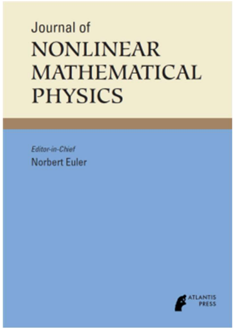

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Decomposition of 2-Soliton Solutions for the Good Boussinesq Equations

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We consider decompositions of two-soliton solutions for the good Boussinesq equation obtained by the Hirota method and the Wronskian technique. The explicit forms of the components are used to study the dynamics of 2-soliton solutions. An interpretation in the context of eigenvalue problems arising from KdV type equations and transport equations is considered. Numerical examples are included.

Keywords: N-Solitons; Wronskian Solutions; KdV, Boussinesq.

2000 Mathematics Subject Classification: 35C08; 35Q35.

1. Introduction

The N-soliton solutions of nonlinear PDE describing shallow water waves (SWW) as the Korteweg-de Vries (KdV), the KP hierarchy, Boussinesq and more recent Camassa-Holm (CH), Depasperies-Procesi (DP) and other PDE have been subject of numerous recent studies. The interest stems from the mathematical and the physical importance of those solutions. Explicit forms of the N-soliton solutions also can provide better modeling of SWW. There are well developed analytical methods for obtaining explicit solutions via recursion operators, Hirota bi-linear method or the Wronskian technique but the physical interpretation is obscured by the complexity of the formulas. A few papers on interpretations and discussion of possible decomposition are [2], [13], [21], [5].

The intend in this work is to study the properties of 2-soliton solutions for one of the simplest and oldest models, the Boussinesq equation. The general form of the Boussinesq equation is

$$u_{tt} + a_1 u_{xx} + a_2 (u^2)_{xx} + a_3 u_{xxxx} = 0,$$

where a_i, $i = 1, 2, 3$ are real numbers and $a_2 a_3 \neq 0$. In the case $a_3 > 0$ it is known as the 'good' Boussinesq equation and is equivalent to

$$v_{tt} + (v^2)_{xx} + v_{xxxx} = 0,$$

under the transformation

$$u(x,t) = -\frac{a_1}{2a_2} + \frac{a_3}{a_2} v(x, \sqrt{a_3} t).$$

We consider the 'good' Boussinesq (gB) equation as defined in [6]

$$u_{tt} - u_{xx} + (u^2)_{xx} + \frac{1}{3} u_{xxxx} = 0. \tag{1.1}$$

The equations has a well known Lax pair

$$\begin{aligned}&\phi_{xxx} + v\phi_x + w\phi = \lambda\phi,\\&\phi_t = -\phi_{xx} - v\phi.\end{aligned}$$

PDE with a Lax pair is usually a definition of integrable system, [11], and can be solved by the inverses scattering transform (IST) introduced by Clifford, Greene, Kruskal, and Muira, [9]. By utilizing IST Ablowitz and Segur, [1] provide an almost linearization of the KdV

$$u_t - 6uu_x + u_{xxx} = 0 \tag{1.2}$$

with respect to the spatial variable in terms of an eigenvalue problem with the Lax pair

$$\begin{aligned}&\phi_{xx} = (\lambda - u)\phi,\\&\phi_t = -4\phi_{xxx} - 3u_x\phi - 6u\phi_x.\end{aligned}$$

Many of the SWW equations posses a single traveling wave of the form $\operatorname{sech}^r(ax-bt)$ for $r = 1, 2$ and a, b independent of x. The duality of a wave and a particle of the one soliton solution is often considered, [5]. The results presented in this work are also interpreted in a direction of that comparison for the case of two-soliton solutions. In order to clarify this we present a brief discussion on the duality of one soliton solution in the context of KdV and gB.

A one soliton solution to (1.2) for $p > 0$ is $k(x,t) = p\operatorname{sech}^2 p(x - 4p^2t - w)$, where w is a real phase shift. On the other hand a one soliton solution to (1.1) is not defined for all p but for a range of p that can be represented by two of the solutions, m_1, m_2 of the cubic equation $\mu^3 - \frac{3}{4}\mu = \lambda$, $|\lambda| \leqslant \frac{1}{4}$ and $p = \frac{1}{2}(m_1 - m_2)$. Let $M = \frac{m_1+m_2}{2}$ then soliton solution to (1.1) corresponding to such p is the function $b(x,t) = p\operatorname{sech}^2 p(x - 2Mt)$, [6]. The time coefficients in the phases of the two 'sech' functions are different and we can not compare k and b for any time t and any x. We will compare k and b at any fixed time $t = t_0$ with a phase shift for KdV $w(t_0) = (4p^2 - M)t_0$, and hence $v(x,t_0) = k(x - w/p, t_0)$.

The two equations describe traveling waves that in turn represent an energy transport. The function $\psi(x,t) = \operatorname{sech} p(x - 4p^2t - w)$ is a solution of the time independent equation from the Lax pair for KdV $\psi_{xx} = (p^2 - k)\psi$ with potential $k = p\psi^2$. The function $\Phi(x,t) = \operatorname{sech} p(x - Mt)$ is a solution of the transport equation

$$\Phi_t + M\Phi_x = 0,$$

and $b = p\Phi^2$. The equation for ψ is an eigenstate equation. The function $\phi(x,t) = (e^{m_1x - m_1^2t} + e^{m_2x - m_2^2t})^{-1}b$ is a solution of the wave equation $\phi_t = \phi_{xx} + b\phi$. Since $k(x - w(t_0)/p, t_0) = b(x,t_0)$ for any t_0 we can consider the function ψ defines the steady state of a 'particle', and $k = p\psi^2$ can be considered as a weighted energy distribution and potential. The function ϕ is a solution of the real time-dependent Schrodinger equation corresponding to the energy state $\psi(x,t)$ so $e^{-(m_1+m_2)x - (m_1^2+m_2^2)t}b$ could describe the probability distribution of the particle in motion. On the other hand the energy is confined into the soliton and the transport equation governs the transfer of energy which is constant in time. Comparing the solutions of KdV and gB is justified by the fact that both are approximations of the same order, [18], of the Euler's equation in Fluid Mechanics.

In the next section we include brief overview of decomposing N-soliton solution into components with an emphasis on KdV type equations.

Section 3 contains decomposition of 2-soliton solutions of gB with components that satisfy similar differential equations as in the case of a single soliton. In sections 4, 5 we compare the resulting solutions to provide interpretation in the context of the discussion for one soliton, including numerical examples. The last section is a discussion of the results presented and further directions.

2. Decomposition for 2-Soliton Solutions

A general natural way method for decomposing a N-soliton solution u_N onto N interacting components (interacting solitons)

$$u_N = \sum_{k=1}^{N} (u_N)_k$$

seems to be by a recursion operator $\Phi(u)$ and a corresponding hierarchy, [8], [4],

$$u_t = \Phi^m(u)u_x = K_m(u), \quad m \in N.$$

In this case for real c_k^m the spatial derivative of $u_N = u_N(\xi_1, \dots, \xi_N)$, $\xi_k = x - c_k^m t + q_k$ is

$$(u_N)_x = \sum_{k=1}^{N} c_k^m w_k, \quad w_k = \frac{\partial u_N}{\partial q_k}.$$

The operators Φ and K_m' are a Lax pair

$$\Phi(u_N)w_k = c_k w_k, \quad (w_k)_t = K_m'[w_k]$$

and the N-soliton solution decomposes as follows

$$u_N = \sum_{k=1}^{N} c_k^m (u_N)_k = \sum c_k^m \partial_x^{-1} w_k = \sum c_k^m \partial_x^{-1} \frac{\partial u_N}{\partial q_k},$$

which can be immediately calculated from the Hirota (or Wronskian) formula. The method of recursion operators has been used in many works, [4], [3], [7], [8], [15], to produce N-soliton decomposition for Burgers, KdV, mKdV, sine-Gordon, cubic Schrodinger and other nonlinear PDE's. For the Boussinesq equation a recursion operator is presented in [20] but no decomposition of N-soliton solution discussed.

The Lax pair (Φ, K_m') is often not the 'standard' Lax pair related to the linear problem

$$L\psi = \lambda\psi, \quad \psi_t = A_m\psi$$

and in general the eigenfunctions ψ_k and w_k as well as the eigenvalues λ_k and c_k are interrelated. For the KdV for example

$$w_k = (\psi_k^2)_x, \quad c_k = \lambda_k = p_k^2.$$

Our interest is in nonlinear PDE modeling SWW. These equations usually are either directly derived from or related to the Fluid Eulers' Equation. The simplest model equations are KdV and Boussinesq equations. They model unidirectional solitons and posses explicit and relatively simple N-soliton solutions. Models of higher order of approximation to the Euler's equations and multi-soliton solutions are Camassa-Holm (CH), [23], Depasperis-Procesi (DP), [17], Novikov (N), [17], and

Sawada-Kotera (SK), [22]. Recursion operators for some of those equation are known but with no simple analytical descriptions.

One of the goals in this work is to investigate decomposition of 2-soliton solution onto components related to lower order linear PDE. We consider known solution and the lower order equation from a Lax pair. For KdV this is the second order eigenproblem but for the Boussinesq L is a third order and we consider the second order wave equation corresponding to A_m.

The Hirota method, with Wronskian solutions, Darboux and Backlund transformations are more often used to model 2-soliton solutions and their interactions in the setting of SWW PDE. For some of them fully explicit 2-soliton solutions are not known. For example, in [23] a 2-soliton solution for Negative order KdV is transformed to express the 2-soliton solution for CH. The 2-soliton solutions for the other shallow water waves present the same difficulty due to the complexity of the functions describing them. All of the equations posses a single soliton solution in $\operatorname{sech}(ax-bt)$ form but the construction of the 2-soliton solutions from two seed solutions is more involved. A common element in the most of the constructions is the Wronskian determinant

$$W(f,g) = \begin{vmatrix} f & g \\ f_x & g_x \end{vmatrix} = fg_x - f_x g,$$

with two generating functions f, g defined as

$$\begin{aligned} f(x,t) &= e^{p_1x-T_1(t)} + e^{-p_1x-T_1(t)} = 2\cosh(p_1x - T_1(t)) \\ g(x,t) &= e^{p_2x-T_2(t))} - e^{-p_2x-T_2(t)} = 2\sinh(p_2x - T_2(t)), \end{aligned}$$

for real p_1, p_2.

The most popular model for SWW is still the Boussinesq eqaution due to its simplicity but even in this case there are difficulties in considering 2-soliton solutions. In [17] a merging of two solitons, fusion, was reported. This is likely due to the fact that the Lax pair for gB has a third order spatial equation. Explicit N-soliton solutions for KdV and gB with elastic interaction, i.e. after the interaction only phase shifts occur, can be obtained by Hirota's method, [10] and Wronskian determinants, [12], [6].

For the 'good' Boussinesq equation f and g are of the form

$$\begin{aligned} f(x,t) &= e^{m_1(x-m_1t)} + e^{m_2(x-m_2t)}, \\ g(x,t) &= e^{n_1(x-n_1t)} + \varepsilon e^{n_2(x-n_2t)}, \end{aligned} \tag{2.1}$$

where m_1, m_2 are two solutions to the equation $\mu^3 - \frac{3}{4}\mu = \lambda_j$, $|\lambda_j| \leqslant \frac{1}{4}$ and $\varepsilon = -1$ corresponds to waves traveling in the same direction while $\varepsilon = 1$ corresponds to waves traveling in opposite directions.

Next by using the Wronskian technique and Hirota's log substitution one can get 2-soliton solutions of KdV and gB equations in the form,

$$w = 2(\ln W)_{xx} = 2\frac{\partial}{\partial x}\left(\frac{W_x}{W}\right).$$

We introduce the notations $c_i(x,t) = \cosh(p_ix - T_i(t))$, $s_i(x,t) = \sinh(p_ix - T_i(t))$, and $t_i = \frac{s_i}{c_i}$ for $i = 1, 2$. The x derivatives of c_i, s_i are $c_{i,x} = p_is_i$, $s_{i,x} = p_ic_i$, and $|t_i| < 1$. We also define the KdV type Wronskian determinants

$$K = \begin{vmatrix} c_1 & s_2 \\ c_{1,x} & s_{2,x} \end{vmatrix} = \begin{vmatrix} c_1 & s_2 \\ p_1 s_1 & p_2 c_2 \end{vmatrix}$$
$$= p_2 c_1 c_2 - p_1 s_1 s_2 = c_1 c_2 (p_2 - p_1 t_1 t_2).$$

In the case of unidirectional solitons $p_2 > p_1 > 0$ and hence $K > 0$. The next result is an extension of results from [2] for characterization of $k = 2(\ln K)_{xx}$.

Lemma 2.1. *If* $\psi_1 = \frac{s_2}{K}$ *and* $\psi_2 = \frac{c_1}{K}$ *then*

$$k = 2(\ln K)_{xx} = 2p_1^2(p_2^2 - p_1^2)\psi_1^2 + 2p_2^2(p_2^2 - p_1^2)\psi_2^2$$

and ψ_i *are solutions of the eigenvalue problem* $\psi_{xx} = (\lambda^2 - k)\psi$ *for the eigenvalues* p_1 *and* p_2 *correspondingly.*

Proof. By the Hirota's substitution with $K_x = (p_2^2 - p_1^2)c_1 s_2$ and $K_{xx} = (p_2^2 - p_1^2)(p_1 s_1 s_2 + p_2 c_1 c_2)$ we calculate directly that

$$k = 2(\ln K)_{xx} = 2\frac{K_{xx}K - (K_x)^2}{K^2} = 2p_1^2(p_2^2 - p_1^2)\psi_1^2 + 2p_2^2(p_2^2 - p_1^2)\psi_2^2.$$

Next, one can differentiate $\psi_1 = \frac{s_2}{K}$ twice with respect to x and get that

$$\psi_{1,xx} = \left(\frac{s_{2,x}K - s_2 K_x}{K^2}\right)_x = \frac{s_{2,xx}K - s_2 K_{xx}}{K^2} - \frac{2}{K}\frac{(s_{2,x}K - s_2 K_x)K_x}{K^2}$$
$$= -2\frac{K_{xx}K - (K_x)^2}{K^2}\frac{s_2}{K} + \frac{p_2^2 s_2 K - s_2 K_{xx} - 2p_2 c_2 K_x}{K^2}.$$

Using the expressions for K_x and K_{xx} we get the identity

$$p_2^2 s_2 K - s_2 K_{xx} - 2p_2 c_2 K_x = p_2^2 s_2 K - s_2(p_2^2 - p_1^2)(p_1 s_1 s_2 + p_2 c_1 c_2)$$
$$- 2p_2 c_2 (p_2^2 - p_1^2) c_1 s_2 = p_1^2 s_2 K$$

and the result for ψ_1 follows. Similarly, ψ_2 is an eigenfunction corresponding to p_2. □

For $T_i(t) = 4p_i^3 t$ we obtain the 2-soliton solution for KdV and the decomposition agrees with the one associated to IST, [9]. The result in lemma 2.1 holds for all functions T_i and we consider the potential function k as a weighted energy distribution of particles in the two states ψ_i with no interactions. There is no immediate PDE that can be associated to k for any choice of T_i and we will refer to k as KdV type potential. Ant advantage of the KdV type potentials is that they are well defined for solitons traveling in opposite directions.

3. 2-Soliton solutions for the 'good' Boussinesq Equation

In this section we derive decomposition of the 2-soliton solution for gB. The Wronskian determinant $V = \begin{vmatrix} f & g \\ f_x & g_x \end{vmatrix}$ where f and g were defined in (2.1), generates a 2-soliton solution $2b = (\ln V)_{xx}$ for gB. The functions f and g are solutions of the following ODEs

$$\begin{aligned} f_{xx}-(m_1+m_2)f_x+m_1m_2f=0,\\ g_{xx}-(n_1+n_2)g_x+n_1n_2g=0, \end{aligned} \tag{3.1}$$

with characteristic polynomials $P_1(\lambda)=(\lambda-m_1)(\lambda-m_2)$ and $P_2(\lambda)=(\lambda-n_1)(\lambda-n_2)$. Let $\theta_1(x,t)=-(m_1+m_2)\frac{x}{2}+(m_1^2+m_2^2)\frac{t}{2}$, $\theta_2(x,t)=-(n_1+n_2)\frac{x}{2}+(n_1^2+n_2^2)\frac{t}{2}$, $D=n_1+n_2-m_1-m_2$, and $L=m_1m_2-n_1n_2$.

Theorem 3.1. *For functions f and g, defined in* (2.1)*, a two soliton solution $b=2(\ln V)_{xx}$ for the gB has the following representation*

$$b=8p_1^2e^{-2\theta_1}\left(Dg_x+Lg\right)\frac{g}{V^2}-8\varepsilon p_2^2e^{-2\theta_2}\left(Df_x+Lf\right)\frac{f}{V^2}. \tag{3.2}$$

The functions $\phi_1=\frac{g}{V}$ and $\phi_2=\frac{f}{V}$ are two different solutions to the wave equation $\phi_t=\phi_{xx}+v\phi$ and

$$b=8p_1^2e^{-2\theta_1}\left(D\frac{g_x}{g}+L\right)\phi_1^2-8\varepsilon p_2^2e^{-2\theta_2}\left(D\frac{f_x}{f}+L\right)\phi_2^2. \tag{3.3}$$

The functions $\Phi_i=e^{-\theta_i}\phi_i$ are solutions of the transport-eigenvalue problem $\Phi_t+2R\Phi_x=\Phi_{xx}+\left(-\lambda^2+v\right)\Phi$ with $R=M$ and $\lambda=p_1$ for Φ_1, and $R=\frac{n_1+n_2}{2}$ and $\lambda=p_2$ for Φ_2 and

$$b=8p_1^2\left(D\frac{g_x}{g}+L\right)\Phi_1^2-8\varepsilon p_2^2\left(D\frac{f_x}{f}+L\right)\Phi_2^2. \tag{3.4}$$

Proof. To compute $(\ln V)_x$ we use a column-wise differentiation of determinants

$$(\ln V)_x=\frac{V_x}{V}=\frac{\begin{vmatrix} f_x & g\\ f_{xx} & g_x\end{vmatrix}+\begin{vmatrix} f & g_x\\ f_x & g_{xx}\end{vmatrix}}{V}=\frac{W(f_x,g)+W(f,g_x)}{V}.$$

Denoting $\nu_1=\frac{W(f_x,g)}{W(f,g)}$ and $\nu_2=\frac{W(f,g_x)}{W(f,g)}$ we see that $b=2\nu_{1,x}+2\nu_{2,x}$. Differentiating ν_1 row-wise with respect to x we get

$$\begin{aligned}\nu_{1,x}&=\frac{\begin{vmatrix} f_x & g\\ f_{xxx} & g_{xx}\end{vmatrix}\begin{vmatrix} f & g\\ f_x & g_x\end{vmatrix}-\begin{vmatrix} f_x & g\\ f_{xx} & g_x\end{vmatrix}\begin{vmatrix} f & g\\ f_{xx} & g_{xx}\end{vmatrix}}{W^2(f,g)}\\ &=\frac{g}{W^2}\begin{vmatrix} f & f_x & g\\ f_x & f_{xx} & g_x\\ f_{xx} & f_{xxx} & g_{xx}\end{vmatrix}.\end{aligned}$$

The last identity can be verified by using the Sylvester's method to expand $W(f,f_x,g)$ along the second and third rows and first and second columns.

Since f and f_x satisfy (3.1), multiplying the first row by m_1m_2 and the second by $-(m_1+m_2)$ and adding them to the last we get that

$$W(f,f_x,g) = \begin{vmatrix} f & f_x & g \\ f_x & f_{xx} & g_x \\ 0 & 0 & g_{xx}-(m_1+m_2)g_x+m_1m_2g \end{vmatrix}.$$

Since, $\begin{vmatrix} f & f_x \\ f_x & f_{xx} \end{vmatrix} = (m_1-m_2)^2e^{-2\theta_1} = 4p_1^2e^{-2\theta_1}$ and $g_{xx}-(m_1+m_2)g_x+m_1m_2g = (n_1+n_2-m_1-m_2)g_x+(m_1m_2-n_1n_2)g = Dg_x+Lg$ then we obtain the the representation for $2\nu_{1,x}$. For the second term the determinant $\begin{vmatrix} g & g_x \\ g_x & g_{xx} \end{vmatrix} = \varepsilon(n_1-n_2)^2e^{-2\theta_2} = 4\varepsilon p_2^2e^{-2\theta_2}$ and $f_{xx}-(n_1+n_2)f_x+n_1n_2f = -(n_1+n_2-m_1-m_2)f_x-(m_1m_2-n_1n_2)f = -(Df_x+Lf)$ and thus (3.2) follows.

Next we show that the functions $\phi_1 = \frac{g}{V}$ and $\phi_2 = \frac{f}{V}$ are solutions of the wave equation. From $v = 2\frac{VV_{xx}-(V_x)^2}{V^2}$ and $V_x = fg_{xx}-f_{xx}g$ it follows that

$$\begin{aligned}\phi_{1,xx} = \left(\frac{g}{V}\right)_{xx} &= \frac{g_{xx}V-gV_{xx}}{V^2} - 2\frac{V_x}{V}\frac{g_xV-gV_x}{V^2}, = -2\frac{g}{V}\frac{VV_{xx}-V_x^2}{V^2} \\ &+ \frac{gV_{xx}-2g_xV_x+g_{xx}V}{V^2} = -v\phi_1 + \frac{gV_{xx}-2g_xV_x+g_{xx}V}{V^2}.\end{aligned}$$

Since $f_t = -f_{xx}$ and $g_t = -g_{xx}$ we have that

$$\phi_{1,t} = \frac{g_tV-gV_t}{V^2} = -\frac{g_{xx}V+gV_t}{V^2},$$

and hence

$$\begin{aligned} V_x &= -fg_t+gf_t, \\ V_{xx} &= -f_xg_t-fg_{xt}+g_xf_t+gf_{xt}, \\ V_t &= f_tg_x+fg_{xt}-gf_{xt}-f_xg_t. \end{aligned}$$

Finally, from the relation

$$\begin{aligned} &g(V_{xx}+V_t)-2g_xV_x+2g_{xx}V \\ &\qquad = g(2f_tg_x-2f_xg_t)-2g_x(f_tg-fg_t)-2g_t(fg_x-f_xg) = 0 \end{aligned}$$

we get the identity for ϕ_1. Similarly we can obtain the result for ϕ_2 and (3.3).

By using the relation $\phi_i = e^{\theta_i}\Phi_i$ and substituting in the wave equation we see that Φ_i are solutions to the corresponding transport-eigenvalue problems and b solves (3.4). □

The decomposition for b holds in the case of solitons traveling in the same or opposite directions. The function g has a zero in the case of opposite directions but Φ_2^2 has a double zero at the same location.

The two components of k from lemma 2.1 could be considered as two energy states of the eigenvalue problem with potential k, the steady state, ψ_2 (has no zeros) and the excited state, ψ_1 (has one simple zero). For gB we obtained two representations, (3.3) with components being solutions either to the wave equation with a potential b but with damping factors $e^{-2\theta_i}$ or in case of (3.4) the transport-eigenvalue problem, with potential b. Next we compare the potential functions k and b

and the corresponding components for the 2-soliton solutions corresponding to gB and KdV type potentials. For $t \to \pm\infty$ both Φ_i and ψ_i, up to a constant shift approach sech_i^2. This can be interpreted as two 'particles' being in a time-varying steady state. At the moment of interaction one remains in a steady state, ψ_2, while the other transition into an excited state, ψ_1.

Let $u = \frac{1}{4}\log\left|\frac{(n_1-m_2)(n_2-m_2)}{(n_2-m_1)(n_1-m_1)}\right|$, $v = \frac{1}{4}\log\left|\frac{(n_2-m_1)(n_2-m_2)}{(n_1-m_1)(n_1-m_2)}\right|$ and define $s_1^*(x,t) = s_1(x-u/p_1,t)$, $s_2^*(x,t) = s_2(x-v/p_2,t)$. Let $P_{1i} = P_1(n_i)$ and $P_{2i} = P_2(m_i)$, $i = 1, 2$. Then we have the following

Theorem 3.2. *Let $\sigma = \sqrt{P_{11}P_{12}}$, $b_2 = \frac{p_2^2}{2}(\sqrt{|P_{22}|} - \sqrt{|P_{21}|})^2$ and in case of chasing solitons, $\varepsilon = 1$, $b_1 = b_1^c = \frac{p_1^2}{2}(\sqrt{P_{12}} - \sqrt{P_{11}})^2$ and in case of solitons traveling in opposite directions, $\varepsilon = -1$, $b_1 = b_1^o = -\frac{p_1^2}{2}(\sqrt{P_{12}} + \sqrt{P_{11}})^2$ then*

$$b = 2\sigma p_1^2\left(\frac{s_2^*}{B}\right)^2 + 2\sigma p_2^2\left(\frac{c_1^*}{B}\right)^2 + (b_2 - b_1)\frac{1}{B^2}. \tag{3.5}$$

Proof. First we notice that in the Hirota substitution the solution for gB can be generated by using the following determinant

$$B = \frac{e^{-(\theta_1+\theta_2)}}{4}V = \begin{vmatrix} c_1 & \frac{1-\varepsilon}{2}s_2 + \frac{1+\varepsilon}{2}c_2 \\ p_1s_1 + Mc_1 & \frac{n_1+\varepsilon n_2}{2}c_2 + \frac{n_1-\varepsilon n_2}{2}s_2 \end{vmatrix}.$$

instead of using V. Indeed, since θ_1 and θ_2 are linear in x we have that

$$2(\log B)_{xx} = 2(\log e^{-(\theta_1+\theta_2)}V)_{xx} = 2(\log V)_{xx} = b.$$

From the identity $f(x,t) = e^{m_1x-m_1^2t} + e^{m_2x-m_2^2t} = 2e^{\theta_1}\cosh p_1(x-2Mt) = 2e^{-\theta_1}c_1$ after differentiating we get that

$$\begin{aligned} f_x(x,t) &= m_1e^{m_1x-m_1^2t} + m_2e^{m_2x-m_2^2t} \\ &= e^{-\theta_1(x,t)}(m_1e^{p_1(x-2Mt)} + m_2e^{-p_1(x-2Mt)}) \\ &= e^{-\theta_1}(m_1(c_1+s_1) + m_2(c_1-s_1)) = 2e^{-\theta_1}(p_1s_1 + Mc_1). \end{aligned}$$

Since $D = n_1 + n_2 - m_1 - m_2$ and $L = m_1m_2 - n_1n_2$ the identities $2Dp_1 = -P_{22} - P_{21}$ and $2(DM + L) = P_{22} - P_{21}$ hold. The second term of b, in (3.1), in both cases is

$$\begin{aligned} &8p_2^2e^{-2\theta_2}(Df_x + Lf)\frac{f}{V^2} \\ &= 32p_2^2e^{-2(\theta_1+\theta_2)}(D(p_1s_1 + Mc_1) + Lc_1)\frac{c_1}{(4e^{-(\theta_1+\theta_2)}B)^2} \\ &= (-(P_2(m_2) + P_2(m_1))c_1^2 + (P_2(m_2) - P_2(m_1)s_1c_1)\frac{p_2^2}{B^2} \\ &= (-(P_{22} + P_{21})c_1^2 + (P_{22} - P_{21})c_1s_1)\frac{p_2^2}{B^2}. \end{aligned}$$

To evaluate the first term of (3.1) we consider separately the cases for ε. For $\varepsilon = -1$ we have $g = e^{n_1x-n_1^2t} - e^{n_2x-n_2^2t} = 2e^{-\theta_2}\sinh p_2(x-Nt) = 2e^{\theta_2}s_2$ with x derivative $g_x = n_1e^{n_1x-n_1^2t} - n_2e^{n_2x-n_2^2t} =$

$2e^{-\theta_2}(p_2c_2+Ns_2)$ and similarly to the case for f the first term of b in (3.1) is

$$8p_1^2e^{-2\theta_1}\,(Dg_x+Lg)\,\frac{g}{V^2}=((P_{11}+P_{12})s_2^2+(P_{11}-P_{12})s_2c_2)\frac{p_1^2}{B^2}.$$

Summing up we get that the two-soliton solution for gB for $\varepsilon=-1$ is

$$b=((P_{11}+P_{12})s_2^2+(P_{11}-P_{12})s_2c_2)\frac{p_1^2}{B^2}+(-(P_{22}+P_{21})c_1^2+(P_{22}-P_{21})c_1s_1)\frac{p_2^2}{B^2}.$$

In the case of solitons traveling in opposite directions $\varepsilon=1$ and $g=e^{n_1x-n_1^2t}+e^{n_2x-n_2^2t}=2e^{-\theta_2}\cosh p_2(x-Nt)=2e^{-\theta_2}s_2$ with x derivative $g_x=n_1e^{n_1x-n_1^2t}+n_2e^{n_2x-n_2^2t}=2e^{-\theta_2}(p_2s_2+Nc_2)$. In the same fashion as for f we have that the first term of b in (3.1) is

$$8p_1^2e^{-2\theta_1}\,(Dg_x+Lg)\,\frac{g}{V^2}=((P_{11}+P_{12})c_2^2+(P_{11}-P_{12})s_2c_2)\frac{p_1^2}{B^2},$$

and summing up we get that the two-soliton solution for gB and $\varepsilon=1$ is

$$b=((P_{11}+P_{12})c_2^2+(P_{11}-P_{12})s_2c_2)\frac{p_1^2}{B^2}+(-(P_{22}+P_{21})c_1^2+(P_{22}-P_{21})c_1s_1)\frac{p_2^2}{B^2}.$$

From the choice of u, v we get that

$$\sinh v=\frac{1}{2}\left(\left(\frac{P_{12}}{P_{11}}\right)^{\frac{1}{4}}-\left(\frac{P_{12}}{P_{11}}\right)^{-\frac{1}{4}}\right)=\frac{1}{2\sqrt{\sigma}}\left(\sqrt{P_{12}}-\sqrt{P_{11}}\right),$$

$\cosh v=\frac{1}{2\sqrt{\sigma}}(\sqrt{P_{12}}+\sqrt{P_{11}})$, and hence

$$\begin{aligned}(s_2^*)^2&=(c_vs_2-s_vc_2)^2=(c_v^2+s_v^2)s_2^2-2c_vs_vs_2c_2+s_v^2\\&=\frac{1}{2\sigma}\left((P_{11}+P_{12})s_2^2-(P_{12}-P_{11})c_2s_2+\frac{1}{2}(\sqrt{P_{12}}-\sqrt{P_{11}})^2\right).\end{aligned}$$

By solving for the first term in the parenthesis

$$\begin{aligned}&\frac{p_1^2}{B^2}\left((P_{11}+P_{12})s_2^2-(P_{12}-P_{11})c_2s_2\right)\\&=\frac{1}{B^2}\left(2p_1^2\sigma(s_2^*)^2-\frac{p_1^2}{2}(\sqrt{P_{12}}-\sqrt{P_{11}})^2\right)=\frac{2\sigma p_1^2}{B^2}(s_2^*)^2-\frac{b_1^c}{B^2}.\end{aligned}$$

Similarly by expressing the terms depending on u and taking into and account that $\varepsilon=-1$ and P_{12} and P_{22} are negative we obtain

$$\frac{p_2^2}{B^2}\left(-(P_{11}+P_{12})c_1^2+(P_{22}-P_{21})c_1s_1\right)=\frac{2\sigma p_2^2}{B^2}(c_1^*)^2+\frac{b_2^c}{B^2}.$$

By adding up the last two identities we get the formula in case of chasing solitons. The derivation for solitons traveling in opposite directions is similar. □

Next we show that the 2-soliton solution for gB obtained in theorem 3.1 is always positive.

Lemma 3.1. *For any x and t the 2-soliton solution b is positive.*

Proof. In the case of colliding solitons the coefficients b_1 and b_2 are always positive so we need to consider only the case of chasing solitons. In this case $n_1 < m_1 < m_2 < n_2$ and hence $P_{11},\ P_{12}$ are positive while $P_{21},\ P_{22}$ are negative. Since $c_1^* \geqslant 1$ we have that

$$B^2 v \geqslant 2\sigma p_2^2 - b_1^2 + b_2^c = 2\sigma p_2^2 - \frac{p_1^2}{2}(\sqrt{P_{12}} - \sqrt{P_{11}})^2 + \frac{p_2^2}{2}(\sqrt{|P_{22}|} - \sqrt{|P_{21}|})^2$$
$$= (p_2^2 - p_1^2)\sigma - \frac{p_1^2}{2}(P_{11} + P_{12}) + \frac{p_2^2}{2}(-P_{21} - P_{22}).$$

Furthermore since $p_2 > |p_1|$ and $\sigma > 0$ we need to show that $-\frac{p_1^2}{2}(P_{11} + P_{12}) + \frac{p_2^2}{2}(-P_{21} - P_{22})$ is positive. The m's and n's are solution of the equation $\mu^3 - \frac{3}{4}\mu = \lambda_j, |\lambda_j| < \frac{1}{4}$ which roots can be expressed in trigonometric form for $0 < \alpha = \frac{1}{3}\arccos(4\lambda_j) \leqslant \frac{\pi}{3}$

$$m_1 = \cos(\alpha),\ \ m_2 = \cos(\alpha + \frac{4\pi}{3}),\ \ m_3 = \cos(\alpha + \frac{2\pi}{3}). \tag{3.6}$$

Thus $-1 \leqslant m_3 \leqslant -\frac{1}{2}$. For $\alpha < \beta \leqslant \frac{\pi}{3}$ we have $n_1 = \cos(\beta) > m_1$ and n_2 is either $\cos(\beta + \frac{2\pi}{3})$ or $\cos(\beta + \frac{4\pi}{3})$. Let assume first that $n_2 = \cos(\beta + \frac{4\pi}{3}), n_3 = \cos(\beta + \frac{2\pi}{3})$ then from Vieta's formulas we get that

$$\begin{aligned} m_3 = -(m_1 + m_2),\ \ m_1 m_2 = m_3^2 - \frac{3}{4},\ \ m_1^2 + m_2^2 = \frac{3}{2} - m_3^2, \\ n_3 = -(n_1 + n_2), n_1 n_2 = n_3^2 - \frac{3}{4}, n_1^2 + n_2^2 = \frac{3}{2} - n_3^2, \\ 4p_1^2 = (m1 - m2)^2 = 3(1 - m_3^2), \\ 4p_2^2 = (n_1 - n_2)^2 = 3(1 - n_3^2), \\ P_{11} + P_{12} = 2m_3^2 - m_3 n_3 - n_3^2, \\ P_{21} + P_{22} = 2n_3^2 - m_3 n_3 - m_3^2. \end{aligned}$$

By direct evaluation we get that

$$-\frac{p_1^2}{2}(P_{11} + P_{12}) + \frac{p_2^2}{2}(-P_{21} - P_{22}) = \frac{3}{4}(m_3 - n_3)^2(2m_3^2 + 3m_3 n_3 + 2n_3^2 - 1) = \frac{3}{4}U(n_3),$$

where $U(\xi) = 2m_3^2 + 3m_3\xi + 2\xi^2 - 1$. The linear function $U'(\xi) = 4\xi + 3m_3$ is negative for any $\xi < m_2$. Indeed, $U'(\xi) \leqslant U'(m_2) = 4\cos(\alpha + \frac{4\pi}{3}) + 3\cos(\alpha + \frac{2\pi}{3}) = \frac{\sqrt{3}\cos\alpha}{2}(\tan\alpha - \frac{7}{\sqrt{3}}) < 0$ since $0 \leqslant \alpha \leqslant \frac{\pi}{3}$. On the other hand from the Vieta's formulas it follows that $U(m_2) = m_1^2 - \frac{1}{4} \geqslant 0$, and hence $U(\xi) > 0$ for any $\xi < m_2$. Both of the choices for n_3, $\cos(\beta + \frac{4\pi}{3})$ and $\cos(\beta + \frac{2\pi}{3})$ are less than m_2 the statement in the lemma is established. □

In the next section we obtain necessary condition on the distribution of m's and n's such that the determinant B is nonzero. We also the properties of $\frac{1}{B^2}$.

4. Analysis of the parameters of the 2-Soliton Solution for gB

In order to have non-singular solutions it is necessary to require that the Wronskian determinant $B(x,t) > 0$ for any real x and t. Initially, we consider separately the cases for ε, starting with chasing solitons i.e. $\varepsilon = -1$,

$$B = Bc = \begin{vmatrix} c_1 & s_2 \\ p_1 s_1 + Mc_1 & p_2 c_2 + N s_2 \end{vmatrix} = c_1 c_2 (p_2 + (N - M)t_2 - p_1 t_1 t_2) = c_1 c_2 Tc(t_1, t_2),$$

where

$$Tc(t_1, t_2) = p_2 + (N - M)t_2 - p_1 t_1 t_2.$$

is a bi-linear function in t_1, t_2. Since t_1 and t_2 are hyperbolic tangents in terms of x and t it is clear that the values of the pair (t_1, t_2) are in the square with vertices

$$E = \{(-1, -1), (-1, 1), (1, -1), (1, 1)\}$$

in the t_1, t_2 plane. The bi-linear function Tc attains its maximum and minimum at the elements of E. Considering the values at the vertices we obtain sufficient conditions for the distributions of the n's and m's. Direct computations lead to

$$\begin{gathered} Tc(-1, -1) = m_2 - n_2, Tc(-1, 1) = n_1 - m_2, \\ Tc(1, -1) = m_1 - n_2, Tc(1, 1) = n_1 - m_1, \end{gathered}$$

and hence if

$$\min(n_1, n_2) < m_1, m_2 < \max(n_1, n_2) \tag{4.1}$$

then all of the values above are positive and since c_1, c_2 are always positive we get that B is positive for any x, t.

In the case of solitons traveling in opposite directions $\varepsilon = 1$ and

$$B = Bo = \begin{vmatrix} c_1 & c_2 \\ p_1 s_1 + Mc_1 & p_2 s_2 + N c_2 \end{vmatrix} = c_1 c_2 (N - M - p_1 t_1 + p_2 t_2) = c_1 c_2 To(t_1, t_2)$$

where the bi-linear function is

$$To(t_1, t_2) = N - M - p_1 t_1 + p_2 t_2.$$

Again, direct computations lead to

$$\begin{aligned} To(-1, -1) &= n_2 - m_2, \qquad & To(-1, 1) &= n_1 - m_2, \\ To(1, -1) &= n_2 - m_1, \qquad & To(1, 1) &= n_1 - m_1, \end{aligned}$$

and hence if

$$\max(m_1, m_2) < \min(n_1, n_2) \tag{4.2}$$

then all of the values are positive and since c_1, c_2 are always positive we get that B is positive for any x, t.

The results from the previous section suggest that b is closely related to the KdV type potentials k, they both are expressed in terms of c_1, c_2. Similarly to the case of one soliton from the introduction we consider the spatial shifts u, v defined in theorem 3.2 in the two seed solitons,

$$\begin{aligned}
K(x,t,u,v) &= \begin{vmatrix} c_1(x-u/p_1,t) & s_2(x-v/p_2,t) \\ p_1 s_1(x-u/p_1,t) & p_2 c_2(x-v/p_2,t) \end{vmatrix} \\
&= \begin{vmatrix} c_u c_1 - s_u s_1 & c_v s_2 - s_v c_2) \\ p_1(c_u s_1 - s_u c_1) & p_2(c_v c_2 - s_v s_2) \end{vmatrix} \\
&= c_1 c_2 \begin{vmatrix} c_u - s_u t_1 & c_v t_2 - s_v) \\ p_1(c_u t_1 - s_u) & p_2(c_v - s_v t_2) \end{vmatrix},
\end{aligned}$$

where as above $s_v = \sinh v$, $s_u = \sinh u$ and c_c, c_u are the hyperbolic cosines. We associate a bi-linear function in the variables t_1, t_2

$$T(t_1,t_2,u,v) = \begin{vmatrix} c_u - s_u t_1 & c_v t_2 - s_v) \\ p_1(c_u t_1 - s_u) & p_2(c_v - s_v t_2) \end{vmatrix}.$$

The values of T at the vertices of E are

$$\begin{aligned}
T(-1,-1,u,v) &= (p_2 - p_1)e^{u+v}, \qquad T(-1,1,u,v) = (p_2 + p_1)e^{u-v}, \\
T(1,-1,u,v) &= (p_2 + p_1)e^{-u+v}, \qquad T(1,1,u,v) = (p_2 - p_1)e^{-u-v}.
\end{aligned}$$

Since $p_1 = \frac{m_1 - m_2}{2}$, $p_2 = \frac{n_1 - n_2}{2}$ in the case of chasing solitons, (4.1), we have $p_2 > p_1 > 0$ and hence T is positive for any choice of the parameters u and v. In the case of solitons traveling in opposite directions, (4.2), $p_1 < 0$ and a necessary condition all values to be positive is $p_2 > |p_1|$. In what follows we assume that indeed $p_2 > |p_1|$.

Next we show that B and K are equivalent in the sense that there exist positive real constants a_1, a_2 such that $a_1 K \leqslant B \leqslant a_2 K$ for any x and t. In such a case we write $B \approx K$.

Lemma 4.1. *Let*

$$\eta_1 = \frac{2\sqrt{|n_2 - m_2||n_1 - m_1|}}{|n_1 - n_2 - m_1 + m_2|}, \quad \eta_2 = \frac{2\sqrt{|n_1 - m_2||n_2 - m_1|}}{|n_1 - n_2 + m_1 - m_2|},$$

then $0 < \theta_m \leqslant \frac{B}{K} \leqslant \Theta_M < \infty$, *or* $B \approx K$, *where* $\theta_m = \min(\eta_1, \eta_2)$ *and* $\Theta_M = \max(\eta_1, \eta_2)$.

Proof. In the case of chasing solitons the positive bi-linear functions Tc and T attain their extreme values at the elements of $E = \{(e_i, e_j)_{i,j=1}^2\}$ and from $\frac{B}{K} = \frac{Tc}{T}$ we get that

$$0 < \frac{\min_E Tc}{\max_E T} \leqslant \frac{B}{K} = \frac{Tc(x,t)}{T(x,t)} \leqslant \frac{\max_E T_b}{\min_E T} < \infty.$$

Direct substitution of u, v show that the ratio $\frac{T_b}{T}$ has only η_1 and η_2 as possible values at E and this determins the choice for θ_m and Θ_M. In the case of solitons traveling in opposite direction we notice that $To(e_i, e_j) = Tc(e_i, e_j)$, i, $j = 1, 2$ and hence we arrive at the same estimate. □

The explicit expressions obtained so far suggest that the main difference between b and k^* is in the ‘interaction’ $\frac{1}{B^2}$. Next we show that the functions $I = \frac{1}{B^2}$ and $J = \frac{1}{K^2}$ indeed exhibit properties of interaction terms.

Lemma 4.2. *For the choices of the parameters as in theorem 3.2, I and J are bell-shaped, $\lim_{x\to\pm\infty} I(x,t) = 0$, $\lim_{x\to\pm\infty} J(x,t) = 0$, with a single maximum tending to zero as $t \to \pm\infty$, and $\frac{1}{K} = W(\psi_1, \psi_2)$.*

Proof. From lemma 3.1 it follows that $2(\ln(B))_{xx} = b > 0$ and hence its anti-derivative $(\ln(B))_x = \frac{B_x}{B}$ is monotone. Furthermore since $B_x(-\infty)B_x(\infty) < 0$ it follows that B_x has exactly one zero on the x-axis or I has a single maximum. For J we have that $J_x = -\frac{2K_x}{K^3} = (p_2^2 - p_1^2)\frac{c_1 s_2}{K^3}$ and has a single maximum when $s_2(x,t) = \sinh(p_2 x - 2p_2 Nt) = 0$ or $x(t) = 2Nt$. Thus $\max_x J(x,t) = \frac{1}{p_2 \cosh^2(2p_1 N - 2p_1 M)t}$ and it tends to zero when $t \to \pm\infty$. Since $B \approx K$ and both are positive, it follows that $\frac{1}{B} \approx \frac{1}{K}$, it follows that the maximum value of I also decreases to 0 when $t \to \pm\infty$. The asymptotic for x is clear from the estimates $I, J \leqslant \frac{1}{c_1 c_2 T^2}$ where T is either Tc or To from lemma 4.1 and in both cases T is positive function uniformly bounded from below. $\square$

For the choice of u, v as above we observe that

$$\left|\frac{\sigma}{p_2^2 - p_1^2}\right| = \eta_1 \eta_2, \tag{4.3}$$

and hence

$$b = 2\sigma p_1^2 \frac{(s_2^*)^2}{B^2} + 2\sigma p2^2 \frac{(c_1^*)^2}{B^2} + \frac{b_2 - b_1}{B^2} = \frac{(K^*)^2}{B^2}\left(\eta_1 \eta_2 k^* + \frac{b_2 - b_1}{(2K^*)^2}\right).$$

Since, $1 \approx \frac{(K^*)^2}{B^2}$ with constants $\frac{1}{\eta_1^2}$ and $\frac{1}{\eta_2^2}$, it follows that $b + \frac{b_1 - b_2}{B^2} \approx k^*$ with constant of equivalency $\frac{\eta_1}{\eta_2}$ and $\frac{\eta_2}{\eta_1}$.

In the next section we consider numerical estimates and examples.

5. Numerical Examples

In this section we derive numerical estimates for dynamics of b and k^* in terms of m_i, n_i. We start by discussing the distribution of the roots of $\mu^3 - \frac{3}{4}\mu = \lambda$, $|\lambda| \leqslant \frac{1}{4}$. In trigonometric form they are

$$\mu_0 = \cos\alpha, \quad \mu_1 = \cos\left(\alpha + \frac{2\pi}{3}\right), \quad \mu_2 = \cos\left(\alpha + \frac{4\pi}{3}\right),$$

for $0 \leqslant \alpha = \frac{1}{3}\arccos(4\lambda) \leqslant \frac{\pi}{3}$ and hence $-1 \leqslant \mu_1 < -\frac{1}{2} < \mu_2 < \frac{1}{2} < \mu_0 \leqslant 1$. If $0 \leqslant \beta < \alpha \leqslant \frac{\pi}{3}$ we have that

$$-1 \leqslant \mu_1(\alpha) < \mu_1(\beta) < \mu_2(\beta) < \mu_2(\alpha) < \mu_0(\alpha) < \mu_0(\beta) \leqslant 1.$$

The parameters m_i, n_i depend on α and β and the η's are nonlinear functions in α, β. When $\alpha = \beta$ the function K^* has always a zero and thus in the numerical examples we consider $\alpha < \beta$. If we further consider $\beta - \alpha > 0.1$ then $\frac{\eta_1}{\eta_2} < \frac{\eta_2}{\eta_1}$ and graphs of the functions (in α, β) are plotted in Fig. 1.

The deviation is bigger for smaller values of both parameters and approaches ∞ when α approaches β.

For the numerical estimates we pick values for β and α.

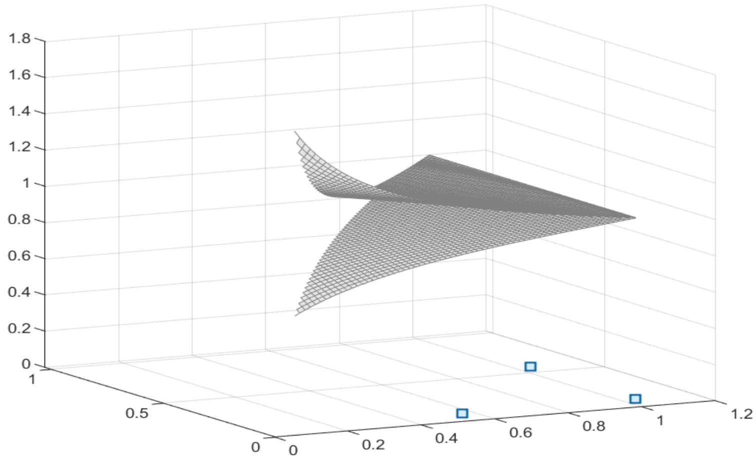

Fig. 1. $\frac{\eta_1}{\eta_2}$ the top and $\frac{\eta_2}{\eta_1}$ the bottom functions as functions of $\alpha > \beta + 0.1$

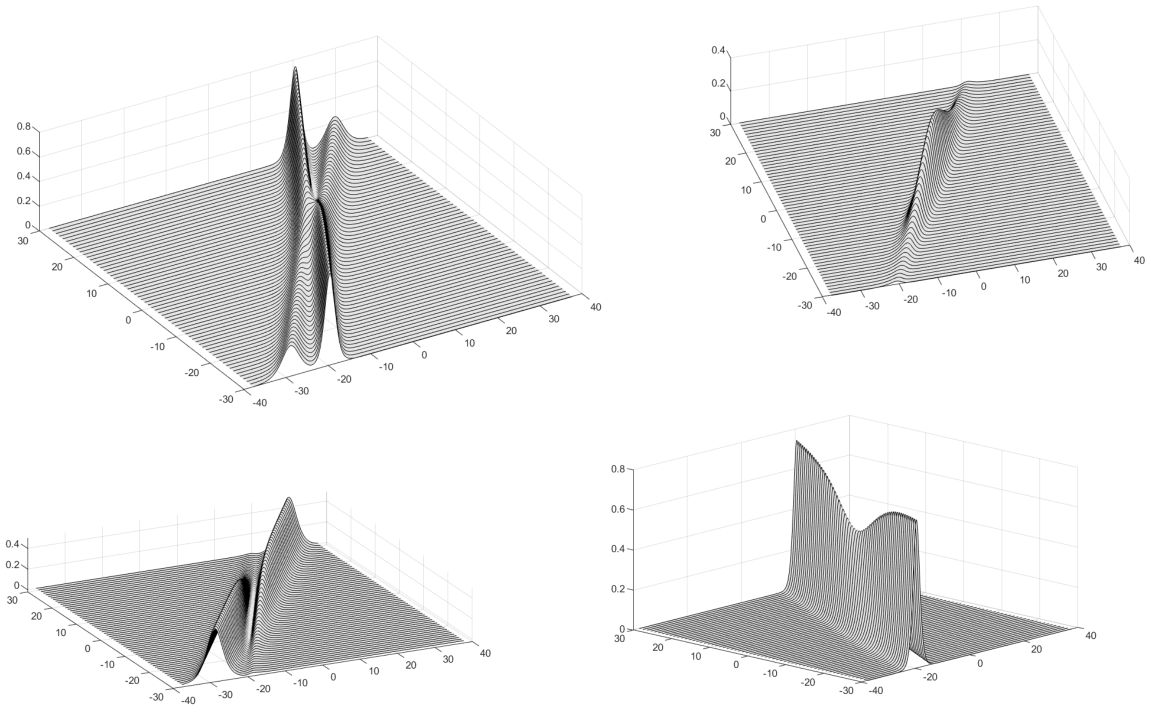

Fig. 2. $\alpha = \frac{\pi}{5}$ and $\beta = \frac{\pi}{12}$ *top left*: superposition of k, lighter cooler, and b, black counter lines; *top right*: superposition of normalized interaction terms; *bottom left*: smaller component of v; *bottom right*: larger component of v.

The case of chasing solitons: We pick $\alpha = \frac{\pi}{5}$ and $\beta = \frac{\pi}{12}$ then

$$n_2 = -0.2588 < m_2 = 0.1045 < m_1 = 0.8090 < n_1 = 0.9659,$$

$p_1 = 0.3522$ and $p_2 = 0.6124$. The shifts are $u = 0.1562$, $v = 0.2636$, and $k = 0.0623\psi_1^2 + 0.1882\psi_2^2$ while $b = 0.0568\phi_1^2 + 0.1718\phi_2^2 + \frac{0.0002}{B^2}$ and $0.9232 \leqslant \frac{K}{B} \leqslant 1.0832$.

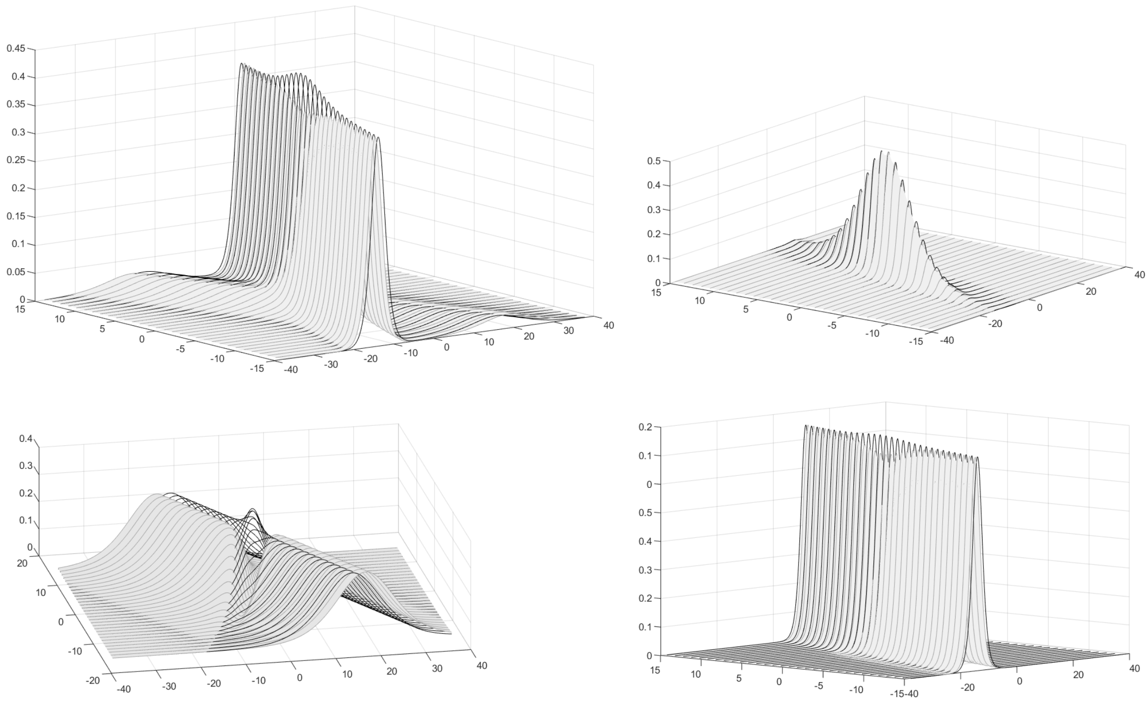

Fig. 3. $\alpha = \frac{\pi}{24}$ and $\beta = \frac{\pi}{6}$ *top left*: superposition of k, lighter cooler, and b, black counter lines; *top right*: superposition of normalized interaction terms; *bottom left*: smaller component of v; *bottom right*: larger component of v.

The case of solitons traveling in opposite directions: We pick $\alpha = \frac{\pi}{24}$ and $\beta = \frac{\pi}{6}$ then

$$m_1 = -0.6088 < m_2 = 0.3827 < n_2 = 0 < n_1 = 0.8660,$$

$p_1 = -0.1130$ and $p_2 = 0.4330$. The shifts are $u = -0.1577$, $v = -0.5169$, and $k = 0.0045\psi_1^2 + 0.0655\psi_2^2$ while $b = 0.0167\phi_1^2 + 0.2456\phi_2^2 + \frac{0.0110}{B^2}$ and $0.7189 \leqslant \frac{K}{B} \leqslant 1.0899$.

The two examples illustrate that the interaction therm $\frac{1}{B^2} \approx \frac{1}{K^2}$ is the significant difference between 2-soliton solution for gB and KdV type potentials. The solitons corresponding to this term are virtual solitons in the context of [8]. Visual inspection of 2-soliton solutions for the Euler's equations, [16] confirmed that the presence of the interaction is relevant.

6. Discussion and Conclusions

In the previous sections we presented decomposition of the KdV type potentials, which include the 2-soliton solutions of KdV, as $k = \gamma_1(\psi_1^*)^2 + \gamma_2(\psi_2^*)^2$ where, γ_i are constants depending only on p_1, p_2. The two functions ψ_i are eigenfunctions of $\psi_{xx} = (\lambda^2 - k^*)\psi$ for eigenvalues p_1^2, p_2^2 and can be considered as two eigenstates, ψ_1^* excited and ψ_2^* steady state. In theorem 3.1, (3.4), we showed that $b = h_1\Phi_1^2 + h_2\Phi_2^2$, where Φ_i are solutions of the transport-eigenvalue equations $\Phi_t + 2R\Phi_x = \Phi_{xx} + (-\lambda^2 + v)\Phi$ and h_1 and h_2 are variable coefficients. In theorem 3.2 we established that $b \approx k^* + \frac{1}{K^2}$. A close examination of the formulas also shows that for the positive components $\phi_i \approx \psi_i$ for $i = 1, 2$. The 2-soliton solution for gB can be considered as an energy transport process of two energy waves with influence on each other via the eigenfunctions ψ_i. For $t \to \pm\infty$ the components Φ_i and ψ_i for $i = 1, 2$ correspondingly are bell-shaped and differ from the single soliton case primarily by a phase shift and the interaction terms tend to 0. From the signs of b_1, b_2 it also can be observed that during the interaction the two waves exchange energy, this might be concluded from the signs of the interaction terms and the fact that Φ_1 has negative values. The dynamics is different for the solitons traveling in the same or opposite directions. In the case of chasing solitons, the two pairs of components are almost identical and the interaction results in loss of energy at the pick of ψ_2^* and energy gain away of it, as also can be seen from Fig. 2. In the case of opposite directions the interactions causes a temporarily appearance of a traveling wave along ψ_1^*, this agrees with the different signs of b_1 and b_2, as also can be seen from Fig. 3.

Another observation that can be made is that ψ_1 away from the interaction is up to a phase shift a single soliton while during the interaction it looks like transition from a steady state to excited state. This also might be in agreement with the fact that after interaction the smaller and slower soliton, ψ_1, exhibits a time delay while the faster and taller, ψ_2 exhibits positive time shift. The horizontal velocity of a particle is approximated by kdv and the slowing of the horizontal velocityof ψ_1 can be explained by the more chaotic movement during the interaction while the acceleration in horizontal direction of ψ_2 might be explained with the fact that the vertical displacement is lesser during interaction while the horizontal is about the same.

An interesting observation is that the results obtained in this work can be extended for gB type potentials $\widetilde{b} = 2(\ln \widetilde{V})_{xx}$. The Wronskian $\widetilde{V} = \begin{vmatrix} \widetilde{f} & \widetilde{g} \\ \widetilde{f}_x & \widetilde{g}_x \end{vmatrix}$ is generated by more general seed solutions

$$\widetilde{f}(x,t) = e^{a_{1,1}x - b_{1,1}t} + \varepsilon_1 e^{a_{1,2}x - b_{1,2}t},$$
$$\widetilde{g}(x,t) = e^{a_{2,1}x - b_{2,1}t} + \varepsilon_2 e^{a_{2,2}x - b_{2,2}t},$$

for arbitrary parameters $(a_{i,j}, b_{i,j})$, ε_j, i, $j = 1, 2$. Straight substitution and computations lead to similar decomposition as in theorem 3.1. The main difference is that the corresponding modes $\frac{\widetilde{f}}{\widetilde{V}}$ and $\frac{\widetilde{g}}{\widetilde{V}}$ are solutions to different type equations. This observation can be used to investigate 2-soliton solution for SWW equations with third order spatial eigenvalues, CH, DP, or N, equations for example.

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