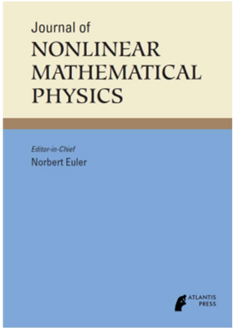


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LETTER TO THE EDITOR

# On the hierarchies of the fully nonlinear Möbius-invariant and symmetry-integrable evolution equations of order three

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This is a follow-up paper to the results published in *Studies in Applied Mathematics* **143**, 139–156 (2019), where we reported a classification of 3rd- and 5th-order semi-linear symmetry-integrable evolution equations that are invariant under the Möbius transformation, which includes a list of fully nonlinear 3rd-order equations that admit these properties. In the current paper we propose a simple method to compute the higher-order equations in the hierarchies for the fully nonlinear 3rd-order equations. We apply the proposed method to compute the 5th-order members of the hierarchies explicitly.

*Keywords*: Symmetry-Integrable Nonlinear Evolution Equations, Fully Nonlinear PDEs, Möbius transformations.

2000 Mathematics Subject Classification: 37K35, 35B06.

## 1. Introduction

In an earlier work [2] we derived all $(1+1)$-dimensional semi-linear evolution equations of order three and order five which are both Möbius-invariant and symmetry-integrable. The classification was done for semi-linear and fully nonlinear 3rd-order equations (meaning nonlinear in the highest derivative) of the form

$$u_t = u_x \Psi(S) \tag{1.1}$$

and for semi-linear 5th-order equations of the form

$$u_t = u_x \Psi(S, S_x, S_{xx}). \tag{1.2}$$

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Here and throughout this paper, $S$ denotes the Schwarzian derivative in terms of $u$, namely

$$S := \frac{u_{xxx}}{u_x} - \frac{3}{2}\left(\frac{u_{xx}}{u_x}\right)^2. \tag{1.3}$$

We remark that $S$, itself, is invariant under the Möbius transformation.That is

$$S(\overline{u}) = S(u), \quad \text{where} \quad \overline{u} = \frac{\alpha_1 u + \beta_1}{\alpha_2 u + \beta_2} \tag{1.4}$$

with $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$. Clearly $u_t/u_x$ is also Möbius invariant, as well as the $x$-derivatives of $S$, so that the $n$th-order equation in $u$,

$$u_t = u_x\Psi(S, S_x, S_{xx}, S_{3x}, \ldots, S_{(n-3)x}), \tag{1.5}$$

is Möbius invariant for any smooth function $\Psi$ with $n \geqslant 3$.

It follows [2] that the only semi-linear equation of the form (1.1) which is symmetry-integrable is the Schwarzian Korteweg-de Vries equation

$$u_t = u_x S. \tag{1.6}$$

When (1.1) is not required to be semi-linear, four additional symmetry-integrable fully nonlinear equations follow, namely [2]

$$u_t = -2\frac{u_x}{\sqrt{S}} \tag{1.7a}$$

$$u_t = \frac{u_x}{(b_1 - S)^2} \tag{1.7b}$$

$$u_t = \frac{u_x}{S^2} \tag{1.7c}$$

$$u_t = u_x\left(\frac{a_1 - S}{(a_1^2 + 3a_2)(S^2 - 2a_1 S - 3a_2)^{1/2}}\right), \tag{1.7d}$$

where the constants $a_1$, $a_2$ and $b_1$ are arbitrary, except for the condition that $a_1^2 + 3a_2 \neq 0$ and $b_1 \neq 0$.

For the 5th-order semi-linear equations

$$u_t = u_x S_{xx} + u_x\Phi_1(S, S_x, S_{xx}) \tag{1.8}$$

we obtained two equations, namely [2]

$$u_t = u_x\left(S_{xx} + \frac{1}{4}S^2\right): \quad \text{the Schwarzian Kupershmidt I equation;} \tag{1.9a}$$

$$u_t = u_x\left(S_{xx} + 4S^2\right): \quad \text{the Schwarzian Kupershmidt II equation.} \tag{1.9b}$$

In addition, the 5th-order Möbius-invariant equation

$$u_t = u_x\left(S_{xx} + \frac{3}{2}S^2\right) \tag{1.10}$$

follows from the 2nd member of the Schwarzian Korteweg-de Vries hierarchy of (1.6).

The following statement is essential for this classification:

**Lemma 1 ([2]).** *The nth-order Möbius-invariant equation*

$$u_t = u_x \Psi(S, S_x, S_{xx}, \ldots, S_{(n-3)x}) \tag{1.11}$$

*can be presented in the form of the Möbius-invariant system*

$$u_t = u_x \Psi(S, S_x, S_{xx}, \ldots, S_{(n-3)x}) \tag{1.12a}$$

$$S_t = \left(D_x^3 + 2SD_x + S_x\right) \Psi(S, S_x, S_{xx}, \ldots, S_{(n-3)x}), \tag{1.12b}$$

*known as the Schwarzian system, where S denotes the Schwarzian derivative in terms of u and* $n \geqslant 3$*.*

*For semi-linear evolution equations with* $n > 3$*, system* (1.12a)–(1.12b) *takes the following form:*

$$u_t = u_x S_{(n-3)x} + u_x \Psi_1(S, S_x, S_{xx}, \ldots, S_{(n-4)x}) \tag{1.13a}$$

$$S_t = S_{nx} + 2SS_{(n-2)x} + S_x S_{(n-3)x} + \left(D_x^3 + 2SD_x + S_x\right) \Psi_1(S, S_x, S_{xx}, \ldots, S_{(n-4)x}). \tag{1.13b}$$

The Möbius-invariant and symmetry-integrable equations listed above then follow from

**Proposition 1 ([2]).** *Let* $R[S]$ *be a recursion operator for* (1.12b)*, such that*

$$Z_j^S = R^j[S] S_t \frac{\partial}{\partial S} \tag{1.14}$$

*are generalized symmetries* (*also known as Lie-Bäcklund symmetries*) *for* (1.12b) *for all* $j \in \mathscr{N}$*. Then*

$$Z_j^u = u_x \Psi(S, S_x, S_{xx}, \ldots, S_{(n-3)x}) \frac{\partial}{\partial u} + R^j[S] S_t \frac{\partial}{\partial S} \tag{1.15}$$

*are generalized symmetries for* (1.12a) *for all* $j \in \mathscr{N}$*. Therefore,* (1.12a) *is symmetry-integrable if* (1.12b) *is symmetry-integrable.*

The hierarchies of higher-order members of the Möbius-invariant and symmetry-integrable equations (1.6), (1.9a) and (1.9b) are well known and are best presented in terms of their recursion operators ([1], [2], [4]). However, for the fully nonlinear equations (1.7a)–(1.7d) one encounters a problem as we have found that these equations do not admit recursion operators of the usual linear form

$$R[u] = \sum_{j=0}^{p} G_j D_x^j + \sum_{k=1}^{q} \eta_k D_x^{-1} \circ \Lambda_k. \tag{1.16}$$

In this paper we propose and alternate approach to compute and present the higher-order members of the fully nonlinear hierarchies.

**Motivation.** The results for 3rd-order and 5th-order semi-linear equations reported in [2] show that the Möbius-invariant systems that are identified by Proposition 1 are exactly those equations that play a central role in the construction of nonlocal and auto-Bäcklund transformations by multipotentialisation, namely the Schwarzian KdV equation (1.6), the Schwarzian Kupershmidt I equation (1.9a) and the Schwarzian Kupershmidt II equation (1.9b) (see [1] for more details). We expect that the Möbius-invariant and symmetry-integrable fully nonlinear equations of 3rd and higher order are of similar importance in the study of fully nonlinear evolution equations.

## 2. Hierarchies of the fully nonlinear evolution equations of order three

Lemma 1 directly leads to the following proposition by which it is relatively easy to compute the higher-order members of the Möbius-invariant and symmetry-integrable hierarchies of the 3rd-order equation $u_t = u_x\Psi(S)$:

**Proposition 2.** *Let*

$$u_t = u_x\Psi(S) \tag{2.1a}$$

$$S_t = (D_x^3 + 2SD_x + S_x)\Psi(S) \tag{2.1b}$$

*be a Möbius-invariant and symmetry-integrable system for some given function $\Psi = \Psi(S)$, where $S$ is the Schwarzian derivative in $u$. Let $R[S]$ be a 2nd-order recursion operator for* (2.1b). *Then the higher-order equations in the hierarchy of the symmetry-integrable equation* (2.1a) *are of the form*

$$u_{t_j} = u_x\Psi_j(S, S_x, S_{xx}, S_{(2j)x}), \qquad j = 0, 1, 2, \dots, \tag{2.2}$$

*where $\Psi_j$ is to be solved for every $j > 0$ from the relation*

$$(D_x^3 + 2SD_x + S_x)\Psi_j(S, S_x, \dots, S_{(2j)x}) = R^j[S](D_x^3 + 2SD_x + S_x)\Psi(S) \tag{2.3}$$

*and $\Psi_0 \equiv \Psi$.*

**Remark 1.** Note that (2.1b) is never fully nonlinear (in the highest derivative of $S$), so that the existence of a recursion operator $R[S]$ of the form (1.16) for the symmetry-integrable equation (2.1b) can be assumed.

**Result.** *Applying Proposition* 2 *we obtain the following* 5*th-order equations that belong to the hierarchies of fully nonlinear* 3*rd-order equations* (1.7a), (1.7b), (1.7c) *and* (1.7d), *respectively:*

$$u_{t_1} = u_x\left(\frac{S_{xx}}{S^{5/2}} - \frac{5}{4}\frac{S_x^2}{S^{7/2}} + \frac{4}{S^{1/2}}\right) \tag{2.4a}$$

$$u_{t_1} = u_x\left(\frac{4S_{xx}}{b_1(b_1-S)^5} + \frac{10S_x^2}{b_1(b_1-S)^6} - \frac{b_1-4S}{b_1(b_1-S)^4}\right) \tag{2.4b}$$

$$u_{t_1} = u_x\left(-\frac{2S_{xx}}{S^5} + \frac{5S_x^2}{S^6} + \frac{2}{S^3}\right) \tag{2.4c}$$

$$\begin{aligned} u_{t_1} = u_x\Bigg(&\frac{S_{xx}}{(S^2-2a_1S-3a_2)^{5/2}} - \frac{5(S-a_1)S_x^2}{2(S^2-2a_1S-3a_2)^{7/2}} \\ &-\frac{a_1S+3a_2}{(a_1^2+3a_2)(S^2-2a_1S-3a_2)^{3/2}}\Bigg). \end{aligned} \tag{2.4d}$$

To discuss the derivation of (2.4a)–(2.4d), we consider the equations (1.7a)–(1.7d) in four separate cases, where we also provide the recursion operators for each $S$-equation associated to (1.7a)–(1.7d):

**Case 1:** We consider the 3rd-order Schwarzian system that is associated with the equation (1.7a), namely

$$u_t = -2\frac{u_x}{\sqrt{S}} \tag{2.5a}$$

$$S_t = S^{-3/2}S_{3x} - \frac{9}{2}S^{-5/2}S_xS_{xx} + \frac{15}{4}S^{-7/2}S_x^3. \tag{2.5b}$$

A recursion operator for (2.5b) is

$$R[S] = \frac{1}{S}D_x^2 - \frac{5}{2}\frac{S_x}{S^2}D_x - 2\frac{S_{xx}}{S^2} + \frac{15}{4}\frac{S_x^2}{S^3} - \frac{1}{2}S_tD_x^{-1}\circ\frac{1}{\sqrt{S}} \tag{2.6}$$

where $R[S]S_x = 0$ and

$$\begin{aligned} S_{t_1} &= R[S]S_t \\ &= S^{-5/2}S_{5x} - 10S^{-7/2}S_xS_{4x} + \frac{455}{8}S^{-9/2}S_x^2S_{3x} - \frac{35}{2}S^{-7/2}S_{xx}S_{3x} \\ &\quad + \frac{315}{4}S^{-9/2}S_xS_{xx}^2 - \frac{3465}{16}S^{-11/2}S_x^3S_{xx} + \frac{3465}{32}S^{-13/2}S_x^5. \end{aligned} \tag{2.7}$$

Applying Proposition 2 we need to find the general solution for $\Psi_1(S,S_x,S_{xx})$ from the relation

$$(D_x^3 + 2SD_x + S_x)\Psi_1(S,S_x,S_{xx}) = R[S]S_t \tag{2.8}$$

with $R[S]S_t$ given by (2.7). This leads to

$$\Psi_1(S,S_x,S_{xx}) = S^{-1/2}\left(S^{-2}S_{xx} - \frac{5}{4}S^{-3}S_x^2 + 4\right), \tag{2.9}$$

so that the 5th-order Schwarzian system in the hierarchy is

$$u_{t_1} = u_x\left(S^{-5/2}S_{xx} - \frac{5}{4}S^{-7/2}S_x^2 + 4S^{-1/2}\right) \tag{2.10a}$$

$$\begin{aligned} S_{t_1} &= R[S]S_t \\ &= S^{-5/2}S_{5x} - 10S^{-7/2}S_xS_{4x} + \frac{455}{8}S^{-9/2}S_x^2S_{3x} - \frac{35}{2}S^{-7/2}S_{xx}S_{3x} \\ &\quad + \frac{315}{4}S^{-9/2}S_xS_{xx}^2 - \frac{3465}{16}S^{-11/2}S_x^3S_{xx} + \frac{3465}{32}S^{-13/2}S_x^5. \end{aligned} \tag{2.10b}$$

**Case 2:** We consider the 3rd-order Schwarzian system that is associated with equation (1.7b), namely

$$u_t = \frac{u_x}{(b_1 - S)^2} \tag{2.11a}$$

$$S_t = \frac{2S_{3x}}{(b_1-S)^3} + \frac{18S_xS_{xx}}{(b_1-S)^4} + \frac{24S_x^3}{(b_1-S)^5} + \frac{(3S+b_1)S_x}{(b_1-S)^3}, \tag{2.11b}$$

where $b_1 \neq 0$. A recursion operator for (2.11b) is

$$R[S] = \frac{1}{b_1}\frac{2}{(b_1-S)^2}D_x^2 + \frac{1}{b_1}\frac{10S_x}{(b_1-S)^3}D_x + \frac{8}{b_1}\left(\frac{S_{xx}}{(b_1-S)^3} + \frac{3S_x^2}{(b_1-S)^4} + \frac{S}{2(b_1-S)^2}\right) + \frac{1}{b_1}S_t D_x^{-1}\circ 1 + \frac{1}{b_1}S_x D_x^{-1}\circ\frac{1}{(b_1-S)^2}, \tag{2.12}$$

whereby $R[S]$ maps the $x$-translation symmetry to the $t$-translation symmetry. That is

$$R[S]S_x = \frac{2S_{xxx}}{(b_1-S)^3} + \frac{18S_xS_{xx}}{(b_1-S)^4} + \frac{24S_x^3}{(b_1-S)^5} + \frac{(3S+b_1)S_x}{(b_1-S)^3} = S_t. \tag{2.13}$$

Calculating $R[S]S_t$ and using Proposition 2 to determine $\Psi_1$, we obtain the following 5th-order Schwarzian system for this hierarchy:

$$u_{t_1} = u_x\left(\frac{4S_{xx}}{b_1(b_1-S)^5} + \frac{10S_x^2}{b_1(b_1-S)^6} - \frac{b_1-4S}{b_1(b_1-S)^4}\right) \tag{2.14a}$$

$$S_{t_1} = R[S]S_t$$

$$= \frac{4S_{5x}}{b_1(b_1-S)^5} + \frac{80S_xS_{4x}}{b_1(b_1-S)^6} + \frac{140S_{xx}S_{3x}}{b_1(b_1-S)^6} + \frac{780S_x^2S_{3x}}{b_1(b_1-S)^7} + \frac{20SS_{3x}}{b_1(b_1-S)^5} + \frac{1080S_xS_{xx}^2}{b_1(b_1-S)^7} + \frac{4620S_x^3S_{xx}}{b_1(b_1-S)^8} + \frac{4(55S+b_1)S_xS_{xx}}{b_1(b_1-S)^6} + \frac{3360S_x^5}{b_1(b_1-S)^9} + \frac{10(35S+13b_1)S_x^3}{b_1(b_1-S)^7} + \frac{(2S^2+5b_1S-b_1^2)S_x}{b_1(b_1-S)^5}. \tag{2.14b}$$

**Case 3:** We consider the 3rd-order Schwarzian system that is associated with the equation (1.7c), namely

$$u_t = u_x\left(\frac{1}{S^2}\right) \tag{2.15a}$$

$$S_t = -2\left(\frac{S_{3x}}{S^3} - \frac{9S_xS_{xx}}{S^4} + \frac{12S_x^3}{S^5} + \frac{3S_x}{2S^2}\right). \tag{2.15b}$$

A recursion operator for (2.15b) is

$$R[S] = \frac{1}{S^2}D_x^2 - \frac{5S_x}{S^3}D_x - \frac{4S_{xx}}{S^3} + \frac{12S_x^2}{S^4} + \frac{2}{S} + \frac{S_t}{2}D_x^{-1}\circ 1 + \frac{S_x}{2}D_x^{-1}\circ\frac{1}{S^2}, \tag{2.16}$$

whereby $R[S]$ maps the $x$-translation symmetry to zero. Calculating $R[S]S_t$ and using Proposition 2 to determine $\Psi_1$, we obtain the following 5th-order Schwarzian system for this hierarchy:

$$u_{t_1} = u_x\left(-\frac{2S_{xx}}{S^5} + \frac{5S_x^2}{S^6} + \frac{2}{S^3}\right) \tag{2.17a}$$

$$\begin{aligned} S_{t_1} &= R[S]S_t \\ &= -\frac{2S_{5x}}{S^5} + \frac{40S_xS_{4x}}{S^6} + \frac{70S_{xx}S_{3x}}{S^6} - \frac{390S_x^2S_{3x}}{S^7} - \frac{10S_{3x}}{S^4} - \frac{540S_xS_{xx}^2}{S^7} \\ &\quad + \frac{2310S_x^3S_{xx}}{S^8} + \frac{110S_xS_{xx}}{S^5} - \frac{175S_x^3}{S^6} - \frac{1680S_x^5}{S^9} - \frac{10S_x}{S^3}. \end{aligned} \tag{2.17b}$$

**Case 4:** We consider the 3rd-order Schwarzian system that is associated with equation (1.7d), namely

$$u_t = u_x\left(\frac{a_1 - S}{(a_1^2+3a_2)(S^2-2a_1S-3a_2)^{1/2}}\right) \tag{2.18a}$$

$$\begin{aligned} S_t &= \frac{S_{3x}}{(S^2-2a_1S-3a_2)^{3/2}} - \frac{9(S-a_1)S_xS_{xx}}{(S^2-2a_1S-3a_2)^{5/2}} \\ &\quad + \frac{3(4S^2-8a_1S+5a_1^2+3a_2)S_x^3}{(S^2-2a_1S-3a_2)^{7/2}} - \frac{(S^3-3a_1S^2-9a_2S+3a_1a_2)S_x}{(a_1^2+3a_2)(S^2-2a_1S-3a_2)^{3/2}}, \end{aligned} \tag{2.18b}$$

where $a_1^2+3a_2 \neq 0$. Note that the case $a_1^2+3a_2 = 0$ is given by Case 2 above. A recursion operator for (2.18b) is

$$\begin{aligned} R[S] &= \frac{1}{S^2-2a_1S-3a_2}D_x^2 + \frac{5S_x(a_1-S)}{(S^2-2a_1S-3a_2)^2}D_x + \frac{4S_{xx}(a_1-S)}{(S^2-2a_1S-3a_2)^2} \\ &\quad + \frac{3S_x^2(4S^2-8a_1S+5a_1^2+3a_2)}{(S^2-2a_1S-3a_2)^3} + \frac{2S}{S^2-2a_1S-3a_2} + \frac{a_1}{a_1^2+3a_2} \\ &\quad + S_tD_x^{-1}\circ\frac{a_1-S}{(S^2-2a_1S-3a_2)^{1/2}} - \frac{S_x}{a_1^2+3a_2}D_x^{-1}\circ 1, \end{aligned} \tag{2.19}$$

whereby $R[S]$ maps the $x$-translation symmetry to zero. Calculating $R[S]S_t$ and using Proposition 2 to determine $\Psi_1$, we obtain the following 5th-order Schwarzian system for this hierarchy:

$$\begin{aligned} u_{t_1} &= u_x\left(\frac{S_{xx}}{(S^2-2a_1S-3a_2)^{5/2}} - \frac{5(S-a_1)S_x^2}{2(S^2-2a_1S-3a_2)^{7/2}}\right. \\ &\quad \left. - \frac{a_1S+3a_2}{(a_1^2+3a_2)(S^2-2a_1S-3a_2)^{3/2}}\right) \end{aligned} \tag{2.20a}$$

$$S_{t_1} = \frac{S_{5x}}{(S^2-2a_1S-3a_2)^{5/2}} - \frac{20(S-a_1)S_xS_{4x}}{(S^2-2a_1S-3a_2)^{7/2}} - \frac{35(S-a_1)S_{xx}S_{3x}}{(S^2-2a_1S-3a_2)^{7/2}}$$

$$+\frac{65(6S^2-12a_1S+3a_2+7a_1^2)S_x^2S_{3x}}{2(S^2-2a_1S-3a_2)^{9/2}}+\frac{(2a_1S^2+a_1^2S+15a_2S-6a_1a_2)S_{3x}}{(a_1^2-3a_2)(S^2-2a_1S-3a_2)^{5/2}}$$

$$+\frac{45(6S^2-12a_1S+3a_2+7a_1^2)S_xS_{xx}^2}{(S^2-2a_1S-3a_2)^{9/2}}$$

$$-\frac{1155(S-a_1)(2S^2-4a_1S+3a_2+3a_1^2)S_x^3S_{xx}}{2(S^2-2a_1S-3a_2)^{11/2}}$$

$$-\frac{(18a_1S^3+a_1^2S^2+165a_2S^2-9a_1^3S-189a_1a_2S+84a_1^2a_2+90a_2^2)S_xS_{xx}}{(a_1^2+3a_2)(S^2-2a_1S-3a_2)^{7/2}}$$

$$+\frac{105(16S^4-64a_1S^3+112a_1^2S^2+48a_2S^2-96a_1^3S-96a_1a_2S)S_x^5}{2(S^2-2a_1S-3a_2)^{13/2}}$$

$$+\frac{105(33a_1^4+54a_1^2a_2+9a_2^2)S_x^5}{2(S^2-2a_1S-3a_2)^{13/2}}+\frac{(945a_2^2S+30a_1^4S-693a_1a_2^2-375a_1^3a_2)S_x^3}{2(a_1^2+3a_2)(S^2-2a_1S-3a_2)^{9/2}}$$

$$+\frac{(48a_1S^4+525a_2S^3-17a_1^2S^3-33a_1^3S^2-963a_1a_2S^2+981a_1^2a_2S)S_x^3}{2(a_1^2+3a_2)(S^2-2a_1S-3a_2)^{9/2}}$$

$$+\frac{3(a_1S^3+5a_2S^2-a_1a_2S+3a_2^2)S_x}{(a_1^2+3a_2)(S^2-2a_1S-3a_2)^{5/2}}. \qquad (2.20b)$$

## 3. Concluding remarks

We have introduced a simple method, given by Proposition 2, by which it is relatively easy to compute the higher-order equations for a hierarchy of Möbius-invariant and symmetry-integrable equations. This is of particular intertest for fully nonlinear equations (1.7a)–(1.7a), where it is difficult to obtain the recursion operators of the equations. Proposition 2 applies to 3rd-order equations, but the extension to higher-order equations is straightforward. Our study of 5th-order quasi-linear and fully nonlinear Möbius-invariant and symmetry-integrable evolution equations is ongoing and will be published in the near future.

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