canonical soft mapping from (F, a) onto $(F, A)/R$.

Proposition 6.5. *Let $R \in SRel_E((F, A))$ and let $f : (F, A) \to (F, A)/R$ be the canonical soft mapping. Then $R = R_f$.*

Proof.

$$\begin{aligned} F(a) \times F(b) \in R &\Leftrightarrow [F(a)] = [F(b)] \text{ [By Result 4.18]} \\ &\Leftrightarrow f(F(a)) = f(F(b)) \text{ [By Proposition 6.5]} \\ &\Leftrightarrow F(a) \times F(b) \in R_f. \text{ [By Proposition 6.4]} \end{aligned}$$

This completes the proof. ☒

Let $f : (F, A) \to (G, B)$ be a soft mapping. Then we can define three soft mappings:

$r : (F, A) \to (F, A)/R_f$ is the canonical soft mapping.

$s : (F, A)/R_f \to \mathrm{ran}f$ is the soft mapping given by $s(F(a)/R_f) = f(F(a)), \forall F(a) \in \mathrm{ran}f$.

Proposition 6.6. *If $f : (F, A) \to (G, B)$ is a soft mapping, then $f = t \circ s \circ r$, where r is surjective, s is bijective and t is injective.*

In this case, the result will be called the canonical decomposition of f.

Proof.
i. Let $F(a)/R_f \in (F, A)/R_f$. Then clearly $F(a) \in (F, A)$. Since r is the canonical soft mapping, $r(F(a)) = F(a)/R_f$. Thus r is surjective.
ii. Let $G(b) \in \mathrm{ran}f$. Then there exists $a \in A$ such that $f(F(a)) = G(b)$. Thus $F(a)/R_f \in (F, A)/R_f$. So, by the definition of s, $s(F(a)/R_f) = f(F(a)) = G(b)$. Hence s is surjective.

Now suppose $s(F(a)/R_f) = s(F(b)/R_f)$. Then $f(F(a)) = f(F(b))$.

Thus $F(a) \times F(b) \in R_f$. So, by Result 4.18, $F(a)/R)f = F(b)/R_f$. Hence is injective. Therefore s is bijective.
iii. For any $F(a), F(b) \in \mathrm{ran}f$, suppose $t(F(a)) = t(F(b))$. Then, by the definition of t, $F(a) = F(b)$. Thus t is injective.
iv. Let $a \in A$. Then

$$\begin{aligned} (t \circ s \circ r)(F(a)) &= (t \circ s)(r(F(a)) \text{ [By Proposition 5.6]} \\ &= (t \circ s)(F(a)/R_f) \text{ [By the definition of } r] \\ &= t(s(F(a)/R_f)) \\ &= t(f(F(a))) \text{ [By the definition of } s] \\ &= f(F(a)). \text{ [By the definition of } t] \end{aligned}$$

Thus $f = t \circ s \circ r$.
This completes the proof.

The following is the immediate result of Proposition 6.6.

Corollary 6.7. If $f : (F, A) \to (G, B)$ is surjective, then $t : \mathrm{ran}f \to (G, B)$ is bijective.

7. CONCLUSIONS

The purpose of the current study was to determine advanced soft relation and soft mapping methods. The most obvious finding to emerge from this study was the relationship between soft equivalence relations and mappings. In particular, we obtained the canonical decomposition of a soft mapping. This research has thrown up many questions in need of further investigation on soft set applications. Based on these results, we can further probe the applications of soft sets.

CONFLICTS OF INTEREST

The authors declare that they have no competing interests.

AUTHORS' CONTRIBUTIONS

Create and conceptualize ideas, J.-G.L and K.H.; writing-original draft preparation, J.-G.L and K.H.; writing-review and editing, G.S.; funding acquisition, J.-G.L. All authors have contributed equally to this paper in all aspects.

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AVAILABILITY OF DATA AND MATERIALS

Not applicable.

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