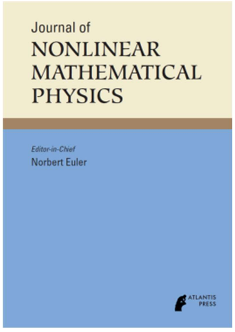

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Integrability conditions of a weak saddle in generalized Liénard-like complex polynomial differential systems

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We consider the complex differential system

$$\dot{x} = x + yf(x), \quad \dot{y} = -y + xf(y),$$

where f is the analytic function $f(z) = \sum_{j=1}^{\infty} a_j z^j$ with $a_j \in \mathbb{C}$. This system has a weak saddle at the origin and is a generalization of complex Liénard systems. In this work we study its local analytic integrability.

Keywords: Integrability problem, weak saddle, Liénard-like complex polynomial differential systems.

2010 Mathematics Subject Classification: 34A05, 34C05, 37C10.

1. Introduction and statement of the main results

The center problem for polynomial vector fields in the real plane with an elementary singular point of the form

$$\dot{x} = -y + \text{h.o.t.}, \quad \dot{y} = x + \text{h.o.t.}$$

(where h.o.t. means higher order terms) has been the subject of many investigations during these last decades, see for instance [4–6, 10, 11, 14, 16] and the references therein. These type of systems can be embedded by the complex change of variables $u = x + iy$ and $v = \overline{u} = x - iy$ into the complex system

$$\dot{u} = u + \text{h.o.t.}, \quad \dot{v} = -u + \text{h.o.t.}$$

The next extension of the above system is to consider analytic vector fields in $\mathbb{C}^2$ of the form

$$\dot{u} = \lambda u + \text{h.o.t.}, \quad \dot{v} = -\mu v + \text{h.o.t.} \tag{1.1}$$

where $\lambda, \mu \in \mathbb{C} \setminus \{0\}$. It was proved by Poincaré (see [2, 15]) that if $\lambda/\mu \notin \mathbb{Q}^+$, then system (1.1) has no local analytic first integrals in a neighborhood of the origin. We recall that system (1.1) has a

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local analytic first integral in a neighborhood of the origin if there exists $H\colon U \subset \mathbb{C}^2 \to \mathbb{C}$ analytic and U a neighborhood of the origin, such that H is constant along the solutions of system (1.1). However, if $\lambda/\mu = p/q \in \mathbb{Q}^+$ with $\gcd(p,q) = 1$ (called the $[p:q]$ *resonant case*) then adding some extra necessary conditions the analytic integrability is sometimes possible. To obtain these conditions note that changing the time, if necessary, the $[p:q]$ resonant case can be written as

$$\dot{u} = pu + \text{h.o.t.}, \quad \dot{v} = -qv + \text{h.o.t.} \tag{1.2}$$

with p, q positive integers. For this system the linear part has the analytic first integral $H_0(u,v) = u^q v^p$ and we can look for conditions on the existence of a formal first integral of the form

$$H(u,v) = H_0(u,v) + \text{h.o.t.}$$

for system (1.2). Doing so, we get the equation

$$\dot{H}(u,v) = \frac{\partial H}{\partial u}\dot{u} + \frac{\partial H}{\partial v}\dot{v} = v_1 H_0^2(u,v) + v_3 H_0^3(u,v) + \text{h.o.t.}$$

and v_i (called the $[p:-q]$ *resonant saddle quantities*) are polynomials of the coefficients of the system in (1.2). If all the resonant saddle quantities v_i are zero we say that we have a *formal analytic resonant saddle* (see for instance [1, 17]) and in this case it follows from [16] that there is a local analytic first integral.

In this paper we give a simple and self-contained proof of the characterization of the local analytic integrability of a complex analytic differential system in $\mathbb{C}^2$ of the form

$$\dot{x} = x + yf(x), \quad \dot{y} = -y + xf(y), \tag{1.3}$$

where f is an analytic function, that is,

$$f(z) = \sum_{j=1}^{\infty} a_j z^j \quad \text{with } a_j \in \mathbb{C}. \tag{1.4}$$

Systems with linear part of the form (1.3) have a weak saddle at the origin and were studied by several authors, see for instance [12, 13] where the nonlinearities only appear in one equation. The integrability of Liénard systems with a weak saddle were considered in [3]. System (1.3) is a generalization of these Liénard systems with a weak saddle at the origin, already started to be studied in [9].

Of course we also obtain the characterization of the C^∞ integrability of system (1.3) because in the proof we will obtain a formal first integral H that can be C^∞ or analytic around the origin. However by results given in [16] the first integral is always analytic. With the method used in the proof we cannot characterize the existence of a less regular first integral, for instance a C^k first integral.

First we consider the case in which $a_1 \neq 0$, and after that when $a_1 = 0$, considering both cases in different theorems.

Theorem 1.1. *Consider system* (1.3) *with* $f = \sum_{j \geqslant n} a_j z^j$ *with* $n \geqslant 1$ *and* $a_n \neq 0$. *If n is odd, system* (1.3) *is locally integrable at the origin if and only if* $a_k = 0$ *for k even, that is, f is odd.*

The proof of Theorem 1.1 is given in section 2.

The case in which $a_1 = 0$ with n even is more involved and we can only solve it completely for each fixed degree of f less than or equal to 8 (remains open the problem for a degree greater than 8 which is outside of our current computing facilities).

Theorem 1.2. *Consider system* (1.3) *with* $f = \sum_{j\geqslant n}^{\infty} a_j z^j$ *with* $n \geqslant 1$. *For each fixed degree of f less than or equal to* 8, *system* (1.3) *is locally integrable at the origin if and only if one of the following conditions holds.*

(i) *n odd and $a_j = 0$ for j even,*
(ii) *$n \geqslant 4$ even and $a_j = 0$ for all j that is not of the form $j = (2i+1)(n-2)+2$ with $i \geqslant 0$.*

Moreover we can give the following sufficient condition of integrability.

Theorem 1.3. *Consider system* (1.3) *with* $f = \sum_{j\geqslant n}^{\infty} a_j z^j$ *with* $n > 1$ *and* $a_n \neq 0$. *If $n \geqslant 4$ is even and $a_j = 0$ for all j that is not of the form $j = (2i+1)(n-2)+2$ with $i \geqslant 0$, then it is locally integrable at the origin.*

The proof of Theorems 1.2 and 1.3 is given in sections 3 and 4, respectively. The results obtained makes us confident to make the following conjecture.

Conjecture 1.1. *System* (1.3) *with* $f = \sum_{j\geqslant n}^{\infty} a_j z^j$ *with* $n > 1$ *and* $a_n \neq 0$ *is locally integrable at the origin if and only if one of the conditions (i) or (ii) holds.*

Note that system (1.3) has a resonant saddle $[1 : -1]$ at the origin. The classical Liénard system is given by

$$\dot{x} = z, \quad \dot{y} = -x - f(x)z.$$

Choosing the variables

$$z = y - F(x) \quad \text{where} \quad F(x) = \int_0^x f(s)\,ds$$

it can be transformed to

$$\dot{x} = y - F(x), \quad \dot{y} = -x.$$

This system has been studied by several authors in the last decades, see [7] and the references therein. Recently in [3] it was given the characterization of the integrable complex analytic differential systems in $\mathbb{C}^2$ of the form

$$\dot{x} = y - F(x), \quad \dot{y} = x.$$

This system has a weak saddle at the origin which corresponds with the $[1 : -1]$ resonance case. Our result is then a generalization of this result because we study the Liénard-like system (1.3) that has also a weak saddle at the origin.

2. Proof of Theorem 1.1

We prove Theorem 1.1 for $n = 1$ and we separate it into the sufficiency and the necessity part. The proof of the general case can be obtained changing a_1 by a_n.

Proof of sufficiency

System (1.3) under the assumptions of Theorem 1.1 takes the form

$$\dot{x} = x + y\sum_{j=1}^{\infty} a_{2j-1}x^{2j-1}, \quad \dot{y} = -y + x\sum_{j=1}^{\infty} a_{2j-1}y^{2j-1}, \tag{2.1}$$

Doing the affine change of variables

$$x = \frac{(1-i)u}{\sqrt{2}} + \frac{(1+i)v}{\sqrt{2}}, \quad y = \frac{(1+i)u}{\sqrt{2}} + \frac{(1-i)v}{\sqrt{2}}, \tag{2.2}$$

system (2.1) takes the form

$$\dot{u} = v + v\mathscr{P}(u,v^2), \qquad \dot{u} = -u + \mathscr{Q}(u,v^2). \tag{2.3}$$

Hence system (2.3) is invariant by the symmetry $(u,v,t) \to (u,-v,-t)$. Taking $z = v^2$ and the scaling of time $dt = v d\tau$ we get a non-singular point at the origin. The first integral which exists around the origin by the Flow-box theorem can be pulled back to a first integral of the form $H(x,y) = xy +$ h.o.t. of the original system. So, sufficiency is proved.

Proof of necessity

Consider $f(z) = \sum_{i=1}^{\infty} a_i z^i$ with $a_1 \neq 0$. As explained in the introduction, to find the saddle quantities we propose a formal first integral of the form

$$H(x,y) = xy + \sum_{k=3}^{\infty} H_k(x,y), \tag{2.4}$$

where $H_k(x,y)$ are homogeneous polynomials that can be written as

$$H_k(x,y) = \sum_{i+j=k} c_{i,j}x^i y^j. \tag{2.5}$$

Now we compute the derivative of H along the vector field associated to system (1.3) and we obtain a linear system for each function H_k. Note that

$$(xy)' = y^2 f(x) + x^2 f(y) = \sum_{k=2}^{\infty} a_k(x^k y^2 + y^k x^2)$$

and

$$c_{i,j}(x^i y^j)' = (i-j)c_{i,j}x^i y^j + ic_{i,j}x^{i-1}y^{j+1}f(x) + jc_{i,j}x^{i+1}y^{j-1}f(y).$$

So, a monomial $R_{l_1,l_2}x^{l_1}y^{l_2}$ can be written as

$$R_{l_1,l_2} = c_{l-1,l_2}(l_1 - l_2) + T_{l_1,l_2}$$

where

(i) $T_{l,0} = T_{0,l} = 0$ for $l \geqslant 1$ and $T_{1,1} = 0$,

(ii) for $l \geqslant 1$,

$$T_{2,l} = \begin{cases} a_1 & \text{if } l = 1, \\ a_2 + 2c_{1,2}a_1 & \text{if } l = 2, \\ a_l + a_2 c_{1,l-2} + 2a_1 c_{2,l-1} + \sum_{j=2}^{l} j c_{1,j} a_{l+1-j} & \text{if } l \geqslant 3, \end{cases}$$

and

$$T_{l,2} = \begin{cases} a_1 & \text{if } l = 1, \\ a_2 + 2c_{1,2}a_1 & \text{if } l = 2, \\ a_l + a_2 c_{l-2,1} + 2a_1 c_{l-1,2} + \sum_{j=2}^{l} j c_{j,1} a_{l+1-j} & \text{if } l \geqslant 3 \end{cases}$$

(we recall that $c_{1,1} = 0$),

(iii) for $l_1 \neq 2$ and $l_2 \neq 2$ with $l_1 + l_2 \geqslant 3$,

$$T_{l_1,l_2} = \sum_{i=1}^{l_1} i c_{i,l_2-1} a_{l_1+1-i} + \sum_{j=1}^{l_2} j c_{l_1-1,j} a_{l_2+1-j},$$

with the convention that if $i = 1$ then $l_2 - 1 \geqslant 2$ and if $j = 1$ then $l_1 - 1 \geqslant 2$.

We first compute H_3. In this case we have the linear system

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} c_{3,0} \\ c_{2,1} \\ c_{1,2} \\ c_{0,3} \end{pmatrix} = \begin{pmatrix} 0 \\ a_1 \\ a_1 \\ 0 \end{pmatrix}. \tag{2.6}$$

From the second and third equations we get

$$c_{2,1} = -c_{1,2} = a_1.$$

The linear system for H_4 is

$$\begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} c_{4,0} \\ c_{3,1} \\ c_{2,2} \\ c_{1,3} \\ c_{0,4} \end{pmatrix} = \begin{pmatrix} 0 \\ c_{2,1}a_1 \\ 2a_2 \\ c_{1,2}a_1 \\ 0 \end{pmatrix} \tag{2.7}$$

Taking into account that $c_{1,2} = -c_{2,1}$ we deduce from the third equation that $a_2 = 0$ and thus obtaining a first condition to have formal integrability of system (1.3). Moreover from the second and third equations we get

$$2c_{3,1} = c_{2,1}a_1 \quad \text{and} \quad -2c_{1,3} = c_{1,2}a_1$$

that is

$$c_{3,1} = \frac{a_1^2}{2} \quad \text{and} \quad c_{1,3} = \frac{a_1^2}{2}.$$

Now we compute the linear system for H_5 and we get

$$\begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5 \end{pmatrix} \begin{pmatrix} c_{5,0} \\ c_{4,1} \\ c_{3,2} \\ c_{2,3} \\ c_{1,4} \\ c_{0,5} \end{pmatrix} = \begin{pmatrix} 0 \\ T_{4,1} \\ T_{3,2} \\ T_{2,3} \\ T_{1,4} \\ 0 \end{pmatrix},$$

where

$$\begin{aligned} T_{4,1} &= c_{3,1}a_1, & T_{1,4} &= c_{1,3}a_1, \\ T_{3,2} &= 3c_{2,1}a_2 + 3c_{3,1}a_1 + c_{2,2}a_2, & T_{2,3} &= 3c_{1,2}a_2 + 3c_{1,3}a_1 + c_{2,2}a_2. \end{aligned}$$

Taking into account that $a_2 = 0$ we get that $T_{2,3} = 3c_{3,1}a_1$ and $T_{3,2} = 3c_{1,3}a_1$. Moreover, since $c_{3,1} = c_{1,3} = \frac{a_1^2}{2}$ we conclude that

$$T_{4,1} = T_{1,4} = \frac{a_1^3}{2} \quad \text{and} \quad T_{2,3} = T_{3,2}.$$

So,

$$c_{5,0} = c_{0,5} = 0, \quad c_{4,1} = -c_{1,4} = \frac{a_1^3}{3!} \quad \text{and} \quad c_{3,2} = -c_{2,3} = T_{3,2}.$$

Now we prove the theorem by induction. Our induction hypothesis will be that for each n, we have

$$c_{n-1,1} = (-1)^n c_{1,n-1} = \frac{a_1^{n-2}}{(n-2)!}, \quad T_{i,j} = (-1)^{i+j+1}T_{j,i} \quad \text{for } i+j = n \geqslant 3$$

and if n is even, then

$$a_2 = a_4 = \cdots = a_n = 0.$$

Note that since for $i \neq j$ we have $c_{i,j} = T_{i,j}/(i-j)$, we can check that

$$c_{i,j} = (-1)^{i+j}c_{j,i} \quad \text{for } i+j = n \geqslant 3$$

(the case in which $i = j$ is trivially satisfied). In fact, until now we have proven the induction hypothesis for $n = 2$ and $n = 3$.

First we observe that it follows from the induction hypotheses that

$$T_{i,j} = (-1)^{i+j+1} T_{j,i} \quad \text{for } i+j = n \geqslant 3. \tag{2.8}$$

Indeed, taking the notation

$$j \in \mathscr{A}_l \text{ if } l+1-j \text{ is odd,}$$

by the induction hypotheses (i.e., $a_2 = 0$ and $a_{l+1-j} = 0$ for $l+1-j$ even) and taking into account that in the definition of $T_{i,j}$ given in (i)–(iii) we get that $i+j < n$, we readily obtain:

$$T_{2,l} = \begin{cases} a_1 & \text{if } l = 1, \\ 2c_{1,2}a_1 & \text{if } l = 2, \\ a_l + 2a_1 c_{2,l-1} + \displaystyle\sum_{j=2,\, j\in\mathscr{A}_l}^{l} j c_{1,j} a_{l+1-j} & \text{if } l \geqslant 3. \end{cases}$$

Hence, if l is odd then

$$\begin{aligned} T_{2,l} &= \begin{cases} a_1 & \text{if } l = 1, \\ a_l + 2a_1 c_{2,l-1} + \displaystyle\sum_{j=2,\, j\in\mathscr{A}_l}^{l} j c_{1,j} a_{l+1-j} & \text{if } l \geqslant 3 \text{ odd} \end{cases} \\ &= \begin{cases} a_1 & \text{if } l = 1, \\ a_l + 2a_1 (-1)^{l+1} c_{l-1,2} + \displaystyle\sum_{j=2,\, j\in\mathscr{A}_l}^{l} j (-1)^{j+1} c_{j,1} a_{l+1-j} & \text{if } l \geqslant 3 \text{ odd} \end{cases} \\ &= \begin{cases} a_1 & \text{if } l = 1, \\ a_l + 2a_1 c_{l-1,2} + \displaystyle\sum_{j=2,\, j\in\mathscr{A}_l}^{l} j c_{j,1} a_{l+1-j} & \text{if } l \geqslant 3 \text{ odd} \end{cases} \\ &= T_{l,2} \end{aligned}$$

and if l is even (since by assumptions $a_l = 0$)

$$\begin{aligned} T_{2,l} &= \begin{cases} 2c_{1,2}a_1 & \text{if } l = 2, \\ 2a_1 c_{2,l-1} + \displaystyle\sum_{j=2,\, j\in\mathscr{A}_l}^{l} j c_{1,j} a_{l+1-j} & \text{if } l \geqslant 4 \text{ even} \end{cases} \\ &= \begin{cases} -2c_{1,2}a_1 & \text{if } l = 2, \\ -2a_1 c_{l-1,2} + \displaystyle\sum_{j=2,\, j\in\mathscr{A}_l}^{l} (-1)^{j+1} j c_{1,j} a_{l+1-j} & \text{if } l \geqslant 4 \text{ even} \end{cases} \\ &= \begin{cases} -2c_{1,2}a_1 & \text{if } l = 2, \\ -2a_1 c_{l-1,2} - \displaystyle\sum_{j=2,\, j\in\mathscr{A}_l}^{l} j c_{1,j} a_{l+1-j} & \text{if } l \geqslant 4 \text{ even} \end{cases} \\ &= -T_{l,2} \end{aligned}$$

Furthermore, proceeding in the same manner, taking the notation $p = l_1 + l_2$, we get

$$\begin{aligned}
T_{l_1,l_2} &= \sum_{i=1, i\in\mathscr{A}_{l_1}}^{l_1} i c_{i,l_2-1} a_{l_1+1-i} + \sum_{j=1, j\in\mathscr{A}_{l_2}}^{l_2} j c_{l_1-1,j} a_{l_2+1-j} \\
&= \sum_{i=1, i\in\mathscr{A}_{l_1}}^{l_1} i(-1)^{l_2-1+i} c_{l_2-1,i} a_{l_1+1-i} + \sum_{j=1, j\in\mathscr{A}_{l_2}}^{l_2} j(-1)^{l_1-1+j} c_{j,l_1-1} a_{l_2+1-j} \\
&= (-1)^p \sum_{i=1, i\in\mathscr{A}_{l_1}}^{l_1} i(-1)^{-l_1-1+i} c_{i,l_2-1} a_{l_1+1-i} \\
&\quad + (-1)^p \sum_{j=1, j\in\mathscr{A}_{l_2}}^{l_2} j(-1)^{-l_2-1+j} c_{l_1-1,j} a_{l_2+1-j} \\
&= (-1)^{p+1} \sum_{i=1, i\in\mathscr{A}_{l_1}}^{l_1} i c_{i,l_2-1} a_{l_1+1-i} + (-1)^p \sum_{j=1, j\in\mathscr{A}_{l_2}}^{l_2} j c_{l_1-1,j} a_{l_2+1-j} \\
&= (-1)^{p+1} T_{l_2,l_1} = (-1)^{l_1+l_2+1} T_{l_2,l_1}.
\end{aligned}$$

This implies that (2.8) is satisfied. Furthermore, for any m we have that the linear system obtained so that H given in (2.4) is a formal first integral is

$$\begin{pmatrix}
m & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & m-2 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & \cdots & \ddots & 0 & 0 & \ddots & \cdots & 0 \\
0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & -1 & 0 & \cdots & 0 \\
0 & \cdots & \ddots & 0 & 0 & \ddots & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & 0 & 2-m & 0 \\
0 & \cdots & 0 & 0 & 0 & 0 & 0 & -m
\end{pmatrix}
\begin{pmatrix}
c_{m,0} \\ c_{m-1,1} \\ \vdots \\ c_{(m+1)/2,(m-1)/2} \\ c_{(m-1)/2,(m+1)/2} \\ \vdots \\ c_{1,m-1} \\ c_{0,m}
\end{pmatrix}
=
\begin{pmatrix}
0 \\ c_{1,m-2} a_1 \\ \vdots \\ T_{(m+1)/2,(m-1)/2} \\ T_{(m-1)/2,(m+1)/2} \\ \vdots \\ c_{m-2,1} a_1 \\ 0
\end{pmatrix},$$

if m is odd, and

$$\begin{pmatrix}
m & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & m-2 & 0 & 0 & 0 & \cdots & 0 \\
0 & \cdots & \ddots & 0 & \ddots & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & \cdots & \ddots & 0 & \ddots & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & 2-m & 0 \\
0 & \cdots & 0 & 0 & 0 & 0 & -m
\end{pmatrix}
\begin{pmatrix}
c_{m,0} \\ c_{m-1,1} \\ \vdots \\ c_{m/2,m/2} \\ \vdots \\ c_{1,m-1} \\ c_{0,m}
\end{pmatrix}
=
\begin{pmatrix}
0 \\ c_{1,m-2} a_1 \\ \vdots \\ T_{m/2,m/2} \\ \vdots \\ c_{m-2,1} a_1 \\ 0
\end{pmatrix},$$

if m is even.

In particular, if n is odd the determinant of the above linear system is different from zero and hence the system is compatible and determined and so we can determine all the coefficients $c_{i,j}$ with

$i+j=n$ in the form

$$c_{i,j}=\frac{1}{i-j}T_{i,j}=\frac{1}{i-j}(-1)^{i+j+1}T_{j,i}=(-1)^{i+j}\frac{1}{j-i}T_{j,i}=(-1)^{i+j}c_{j,i}.$$

In particular, for either $i=1$ or $j=1$ we get (see the definition of $T_{l_1,1}$)

$$c_{n-1,1}=\frac{1}{n-2}T_{n-1,1}=\frac{1}{n-2}a_1c_{n-2,1}=\frac{a_1^{n-2}}{(n-2)!}.$$

On the other hand, for $m=n$ even, the determinant of the corresponding linear system is zero, so we have that all the coefficients $c_{i,j}$ where $i+j=n$ with $(i,j)\neq(n/2,n/2)$ are completely determined and we have the extra condition

$$0=T_{n/2,n/2}.$$

Note that the conditions with $i+j=n$ with $(i,j)\neq(n/2,n/2)$ follow exactly as in the case n odd and so we obtain that

$$c_{i,j}=(-1)^{i+j}c_{j,i}\quad\text{and}\quad c_{n-1,1}=\frac{a_1^{n-2}}{(n-2)!}.$$

Now take $m=2n$. Then proceeding as above we have the condition

$$\begin{aligned}0=T_{n,n}&=\sum_{i=1}^{n}i(c_{i,n-1}+c_{n-1,i})a_{n+1-i}\\&=(c_{1,n-1}+c_{n-1,1})a_n+\sum_{i=2}^{n}i(c_{i,n-1}+c_{n-1,i})a_{n+1-i}\\&=c_{n-1,1}(1+(-1)^n)a_n+\sum_{i=2,i\in\mathscr{A}_n}^{n}ic_{n-1,i}(1+(-1)^{n-1+i})a_{n+1-i}\\&=\frac{a_1^{n-2}}{(n-2)!}(1+(-1)^n)a_n+\sum_{i=2,i\in\mathscr{A}_n}^{n}ic_{n-1,i}(1+(-1)^{n+1-i})a_{n+1-i}\\&=\frac{a_1^{n-2}}{(n-2)!}(1+(-1)^n)a_n.\end{aligned}$$

So, if n is odd we get the identity $0=0$ and if n is even we get the identity

$$0=\frac{2a_1^{n-2}}{(n-2)!}a_n\quad\text{which implies}\quad a_n=0,$$

as we wanted to show. This shows that when $a_1\neq0$ system (1.3) has a formal first integral when f is odd. Hence we have proved the necessity, completing the proof of Theorem 1.1.

3. Proof of Theorem 1.2

We first prove the sufficiency of Theorem 1.2. System (1.3) under the assumptions (i) of Theorem 1.2 takes the form (2.1) with $a_1\neq0$, or with $a_1=0$. Both cases are included in Theorem 1.1. Hence the proof given in Theorem 1.1 is also valid for this case.

System (1.3) with f of degree less than or equal to 8 under the assumptions (ii) of Theorem 1.2 splits in two cases:

(a) $a_k = 0$ for $k \neq 4m$ with $m \in \mathbb{N}$,
(b) $a_i = 0$ for all $i \leqslant 8$ except for $i = 6$.

In fact case (a) is valid for any degree of f so we present the general proof of this case. Under the assumptions of case (a) system (1.3) takes the form

$$\dot{x} = x + y\sum_{j=1}^{\infty} a_{4j}x^{4j}, \quad \dot{y} = -y + x\sum_{j=1}^{\infty} a_{4j}y^{4j}. \tag{3.1}$$

Doing the affine change of variables in (2.2), system (3.1) becomes system (2.3). Now proceeding as in the proof of the sufficiency part of Theorem 1.1 and we conclude that there exists a first integral of the form $H(x,y) = xy + \text{h.o.t.}$ of the original system.

In the case (b) system (1.3) with f of degree less than or equal to 8 becomes

$$\dot{x} = x(1 + a_6x^5y), \qquad \dot{y} = y(-1 + a_6xy^5). \tag{3.2}$$

System (3.2) has the analytic first integral

$$H(x,y) = \frac{x^5y^5}{4 + 5a_6x^5y - 5a_6xy^5},$$

that is well-defined around the origin. Hence we have an integrable saddle at the origin.

Now we prove the necessity of Theorem 1.2. As in the proof of Theorem 1.1, to find the saddle quantities we propose a formal first integral of the form in (2.4)–(2.5). We first compute H_3. In this case we have the linear system (2.6) with $a_1 = 0$. Since the determinant of that linear system is compatible and determined, we get that $c_{i,j} = 0$ for $i + j = 3$. The linear system for H_4 is the one given in (2.7) with $a_1 = 0$. From the third equation we get that $a_2 = 0$. So, a condition to have a formal first integral in this case is $a_2 = 0$. This proves that a necessary condition for system (1.3) with $a_1 = 0$ to have a formal first integral is $a_2 = 0$. The linear system for H_5 is

$$\begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5 \end{pmatrix} \begin{pmatrix} c_{5,0} \\ c_{4,1} \\ c_{3,2} \\ c_{2,3} \\ c_{1,4} \\ c_{0,5} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -a_3 \\ -a_3 \\ 0 \\ 0 \end{pmatrix}.$$

From here we have $c_{5,0} = c_{4,1} = c_{1,4} = c_{0,5} = 0$, $c_{3,2} = -a_3$ and $c_{2,3} = a_3$. The linear system for H_6 gives the following result $c_{6,0} = c_{5,1} = c_{3,3} = c_{1,5} = c_{0,6} = 0$, $c_{4,2} = -a_4/2$ and $c_{2,4} = a_4/2$. Solving the different linear systems for H_7, H_8, H_9, H_{10} and H_{11} we do not find any extra necessary condition. The linear system for H_{12} gives the condition $a_3^2a_4$. However to go further with this method is very difficult and we do not see how to prove by induction the conditions for any degree of f.

So, now we have fixed the degree of f less than or equal to 8 (due to the fact that the machine does not allow us to go further). In order to compute the necessity in this case we use the following method. Using the change of variables $X = x + iy$, $Y = x - iy$ and the scaling of time $t \mapsto -t/i$

system (1.3) takes the form

$$\dot{X} = -Y + F(X,Y), \qquad \dot{Y} = X + G(X,Y), \tag{3.3}$$

Taking polar coordinates $X = r\cos\theta$ and $Y = r\sin\theta$ system (3.3) becomes

$$\dot{r} = \sum_{s=2}^{\infty} P_s(\theta)\, r^s, \quad \dot{\theta} = 1 + \sum_{s=2}^{\infty} Q_s(\theta)\, r^{s-1}, \tag{3.4}$$

where P_s and Q_s are trigonometric polynomials of degree s. Now we propose a formal series of the form

$$H(r,\theta) = \sum_{m=2}^{\infty} H_m(\theta) r^m,$$

where $H_2(\theta) = 1/2$ and $H_m(\theta)$ are homogeneous trigonometric polynomials with respect to θ of degree m. Imposing that this power series is a formal first integral of system (3.4) we obtain

$$\dot{H}(r,\theta) = \sum_{k=2}^{\infty} V_{2k} r^{2k},$$

where V_{2k} are in fact the *saddle constants* that depend on the parameters of the original system (1.3). If we fix the degree of f equal to 8 and we compute the saddle constants for V_4 to V_{84}, and we compute the decomposition of the ideal generated by these constants we find the conditions given in Theorem 1.2. This completes the proof of the theorem.

4. Proof of Theorem 1.3

System (1.3) under the assumptions of Theorem 1.3 takes the form

$$\begin{aligned} \dot{x} &= x + y \sum_{i=0}^{\infty} a_{(2i+1)(n-2)+2}\, x^{(2i+1)(n-2)+2}, \\ \dot{y} &= -y + x \sum_{i=0}^{\infty} a_{(2i+1)(n-2)+2}\, y^{(2i+1)(n-2)+2}. \end{aligned} \tag{4.1}$$

Now we propose the change of variables

$$X = xy \quad \text{and} \quad Y = x^{n-2} - y^{n-2}.$$

Doing this change we obtain

$$\dot{X} = y^2 f(x) + x^2 f(y), \quad \dot{Y} = (n-2)x^{n-3}(x + y f(x)) - (n-2)y^{n-3}(-y + x f(y)),$$

that becomes

$$\begin{aligned} \dot{X} &= x^2 y^2 \sum_{i=0}^{\infty} a_{(2i+1)(n-2)+2}\, (x^{(2i+1)(n-2)} + y^{(2i+1)(n-2)}), \\ \dot{Y} &= (n-2)\left[(x^{n-2} + y^{n-2}) + xy \sum_{i=0}^{\infty} a_{(2i+1)(n-2)+2} (x^{(2i+2)(n-2)} - y^{(2i+2)(n-2)}) \right]. \end{aligned}$$

Note that

$$x^{(2i+1)(n-2)}+y^{(2i+1)(n-2)}=(x^{n-2}+y^{n-2})\sum_{j=0}^{2i}(-1)^j x^{(2i-j)(n-2)}y^{j(n-2)} \tag{4.2}$$

and

$$x^{(2i+2)(n-2)}-y^{(2i+2)(n-2)}=(x^{n-2}+y^{n-2})\sum_{j=0}^{2i+1}(-1)^j x^{(2i+1-j)(n-2)}y^{j(n-2)}. \tag{4.3}$$

Now we claim that

$$x^{(n-2)j}+y^{(n-2)j}=P_j(X,Y) \quad \text{for } j\geqslant 2 \text{ even} \tag{4.4}$$

where P_j is a polynomial for each j even and

$$x^{(n-2)j}-y^{(n-2)j}=Q_j(X,Y) \quad \text{for } j\geqslant 1 \text{ odd} \tag{4.5}$$

where Q_j is a polynomial for each j odd.

The proofs of (4.4) and (4.5) will be done by induction on j. Since

$$x^{2(n-2)}+y^{2(n-2)}=(x^{n-2}-y^{n-2})^2+2(xy)^{n-2}=Y^2+2X^{n-2}=P_2(X,Y),$$

equation (4.4) holds for $j=2$ and clearly (4.5) holds for $j=1$ with $Q_1(X,Y)=Y$. Now assume that (4.4) is true for $j=\ell-2$ with ℓ even and we shall prove it for $j=\ell$. Note that by Newton's binomial formula

$$\begin{aligned}(x^{(n-2)}-y^{(n-2)})^{\ell}&=\sum_{k=0}^{\ell}\binom{\ell}{k}(-1)^k x^{(\ell-k)(n-2)}y^{k(n-2)}\\&=x^{\ell(n-2)}+y^{\ell(n-2)}+\sum_{k=1}^{\ell-1}\binom{\ell}{k}(-1)^k x^{(\ell-k)(n-2)}y^{k(n-2)}.\end{aligned}$$

So,

$$
\begin{aligned}
x^{\ell(n-2)}+y^{\ell(n-2)} &= Y^{\ell}-\sum_{k=1}^{\ell-1}\binom{\ell}{k}(-1)^k x^{(\ell-k)(n-2)}y^{k(n-2)}\\
&= Y^{\ell}-\sum_{k=1}^{\ell/2-1}\binom{\ell}{k}(-1)^k x^{(\ell-k)(n-2)}y^{k(n-2)}-\sum_{k=\ell/2+1}^{\ell}\binom{\ell}{k}(-1)^k x^{(\ell-k)(n-2)}y^{k(n-2)}\\
&\quad -\binom{\ell}{\ell/2}(-1)^{\ell/2}X^{\ell(n-2)/2}\\
&= Y^{\ell}-\binom{\ell}{\ell/2}(-1)^{\ell/2}X^{\ell(n-2)/2}\\
&\quad -\sum_{k=1}^{\ell/2-1}\binom{\ell}{k}(-1)^k(x^{(\ell-k)(n-2)}y^{k(n-2)}+x^{k(n-2)}y^{(\ell-k)(n-2)})\\
&= Y^{\ell}-\binom{\ell}{\ell/2}(-1)^{\ell/2}X^{\ell(n-2)/2}\\
&\quad -\sum_{k=1}^{\ell/2-1}\binom{\ell}{k}(-1)^k(xy)^{k(n-2)}(x^{(\ell-2k)(n-2)}+y^{(\ell-2k)(n-2)})\\
&= Y^{\ell}-\binom{\ell}{\ell/2}(-1)^{\ell/2}X^{\ell(n-2)/2}-\sum_{k=1}^{\ell/2-1}\binom{\ell}{k}(-1)^k X^{k(n-2)}P_{\ell-2k}(X,Y),
\end{aligned}
$$

where in the last step we have used the induction hypothesis. So,

$$x^{\ell(n-2)}+y^{\ell(n-2)}=P_{\ell}(X,Y),$$

and claim (4.4) is proved.

Proceeding in the same manner we have that for ℓ odd

$$
\begin{aligned}
x^{\ell(n-2)}-y^{\ell(n-2)} &= Y^{\ell}-\sum_{k=1}^{\ell-1}\binom{\ell}{k}(-1)^k x^{(\ell-k)(n-2)}y^{k(n-2)}\\
&= Y^{\ell}-\sum_{k=1}^{(\ell-1)/2}\binom{\ell}{k}(-1)^k x^{(\ell-k)(n-2)}y^{k(n-2)}\\
&\quad -\sum_{k=(\ell+1)/2}^{\ell-1}\binom{\ell}{k}(-1)^k x^{(\ell-k)(n-2)}y^{k(n-2)}\\
&= Y^{\ell}-\sum_{k=1}^{(\ell-1)/2}\binom{\ell}{k}(-1)^k(x^{(\ell-k)(n-2)}y^{k(n-2)}+x^{k(n-2)}y^{(\ell-k)(n-2)})\\
&= Y^{\ell}-\sum_{k=1}^{(\ell-1)/2}\binom{\ell}{k}(-1)^k(xy)^{k(n-2)}(x^{(\ell-2k)(n-2)}+y^{(\ell-2k)(n-2)})\\
&= Y^{\ell}-\sum_{k=1}^{(\ell-1)/2}\binom{\ell}{k}(-1)^k X^{k(n-2)}Q_{\ell-2k}(X,Y),
\end{aligned}
$$

where in the last step we have used the induction hypothesis. So,

$$x^{\ell(n-2)}-y^{\ell(n-2)}=Q_{\ell}(X,Y),$$

and claim (4.5) is proved.

Note that proceeding as above,

$$\sum_{j=0}^{2i}(-1)^j x^{(2i-j)(n-2)}y^{j(n-2)} = x^{2i(n-2)}+y^{2i(n-2)}+\sum_{j=1}^{2i-1}(-1)^j x^{(2i-j)(n-2)}y^{j(n-2)}$$

$$= x^{2i(n-2)}+y^{2i(n-2)}+\sum_{j=1}^{i-1}(-1)^j x^{(2i-j)(n-2)}y^{j(n-2)}$$

$$+\sum_{j=i+1}^{2i-1}(-1)^j x^{(2i-j)(n-2)}y^{j(n-2)}+(-1)^i X^{i(n-2)}$$

$$= P_{2i}(X,Y)+(-1)^i X^{i(n-2)}+\sum_{j=1}^{i-1}(-1)^j\left(x^{(2i-j)(n-2)}y^{j(n-2)}+x^{j(n-2)}y^{(2i-j)(n-2)}\right)$$

$$= P_{2i}(X,Y)+(-1)^i X^{i(n-2)}+\sum_{j=1}^{i-1}(-1)^j (xy)^{j(n-2)}\left(x^{(2i-2j)(n-2)}+y^{(2i-2j)(n-2)}\right)$$

$$= P_{2i}(X,Y)+(-1)^i X^{i(n-2)}+\sum_{j=1}^{i-1}(-1)^j X^{j(n-2)}P_{2i-2j}(X,Y) = \widetilde{P}_{2i}(X,Y),$$

where $\widetilde{P}_{2i}$ is a polynomial. Hence, by equation (4.2) we have

$$x^{(2i+1)(n-2)}+y^{(2i+1)(n-2)} = (x^{n-2}+y^{n-2})\widetilde{P}_{2i}(X,Y) \tag{4.6}$$

Analogously we have

$$\sum_{j=0}^{2i+1}(-1)^j x^{(2i+1-j)(n-2)}y^{j(n-2)} = (x^{n-2}+y^{n-2})\widetilde{Q}_{2i+1}(X,Y)$$

where $\widetilde{Q}_{2i+1}$ is a polynomial. Hence, by equation (4.3) we get

$$x^{(2i+2)(n-2)}-y^{(2i+2)(n-2)} = (x^{n-2}+y^{n-2})\widetilde{Q}_{2i+1}(X,Y) \tag{4.7}$$

and so,

$$\dot{X} = x^2y^2(x^{n-2}+y^{n-2})\sum_{i=0}^{\infty}a_{(2i+1)(n-2)+2}\widetilde{P}_{2i}(X,Y),$$
$$\dot{Y} = (n-2)(x^{n-2}+y^{n-2})\left[1+X\widetilde{Q}_{2i+1}(X,Y)\right].$$

Next we make a rescaling of time of the form $d\tau = (x^{n-2}+y^{n-2})dt$. In this way the above system becomes

$$X' = x^2y^2\sum_{i=0}^{\infty}a_{(2i+1)(n-2)+2}\widetilde{P}_{2i}(X,Y),$$
$$Y' = (n-2)\left[1+X\widetilde{Q}_{2i+1}(X,Y)\right],$$

where the dot means derivative in the new time τ. The above system does not have a singular point at the origin. Hence since it is polynomial in the variables (X,Y) it has an analytic first integral around the origin by the Flow-box theorem. This first integral can be pulled back to a first integral of the form $H(x,y) = xy+\text{h.o.t.}$ of the original system. This completes the proof of the theorem.

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