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Hierarchies of q -discrete Painlevé equations

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In this paper, we construct a new hierarchy based on the third q -discrete Painlevé equation (qP_{III}) and also study the hierarchy of the second q -discrete Painlevé equation (qP_{II}). Both hierarchies are derived by using reductions of the general lattice modified Korteweg-de Vries equation. Our results include Lax pairs for both hierarchies and these turn out to be higher degree expansions of the non-resonant ones found by Joshi and Nakazono [29] for the second-order cases. We also obtain Bäcklund transformations for these hierarchies. Special q -rational solutions of the hierarchies are constructed and corresponding q -gamma functions that solve the associated linear problems are derived.

Keywords: q -discrete Painlevé equations; Lax pair; Hierarchies; Bäcklund transformations.

2000 Mathematics Subject Classification: 33E17, 34M55, 37K10, 34K17

1. Introduction

Following widespread interest in the Painlevé equations, it is natural to ask whether their integrable discrete versions share their fundamental properties. In this paper, we consider and answer the following question about multiplicative or q -discrete Painlevé equations, namely the construction of new associated infinite sequences of discrete equations called hierarchies. The term ‘hierarchy’ here refers to a sequence of q -difference equations sharing a linear problem.

In the literature, discrete hierarchies are known for some additive discrete Painlevé [10, 16], and q -discrete Painlevé equations [21, 32, 45] but not for most of the known discrete Painlevé equations. Our paper extends the class of known hierarchies of q -discrete Painlevé equations by providing a new hierarchy, associated with the q -discrete third Painlevé equation.

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Our approach starts with an integrable partial difference equation, also known as a lattice equation, and considers higher-order reductions than those that have been constructed before. (See for example [19, 25, 34, 37–39].) Our starting point is equation (3.1), which is a slightly more general (multi-parameter) form of a standard lattice equation denoted by $H_3^{\delta=0}$ in the ABS classification [3, 4]. Further background information about lattice equations is given for the interested reader in §1.1.

In particular, we obtain two q -discrete hierarchies, whose starting points are the q -discrete second and third discrete Painlevé equations, denoted by qP_{II} and qP_{III} respectively below. We will refer to the n^{th} member of the respective hierarchies by $qP_{II}^{(n)}$, $qP_{III}^{(n)}$. In fact, we also obtain additional hierarchies, but we focus on qP_{II} and qP_{III} here to provide a self-contained exposition. The term hierarchy is used because each sequence shares a linear problem, which is one of a pair of linear problems known as Lax pairs. The corresponding Lax pairs have the form

$$\Phi(qx, t) = A(x, t)\Phi(x, t), \tag{1.1a}$$

$$\mathcal{T}(\Phi(x, t)) = B(x, t)\Phi(x, t), \tag{1.1b}$$

where \mathcal{T} is a time-deformation operator, whose action iterates the Painlevé variable t in the resulting hierarchy. The variables x and t are often called spectral and Painlevé variables respectively. The matrix A is polynomial in x and given by equations (1.3), while B is given by equation (1.6) or (1.9). (See our main result Theorem 1.1 in §1.2.)

For Equations (1.1a) and (1.1b) to be compatible, we must have

$$\mathcal{T}(A(x, t))B(x, t) = B(qx, t)A(x, t). \tag{1.2}$$

For each $n \in \mathbb{N}$ corresponding to the degree of A in x , we show that equation (1.2) gives rise to either $qP_{II}^{(n)}$ or $qP_{III}^{(n)}$ (corresponding to the choice of B). Equations (1.8) and (1.11) in Section 2 give the complete form of these hierarchies. In the same way that qP_{II} is a symmetric case (or projective reduction) of qP_{III} , we can show that the n^{th} member of the qP_{II} hierarchy can be obtained from the n^{th} member of the qP_{III} hierarchy.

Previous approaches [10, 21] for constructing hierarchies have started by extending the degree of the matrix A in x and using the resulting compatibility conditions to derive new equations. The key idea here relies on fixing one matrix while we change the degree of the other one. However, the calculations are not straightforward, and become more technical in the case of q -discrete equations. Instead, we pursue a simpler approach given by higher-order reductions of systems of lattice equations, as explained in Section 3. In finding the appropriate reductions, we were motivated by the Lax pairs and the actions of the time-deformation operator \mathcal{T} on parameters given in [29], which allow us to hypothesize the lattice parameters that we use for the reductions.

Given an integer d , a well-known procedure for finding reductions of lattice equations governing a function $x(l, m)$ is to assume a periodic $(d, 1)$ -reduction, i.e., impose $x_{l, m+1} = x_{l+d, m}$ or $x_{l, m+1} = 1/x_{l+d, m}$. Such periodic reductions of the lattice mKdV (modified Korteweg-de Vries) equation, also known as H_3 , were obtained for some integer d in [21]. We note that this led to a qP_{II} hierarchy, which we also find here. However, the calculations are intricate and further assumptions were required to find Lax pairs. In contrast, our approach immediately provides hierarchies as well as their Lax pairs, without intermediate assumptions.

The 2×2 Lax pairs we find have certain advantages for analysis. The most important properties are that each matrix A corresponding to a member of the hierarchy is non-singular at $x = 0$

and ∞ and explicitly factorisable into linear factors in x . The former property enables Birkhoff and Carmichael's classical theory of non-resonant q -linear difference equations [6, 9] to be applied immediately.

1.1. Background

In this section, we provide some background material about Painlevé and discrete Painlevé equations, and integrable lattice equations.

The Painlevé equations appear widely in physical applications (see for example [46]). They were discovered originally by Painlevé [40], Gambier [14] and Fuchs [13] in the search for new transcendental functions that satisfy ordinary differential equations (ODEs) and their general solutions are known to be higher transcendental functions, called the Painlevé transcendents. Later, they were found as reductions of completely integrable partial differential equations (PDEs), such as the Korteweg-de Vries equation [1, 2]. More recently, integrable difference equations with properties that are very similar to those of the classical Painlevé equations have been identified. They are now known as discrete Painlevé equations and there are three types. We focus here on discrete Painlevé equations of q -discrete or multiplicative type; see Sakai [43].

The search for discrete versions of integrable PDEs has also been a very active area of recent research. Many partial difference equations with properties that are very similar to those of integrable PDEs are known to share a geometric property called consistency around a cube (CAC) or multi-dimensional consistency [35,36]. Classifications of scalar equations with this property [3,4,7] led to a list of scalar partial difference equations. By convention, they are denoted as A -, D -, H - or Q -type. In this paper, we focus on an example of H -type, denoted H_3 , which is also known as a lattice mKdV equation.

Many reductions of lattice equations to the q -discrete Painlevé equations are already known [8, 12, 19, 22, 23, 26, 34, 37–39]. For example, the qP_{II} and qP_{III} equations were derived from a so-called geometric reduction of H_3 and a special case of an equation labelled D_4 in [7]; see [30].

Motivated by the existence of hierarchies (infinite sequences) of PDEs associated with each integrable PDE, hierarchies of Painlevé equations have also been found [15, 28]. Correspondingly, hierarchies for a restricted set of discrete Painlevé equations have also been obtained [10, 21, 33]. However, to the best of our knowledge, very few q -discrete Painlevé hierarchies have been constructed.

Although generic solutions of discrete Painlevé equations are higher transcendental functions, there also exist special function and rational solutions for special cases of parameters. Each such special-parameter case can be mapped to another through transformations called Bäcklund transformations [24]. Explicit solutions of qP_{II} were studied and corresponding solutions of its Lax pair found in [31].

Starting with a seed solution corresponding to an initial parameter, one can generate an infinite series of solutions of the same equation with different parameters, provided that the Bäcklund transformation we use does not terminate. Confusingly, the word hierarchy may also be used in this context to refer to an infinite sequence of solutions generated by a Bäcklund transformation when successively applied to a seed solution.

1.2. Main Result

Our main result shows that each member of the hierarchy of qP_{II} and qP_{III} shares one spectral linear problem equation (1.1a), with the coefficient matrix given by

$$A(x, t) = \prod_{j=0}^n A_{n-j}, \quad n \geq 2, \tag{1.3a}$$

where

$$A_l := \begin{pmatrix} -\frac{ib_l \lambda}{h_l} x & 1 \\ -1 & -\frac{ib_l h_l}{\lambda} x \end{pmatrix}. \tag{1.3b}$$

Here, $b_l, l = 0, 1, \dots, n, \lambda$ and q are the non-zero complex parameters and $h_l, l = 0, 1, \dots, n$ are dependent variables. We find that the product of all h_l turns out to be constant in the case of the qP_{II} and qP_{III} hierarchies. We take this constant to be λ^2 without loss of generality, to match with the second-order case given in [29], that is,

$$\prod_{j=0}^n h_j = \lambda^2. \tag{1.4}$$

We also take

$$b_n = q \tag{1.5}$$

to match with the base cases given in the same paper [29]. In fact, one can take b_n to be any constant from the construction in Section 3.

We use the following notation. Given $2 \leq n \in \mathbb{N}$, the n^{th} member of the hierarchy will be described by using a deformation operator referred to as \tilde{T} and \widehat{T} for the case associated with qP_{II} and qP_{III} respectively. These operators act on a set of parameters and independent variables denoted by $b_0, b_1, \dots, b_n, \lambda, q, x$ and on the dependent variables, which we will denote by h_0, \dots, h_n .

The following theorem collects our main results.

Theorem 1.1. *Let $n \in \mathbb{N}, n \geq 2$. The compatibility condition (1.2) is satisfied on solutions of the qP_{II} and qP_{III} hierarchy respectively when $\{\mathcal{T}, B\}$ is replaced by one of the following choices:*

(1) $\{\tilde{T}, B_{II}\}$ and (2) $\{\widehat{T}, B_{III}\}$. The qP_{II} and qP_{III} hierarchies are given as follows.

(1) *Take*

$$B_{II} = \begin{pmatrix} -\frac{ib_0 \lambda}{h_0} x & 1 \\ -1 & -\frac{ib_0 h_0}{\lambda} x \end{pmatrix}, \tag{1.6}$$

and define \tilde{T} by the action

$$\tilde{T} : (b_0, b_1, \dots, b_j, \dots, b_{n-1}, b_n, \lambda, q, x) \rightarrow (b_1, b_2, \dots, b_{j+1}, \dots, qb_0, b_n, \lambda, q, x). \tag{1.7}$$

Then the resulting hierarchy of equations is given by

$$\begin{cases} \tilde{T}(h_r) = h_{r+1}, & 0 \leq r \leq n-2, \\ \tilde{T}(h_{n-1}) = \frac{\lambda^2(1 + b_0 \prod_{j=1}^{n-1} h_j)}{\prod_{j=0}^{n-1} h_j (b_0 + \prod_{j=1}^{n-1} h_j)}. \end{cases} \quad (1.8)$$

(2) Take

$$B_{\text{III}} = \begin{pmatrix} -\frac{ib_1\lambda}{h_1}x & 1 \\ -1 & -\frac{ib_1h_1}{\lambda}x \end{pmatrix} \begin{pmatrix} -\frac{ib_0\lambda}{h_0}x & 1 \\ -1 & -\frac{ib_0h_0}{\lambda}x \end{pmatrix}, \quad (1.9)$$

and define \hat{T} by the action

$$\hat{T} : (b_0, b_1, \dots, b_j, \dots, b_{n-2}, b_{n-1}, b_n, \lambda, q, x) \rightarrow (b_2, b_3, \dots, b_{j+2}, \dots, b_0q, b_1q, b_n, \lambda, q, x), \quad (1.10)$$

Then the resulting hierarchy of equations is given by

$$\begin{cases} \hat{T}(h_r) = h_{r+2}, & 0 \leq r \leq n-3, \\ \hat{T}(h_{n-2}) = \frac{\lambda^2(1 + b_0 \prod_{j=1}^{n-1} h_j)}{\prod_{j=0}^{n-1} h_j (b_0 + \prod_{j=1}^{n-1} h_j)}, \\ \hat{T}(h_{n-1}) = \frac{\lambda^2(1 + b_1(\prod_{j=2}^{n-1} h_j)\hat{T}(h_{n-2}))}{(\prod_{j=1}^{n-1} h_j)\hat{T}(h_{n-2})(b_1 + (\prod_{j=2}^{n-1} h_j)\hat{T}(h_{n-2}))}. \end{cases} \quad (1.11)$$

Remark 1.1. Variables h_l ($l = 0, 1, \dots, n$) are functions of t on which these operators \tilde{T} and \hat{T} act respectively as follows

$$\begin{cases} t \rightarrow pt = q^{\frac{1}{n}}t, & \text{for equation (1.8) by taking } b_0 = t, b_1 = q^{\frac{1}{n}}t, \\ t \rightarrow pt = q^{\frac{2}{n}}t, & \text{for equation (1.11) by taking } b_0 = t, b_1 = bt, b_2 = q^{\frac{2}{n}}t, b_3 = bq^{\frac{2}{n}}t \text{ for even } n, \end{cases}$$

where b is a constant for the latter case. For the former case, one can easily see that $b_1 = pt = \tilde{T}(t) = \tilde{T}(b_0)$. Similarly, for the latter case we have $b_2 = \hat{T}(b_0)$, $b_3 = \hat{T}(b_1)$. Using these operators, we find that

$$\begin{cases} b_j = q^{\frac{j}{n}}t \text{ for } 0 \leq j \leq n-1, & \text{for equation (1.8),} \\ b_{2j} = q^{\frac{2j}{n}}t, b_{2j+1} = bq^{\frac{2j}{n}}t, \text{ for even } n \text{ and } 0 \leq j \leq n/2, & \text{for equation (1.11).} \end{cases}$$

Remark 1.2. The results of Theorem 1.1 can be separated into two hierarchies in each case. These are distinguished by whether n is odd or even. The cases corresponding to odd n reduce to one of lower order in each case. At the base level, we get degenerate limits of qP_{II} and qP_{III} . See further details for the case $n = 3$ in Remark 2.1.

Remark 1.3. It is intriguing to note that the Lax pair $A(x, t)$ given in (1.3) has a form that is analogous to the one given by Kajiwara et al [32]. This is because these matrices both come from periodic reductions of lattice equations.^a

Because of the observations in Remarks 1.2 and 2.1, we will assume that n is even in the remainder of the paper. (Nevertheless, all the results are satisfied also for the case when n is odd.)

^aWe would like to thank the referee for this observation.

1.3. Outline of the paper

This paper is organized as follows. In Section 2, we provide examples of the first and second members of each hierarchy. In Section 3, we use periodic reductions of the general mKdV to derive the hierarchies of q -discrete second and third Painlevé equations. Their associated Lax pairs and Bäcklund transformations are obtained automatically from this method, in Sections 3 and 4. In Section 5 we describe the rational solutions of the hierarchies and deduce the corresponding solutions of their associated Lax pairs.

2. Second- and Fourth-order members of the hierarchies

In this section, we consider the cases $n = 2, 3, 4$ in Theorem 1.1 explicitly. The case $n = 2$ corresponds to the well-known qP_{II} and qP_{III} equations. The case $n = 4$ is the next member of the hierarchy corresponding to each of these equations. The odd case where $n = 3$ is deduced here for illustrative purposes, to show that the result is a second-order equation that is a degenerate version of qP_{II} and qP_{III} . Thereafter, we will refer to the iteration of the variable h_l ($l = 0, \dots, n$) under the deformation operators \tilde{T} and \hat{T} by \tilde{h}_l and \hat{h}_l , respectively.

2.1. The case $n = 2$

In this subsection, we consider the case $n = 2$ in Theorem 1.1. Recall that equations (1.5) and (1.4) give the constraints

$$h_0 h_1 h_2 = \lambda^2 \quad \text{and} \quad b_2 = q.$$

Note that the deformation operators are given as follows

$$\begin{aligned} \tilde{T} &: (b_0, b_1, b_2, \lambda, q) \rightarrow (b_1, qb_0, b_2, \lambda, q), \\ \hat{T} &: (b_0, b_1, b_2, \lambda, q) \rightarrow (qb_0, qb_1, b_2, \lambda, q). \end{aligned}$$

So the first members of the qP_{II} and qP_{III} hierarchies are given by following equations

$$qP_{II}^{(2)} : \begin{cases} \tilde{h}_0 = h_1, \\ \tilde{h}_1 = \frac{\lambda^2(1 + b_0 h_1)}{h_0 h_1 (b_0 + h_1)}, \end{cases} \tag{2.1}$$

$$qP_{III}^{(2)} : \begin{cases} \hat{h}_0 = \frac{\lambda^2(1 + b_0 h_1)}{h_0 h_1 (b_0 + h_1)}, \\ \hat{h}_1 = \frac{\lambda^2(1 + b_1 \hat{h}_0)}{\hat{h}_0 h_1 (b_1 + \hat{h}_0)}. \end{cases} \tag{2.2}$$

Equations (2.1) and (2.2) are the second order of the q -discrete second and third Painlevé equations given in [29].

2.2. The case $n = 3$

Remark 2.1. For the case where n is odd, we can always reduce the order of the associated equations by unity, and that leads to a degenerate version of the hierarchy.

We consider the case $n = 3$ to illustrate the argument. Recall that equations (1.4) and (1.5) give the constraints

$$h_0 h_1 h_2 h_3 = \lambda^2 \text{ and } b_3 = q.$$

Note that the deformation operators are given as follows

$$\begin{aligned} \tilde{T} &: (b_0, b_1, b_2, b_3, \lambda, q) \rightarrow (b_1, b_2, qb_0, b_3, \lambda, q), \\ \hat{T} &: (b_0, b_1, b_2, b_3, \lambda, q) \rightarrow (b_2, qb_0, qb_1, b_3, \lambda, q). \end{aligned}$$

We consider each case listed in Theorem 1.1 separately below.

(1) We obtain

$$\tilde{h}_0 = h_1, \quad \tilde{h}_1 = h_2, \quad \tilde{h}_2 = \frac{\lambda^2(1 + b_0 h_1 h_2)}{h_0 h_1 h_2 (b_0 + h_1 h_2)},$$

which is equivalent to

$$\tilde{\tilde{h}}_0 = \frac{\lambda^2(1 + b_0 \tilde{h}_0 \tilde{h}_0)}{h_0 \tilde{h}_0 \tilde{h}_0 (b_0 + \tilde{h}_0 \tilde{h}_0)}. \tag{2.3}$$

Defining $\tilde{h}_0 \tilde{\tilde{h}}_0 = f$, equation (2.3) becomes

$$\tilde{f} \tilde{f} = \frac{\lambda^2(1 + b_0 f)}{(b_0 + f)}. \tag{2.4}$$

The resulting equation is a degenerate version of the qP_{II} equation that was first obtained in [42]; for details see [17, 18, 42].

(2) We obtain

$$\hat{h}_0 = h_2, \quad \hat{h}_1 = \frac{\lambda^2(1 + b_0 h_1 h_2)}{h_0 h_1 h_2 (b_0 + h_1 h_2)} \quad \text{and} \quad \hat{h}_2 = \frac{\lambda^2(1 + b_1 h_2 \hat{h}_1)}{h_1 h_2 \hat{h}_1 (b_1 + h_2 \hat{h}_1)},$$

which is equivalent to

$$\hat{h}_1 \hat{h}_1 = \frac{\lambda^2(1 + b_0 \hat{h}_1 \hat{h}_0)}{h_0 \hat{h}_0 (b_0 + \hat{h}_1 \hat{h}_0)} \quad \text{and} \quad \hat{h}_0 \hat{h}_0 = \frac{\lambda^2(1 + b_1 \hat{h}_1 \hat{h}_0)}{h_1 \hat{h}_1 (b_1 + \hat{h}_1 \hat{h}_0)}. \tag{2.5}$$

Defining $f = h_0 h_1$ and $g = \hat{h}_0 h_1$, equation (2.5) becomes

$$\hat{f} \hat{f} = \frac{\lambda^2(1 + b_0 g)}{(b_0 + g)}, \quad \hat{g} \hat{g} = \frac{\lambda^2(1 + b_1 \hat{f})}{(b_1 + \hat{f})}. \tag{2.6}$$

The cases listed above cover all the possibilities for $n = 3$. These illustrate the assertion made in Remark 2.1. The general case of odd n will be consider in a separate paper.

2.3. The case $n = 4$

Here we consider the second member (fourth-order) of the qP_{II} and qP_{III} hierarchies. Recall that equations (1.4) and (1.5) give the constraints

$$h_0 h_1 h_2 h_3 h_4 = \lambda^2 \text{ and } b_4 = q.$$

Noting that the deformation operators are given by

$$\begin{aligned} \tilde{T} &: (b_0, b_1, b_2, b_3, b_4, \lambda, q) \rightarrow (b_1, b_2, b_3, qb_0, b_4, \lambda, q), \\ \hat{T} &: (b_0, b_1, b_2, b_3, b_4, \lambda, q) \rightarrow (b_2, b_3, qb_0, qb_1, b_4, \lambda, q), \end{aligned}$$

we obtain the second member of the qP_{II} and qP_{III} hierarchies as follows

$$qP_{II}^{(4)} : \begin{cases} \tilde{h}_0 = h_1, & \tilde{h}_1 = h_2, & \tilde{h}_2 = h_3, \\ \tilde{h}_3 = \frac{\lambda^2(1 + b_0 h_1 h_2 h_3)}{h_0 h_1 h_2 h_3 (b_0 + h_1 h_2 h_3)}, \end{cases} \tag{2.7}$$

or, equivalently

$$\tilde{h}_2 \tilde{h}_2 = \frac{\lambda^2(1 + b_0 \tilde{h}_2 \tilde{h}_2)}{\tilde{h}_2 \tilde{h}_2 (b_0 + \tilde{h}_2 \tilde{h}_2)}. \tag{2.8}$$

Moreover, we have

$$qP_{III}^{(4)} : \begin{cases} \hat{h}_0 = h_2, & \hat{h}_1 = h_3, \\ \hat{h}_2 = \frac{\lambda^2(1 + b_0 h_1 h_2 h_3)}{h_0 h_1 h_2 h_3 (b_0 + h_1 h_2 h_3)}, \\ \hat{h}_3 = \frac{\lambda^2(1 + b_1 h_2 h_3 \hat{h}_2)}{h_1 h_2 h_3 \hat{h}_2 (b_1 + h_2 h_3 \hat{h}_2)}, \end{cases} \tag{2.9}$$

or, equivalently

$$\begin{cases} \hat{h}_2 \hat{h}_2 = \frac{\lambda^2(1 + b_0 \hat{h}_2 \hat{h}_2)}{\hat{h}_2 \hat{h}_2 (b_0 + \hat{h}_2 \hat{h}_2)}, \\ \hat{h}_3 \hat{h}_3 = \frac{\lambda^2(1 + b_1 \hat{h}_3 \hat{h}_3)}{h_2 h_3 \hat{h}_2 (b_1 + h_2 h_3 \hat{h}_2)}. \end{cases} \tag{2.10}$$

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1 by applying a periodic reduction, often called a staircase method, to a lattice equation, which is a multi-parametric version of the discrete modified mKdV equation, given by

$$Q(w_0, w_1, w_2, w_{12}; \alpha_i, \beta_i) = \alpha_1 w_0 w_2 - \alpha_2 w_1 w_{12} - \beta_1 w_0 w_1 + \beta_2 w_2 w_{12} = 0, \tag{3.1}$$

where the arguments w_0, w_1, w_2, w_{12} are associated with vertices of a quadrilateral. Interpreting these as points on a lattice with directions l and m , we assume $w_0 = w_{l,m}, w_1 = w_{l+1,m}, w_2 = w_{l,m+1}$, and $w_{12} = w_{l+1,m+1}$, as shown in Figure 3.1.



Fig. 3.1 Quad equation and CAC

Note that α_i and β_i , $i = 1, 2$ are parameters, which may also depend on l and m . We assume that the parameters (α_1, α_2) and (β_1, β_2) correspond to the l and m directions respectively on the lattice and embed this equation in a cube with a third direction associated with parameters (γ_1, γ_2) , see Figure 3.1 cf. [47].

It is easy to check that if we put the equation with respective associated parameters on each face of the cube and initial values are given at vertices w_0, w_1, w_2, w_3 , we obtain the same value of w_{123} from the three possible ways of computing it [3, 24]. This property is called consistency around a cube, or CAC.

The Lax pair of the lattice mKdV equation is well known [24]. In Appendix A, we provide a derivation of the following Lax pair whose compatibility condition, namely $\mathbf{M}(w_1, w_{12}, \beta_1, \beta_2, x)\mathbf{L}(w_0, w_1, \alpha_1, \alpha_2, x) = \mathbf{L}(w_2, w_{12}, \alpha_1, \alpha_2, x)\mathbf{M}(w_0, w_2, \beta_1, \beta_2, x)$ is the multi-parametric equation (3.1):

$$\begin{cases} \mathbf{L}(w_0, w_1, \alpha_1, \alpha_2, x) = \begin{pmatrix} -\frac{\alpha_1 w_0}{w_1} x & l_1 \\ l_2 & -\frac{\alpha_2 w_1}{w_0} x \end{pmatrix}, \\ \mathbf{M}(w_0, w_2, \beta_1, \beta_2, x) = \begin{pmatrix} -\frac{\beta_1 w_0}{w_2} x & l_1 \\ l_2 & -\frac{\beta_2 w_2}{w_0} x \end{pmatrix}, \end{cases} \quad (3.2)$$

where we consider l_1 and l_2 as constants and x as a spectral variable.

3.1. qP_{II} hierarchy

In this section, suppose $n \geq 2$ is a given, fixed integer. We will derive the qP_{II} hierarchy by using the $(n, 1)$ -reduction of equation (3.1) i.e., by imposing $w_{l,m} = w_{l+n,m+1}$ and taking $u_i = w_{l,m}$ where $i = l - nm$ [37, 41].

Figure 3.2 represents this reduction with associated parameters, which are different on each edge. Consider the last edge on the horizontal line in this figure, which joins u_n to u_{n+1} . We assume that the corresponding parameters are given by $(q\alpha_{0,1}, q\alpha_{0,2})$. This can be seen as a non-autonomous reduction of the general mKdV.

On the quadrilateral on the right of Figure 3.2, we have the equation $Q(u_n, u_{n+1}, u_0, u_1; q\alpha_{0,1}, q\alpha_{0,2}, \beta_1, \beta_2) = 0$, which is given by

$$q\alpha_{0,1}u_nu_0 - q\alpha_{0,2}u_{n+1}u_1 - \beta_1u_nu_{n+1} + \beta_2u_0u_1 = 0.$$

We can solve for u_{n+1} from this equation to find the shift map

$$\begin{aligned} S : (u_0, u_1, \dots, u_{n-1}, u_n; \alpha_{0,i}, \dots, \alpha_{n-1,i}) \\ \longmapsto (u_1, u_2, \dots, u_n, u_{n+1}; \alpha_{1,i}, \dots, \alpha_{n-1,i}, q\alpha_{0,i}) \end{aligned} \quad (3.3)$$

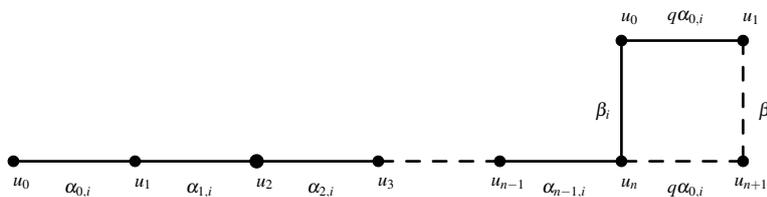


Fig. 3.2 The $(n, 1)$ -reduction of mKdV associated with the qP_{II} hierarchy.

where $i = 1, 2$ and

$$u_{n+1} = \frac{u_0 (q\alpha_{0,1}u_n + \beta_2 u_1)}{q\alpha_{0,2}u_1 + \beta_1 u_n}.$$

Let $h_j = u_{j+1}/u_j$ for $j = 0, 1, 2, \dots, n - 1$, then we obtain the map

$$\begin{aligned} \tilde{T} : (h_0, h_1, \dots, h_{n-1}, \alpha_{0,i}, \dots, \alpha_{n-1,i}) \\ \mapsto (h_1, h_2, \dots, h_{n-1}, \tilde{T}(h_{n-1}), \alpha_{1,i}, \dots, \alpha_{n-1,i}, q\alpha_{0,i}), \end{aligned} \tag{3.4}$$

where

$$\tilde{T}(h_{n-1}) := \tilde{h}_{n-1} = \frac{q\alpha_{0,1}h_1h_2 \dots h_{n-1} + \beta_2}{h_0h_1 \dots h_{n-1} (q\alpha_{0,2} + \beta_1h_1h_2 \dots h_{n-1})}. \tag{3.5}$$

We note that the map (3.4) with (3.5) is a general version of the qP_{II} hierarchy given in Theorem 1.1.

Now we will construct the Lax pair for the map (3.4) from the reduction described in Figure 3.2. We start by constructing a monodromy matrix associated with the $(n, 1)$ -reduction (see [41]). Walking along the vertices labelled by u_0 in Figure 3.2, we have steps taken along horizontal edges, which are represented by \mathbf{L} , and one step up in the vertical, which is represented by \mathbf{M} , leading to the composition:

$$\begin{aligned} \mathcal{L} = (\mathbf{M}(u_n, u_0, \beta_1, \beta_2, x))\mathbf{L}(u_{n-1}, u_n, \alpha_{n-1,1}, \alpha_{n-1,2}, x)\mathbf{L}(u_{n-2}, u_{n-1}, \alpha_{n-2,1}, \alpha_{n-2,2}, x) \\ \dots \mathbf{L}(u_1, u_2, \alpha_{1,1}, \alpha_{1,2}, x)\mathbf{L}(u_0, u_1, \alpha_{0,1}, \alpha_{0,2}, x). \end{aligned} \tag{3.6}$$

This matrix \mathcal{L} is called a monodromy matrix.

Applying the deformation \tilde{T} on \mathcal{L} we have

$$\begin{aligned} \tilde{T}(\mathcal{L}) = (\mathbf{M}(u_{n+1}, u_1, \beta_1, \beta_2, x))\mathbf{L}(u_n, u_{n+1}, q\alpha_{0,1}, q\alpha_{0,2}, x)\mathbf{L}(u_{n-1}, u_n, \alpha_{n-1,1}, \alpha_{n-1,2}, x) \\ \dots \mathbf{L}(u_2, u_3, \alpha_{2,1}, \alpha_{2,2}, x)\mathbf{L}(u_1, u_2, \alpha_{1,1}, \alpha_{1,2}, x). \end{aligned} \tag{3.7}$$

The Lax pair for the quad-equation $Q(u_n, u_{n+1}, u_0, u_1, q\alpha_{0,1}, q\alpha_{0,2}, \beta_1, \beta_2) = 0$ is given by

$$\begin{aligned} \mathbf{M}(u_{n+1}, u_1, \beta_1, \beta_2, x)\mathbf{L}(u_n, u_{n+1}, q\alpha_{0,1}, q\alpha_{0,2}, x) \\ = \mathbf{L}(u_0, u_1, q\alpha_{0,1}, q\alpha_{0,2}, x)\mathbf{M}(u_n, u_0, \beta_1, \beta_2, x). \end{aligned}$$

Substitute this in equation (3.7), we get

$$\begin{aligned} \tilde{T}(\mathcal{L}) &= \mathbf{L}(u_0, u_1, q\alpha_{0,1}, q\alpha_{0,2}, x)\mathbf{M}(u_n, u_0, \beta_1, \beta_2, x)\mathbf{L}(u_{n-1}, u_n, \alpha_{n-1,1}, \alpha_{n-1,2}, x) \\ &\quad \dots \mathbf{L}(u_2, u_3, \alpha_{2,1}, \alpha_{2,2}, x)\mathbf{L}(u_1, u_2, \alpha_{1,1}, \alpha_{1,2}, x), \\ &= \mathbf{L}(u_0, u_1, q\alpha_{0,1}, q\alpha_{0,2}, x)\mathcal{L}(\mathbf{L}(u_0, u_1, \alpha_{0,1}, \alpha_{0,2}, x))^{-1}, \\ &= \mathbf{L}(u_0, u_1, \alpha_{0,1}, \alpha_{0,2}, qx)\mathcal{L}(\mathbf{L}(u_0, u_1, \alpha_{0,1}, \alpha_{0,2}, x))^{-1}, \end{aligned}$$

where we have used $\mathbf{L}(\alpha_i, qx) = \mathbf{L}(q\alpha_i, x)$.

Thus, we get

$$\tilde{T}(\mathcal{L})\mathbf{L}(u_0, u_1, \alpha_{0,1}, \alpha_{0,2}, x) = \mathbf{L}(u_0, u_1, \alpha_{0,1}, \alpha_{0,2}, qx)\mathcal{L}.$$

Letting $A = \mathcal{L}$ and $B = \mathbf{L}(u_0, u_1, \alpha_{0,1}, \alpha_{0,2}, x)$, we obtain a Lax pair of qP_{II} hierarchy.

Taking the parameters as the following

$$\beta_1 = ib_n/\lambda, \quad \beta_2 = ib_n\lambda, \quad \alpha_{l,1} = ib_l\lambda, \quad \text{and} \quad \alpha_{l,2} = ib_l/\lambda, \quad (3.8)$$

where $l = 0, 1, \dots, n - 1$, and $b_n = q$, we obtain the hierarchy of qP_{II} given by (1.8).

3.2. qP_{III} hierarchy

First of all, we notice that the deformation operator of qP_{III} (\widehat{T}) is a two-fold composition of the deformation operator of qP_{II} (\tilde{T}), (i.e. $\widehat{T} = \tilde{T}^2$). Hence, we use the same methods with one extra iteration of the reduction and the corresponding Lax matrices \mathbf{L} and \mathbf{M} .

We consider the hierarchy of qP_{III} starting with $(n, 1)$ -reduction, and it can be described by the following Figure:

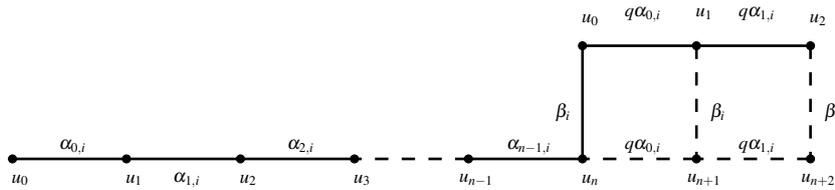


Fig. 3.3 The $(n, 1)$ -reduction of mKdV associated with the qP_{III} hierarchy.

On the quadrilateral on the right of Figure 3.3, we have the following equations

$$\begin{aligned} Q(u_n, u_{n+1}, u_0, u_1; q\alpha_{0,i}, \beta_i) &= 0, \\ Q(u_{n+1}, u_{n+2}, u_1, u_2; q\alpha_{1,i}, \beta_i) &= 0, \end{aligned}$$

which are given by

$$q\alpha_{0,1}u_nu_0 - q\alpha_{0,2}u_{n+1}u_1 - \beta_1u_nu_{n+1} + \beta_2u_0u_1 = 0, \quad (3.9)$$

$$q\alpha_{1,1}u_{n+1}u_1 - q\alpha_{1,2}u_{n+2}u_2 - \beta_1u_{n+1}u_{n+2} + \beta_2u_1u_2 = 0. \quad (3.10)$$

We can solve for u_{n+1} and u_{n+2} from these two equations to find the map

$$\begin{aligned} &(u_0, u_1, \dots, u_{n-1}, u_n; \alpha_{0,i}, \alpha_{1,i}, \dots, \alpha_{n-1,i}) \\ &\longmapsto (u_2, u_3, \dots, u_{n+1}, u_{n+2}; \alpha_{2,i}, \alpha_{3,i}, \dots, \alpha_{n-1,i}, q\alpha_{0,i}, q\alpha_{1,i}). \end{aligned}$$

Similar to the hierarchy of qP_{II} , we define $h_j = \frac{u_{j+1}}{u_j}$ where $j = 0, 1, 2, \dots, n - 1$. Taking parameters as (3.8), where $b_n = q$, then we obtain the map

$$\begin{aligned} \widehat{T} &: (h_0, h_1, \dots, h_{n-2}, h_{n-1}; b_0, b_1, \dots, b_{n-2}, b_{n-1}, b_n) \\ &\longmapsto (h_2, h_3, \dots, \widehat{T}(h_{n-2}), \widehat{T}(h_{n-1}); b_2, b_3, \dots, qb_0, qb_1, b_n), \end{aligned} \quad (3.11)$$

where

$$\begin{aligned} \widehat{T}(h_{n-2}) &:= \widehat{h}_{n-2} = \frac{\lambda^2(b_0 \prod_{i=1}^{n-1} h_i + 1)}{\prod_{i=0}^{n-1} h_i (b_0 + \prod_{i=1}^{n-1} h_i)}, \\ \widehat{T}(h_{n-1}) &:= \widehat{h}_{n-1} = \frac{\lambda^2(b_1 \widehat{h}_{n-2} \prod_{i=2}^{n-1} h_i + 1)}{\widehat{h}_{n-2} \prod_{i=1}^{n-1} h_i (b_1 + \widehat{h}_{n-2} \prod_{i=2}^{n-1} h_i)}. \end{aligned} \tag{3.12}$$

We note that the map (3.11) with (3.12) is the qP_{III} hierarchy given in Theorem 1.1.

Now as in the qP_{II} hierarchy, we will construct the Lax pair for the map (3.11) from the reduction described in Figure 3.3. Similarly, the monodromy matrix associated with the $(n, 1)$ -reduction is given by

$$\begin{aligned} \mathcal{L} &= \mathbf{M}(u_n, u_0, \beta_1, \beta_2, x) \mathbf{L}(u_{n-1}, u_n, \alpha_{n-1,1}, \alpha_{n-1,2}, x) \mathbf{L}(u_{n-2}, u_{n-1}, \alpha_{n-2,1}, \alpha_{n-2,2}, x) \\ &\dots \mathbf{L}(u_1, u_2, \alpha_{1,1}, \alpha_{1,2}, x) \mathbf{L}(u_0, u_1, \alpha_{0,1}, \alpha_{0,2}, x). \end{aligned} \tag{3.13}$$

Applying the deformation \widehat{T} on \mathcal{L} we have

$$\begin{aligned} \widehat{T}(\mathcal{L}) &= \mathbf{M}(u_{n+2}, u_2, \beta_1, \beta_2, x) \mathbf{L}(u_{n+1}, u_{n+2}, q\alpha_{1,1}, q\alpha_{1,2}, x) \mathbf{L}(u_n, u_{n+1}, q\alpha_{0,1}, q\alpha_{0,2}, x) \\ &\dots \mathbf{L}(u_3, u_4, \alpha_{3,1}, \alpha_{3,2}, x) \mathbf{L}(u_2, u_3, \alpha_{2,1}, \alpha_{2,2}, x). \end{aligned} \tag{3.14}$$

The Lax pairs for the quad-equations

$$Q(u_n, u_{n+1}, u_0, u_1; q\alpha_{0,1}, q\alpha_{0,2}, \beta_1, \beta_2) = 0,$$

and

$$Q(u_{n+1}, u_{n+2}, u_1, u_2; q\alpha_{1,1}, q\alpha_{1,2}, \beta_1, \beta_2) = 0,$$

satisfy

$$\begin{aligned} \mathbf{M}(u_{n+1}, u_1, \beta_1, \beta_2, x) \mathbf{L}(u_n, u_{n+1}, q\alpha_{0,1}, q\alpha_{0,2}, x) \\ = \mathbf{L}(u_0, u_1, q\alpha_{0,1}, q\alpha_{0,2}, x) \mathbf{M}(u_n, u_0, \beta_1, \beta_2, x), \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} \mathbf{M}(u_{n+2}, u_2, \beta_1, \beta_2, x) \mathbf{L}(u_{n+1}, u_{n+2}, q\alpha_{1,1}, q\alpha_{1,2}, x) \\ = \mathbf{L}(u_1, u_2, q\alpha_{1,1}, q\alpha_{1,2}, x) \mathbf{M}(u_{n+1}, u_1, \beta_1, \beta_2, x). \end{aligned} \tag{3.16}$$

Using (3.16) and (3.15) in equation (3.14), we get

$$\begin{aligned} \widehat{T}(\mathcal{L}) &= \mathbf{L}(u_1, u_2, q\alpha_{1,1}, q\alpha_{1,2}, x) \mathbf{M}(u_{n+1}, u_1, \beta_1, \beta_2, x) \mathbf{L}(u_n, u_{n+1}, q\alpha_{0,1}, q\alpha_{0,2}, x) \\ &\dots \mathbf{L}(u_3, u_4, \alpha_{3,1}, \alpha_{3,2}, x) \mathbf{L}(u_2, u_3, \alpha_{2,1}, \alpha_{2,2}, x), \\ &= L(u_1, u_2, q\alpha_{1,1}, q\alpha_{1,2}, x) \mathbf{L}(u_0, u_1, q\alpha_{0,1}, q\alpha_{0,2}, x) \mathbf{M}(u_n, u_0, \beta_1, \beta_2, x) \\ &\quad \mathbf{L}(u_{n-1}, u_n, \alpha_{n-1,1}, \alpha_{n-1,2}, x) \dots \mathbf{L}(u_2, u_3, \alpha_{2,1}, \alpha_{2,2}, x) \\ &= \mathbf{L}(u_1, u_2, q\alpha_{1,1}, q\alpha_{1,2}, x) \mathbf{L}(u_0, u_1, q\alpha_{0,1}, q\alpha_{0,2}, x) \mathcal{L} (\mathbf{L}(u_0, u_1, \alpha_{0,1}, \alpha_{0,2}, x))^{-1} \\ &\quad (\mathbf{L}(u_1, u_2, \alpha_{1,1}, \alpha_{1,2}, x))^{-1}, \end{aligned}$$

where we have used $\mathbf{L}(\alpha_i, qx) = \mathbf{L}(q\alpha_i, x)$.

Thus, we obtain

$$\widehat{T}(\mathcal{L})\mathbf{L}(u_1, u_2, \alpha_{1,1}, \alpha_{1,2}, x)\mathbf{L}(u_0, u_1, \alpha_{0,1}, \alpha_{0,2}, x) = \mathbf{L}(u_1, u_2, \alpha_{1,1}, \alpha_{1,2}, qx)\mathbf{L}(u_0, u_1, \alpha_{0,1}, \alpha_{0,2}, qx)\mathcal{L}.$$

If we take $A = \mathcal{L}$, $B = \mathbf{L}(u_1, u_2, \alpha_{1,1}, \alpha_{1,2}, x)\mathbf{L}(u_0, u_1, \alpha_{0,1}, \alpha_{0,2}, x)$, we have the Lax pair of qP_{III} hierarchy.

Remark 3.1. The entries (1,2) and (2,1) of matrix A_j , which we choose to be 1 and -1 in the matrices, can be replaced with any constants and the compatibility condition still holds.

4. Bäcklund Transformations of the hierarchies

One of the interpretations of CAC property is a connection with Bäcklund transformations. The CAC property can be regarded as a Bäcklund transformation between a top and a bottom equation in a cube [5]. It can be described as follows.

We consider the following quad-equation which is CAC

$$Q(u_0, u_1, u_2, u_{12}; \alpha, \beta) = 0. \tag{4.1}$$

We then embed this equation (4.1) in the third direction associated with variables v and lattice parameters γ , this parameter will be a Bäcklund transformation parameter (see Figure 3.1 where w_3 is replaced with v). A Bäcklund transformation between two equations which are depicted by the top and bottom faces in the cube is given by

$$\begin{aligned} Q(u_0, u_1, v_0, v_1; \alpha, \gamma) &= 0, \\ Q(u_0, u_2, v_0, v_2; \beta, \gamma) &= 0. \end{aligned}$$

We note that this is an auto-Bäcklund transformation as the top and bottom equations are the same.

In this section, we use the Bäcklund transformation of the lattice mKdV to derive the Bäcklund transformation for the qP_{II} and qP_{III} hierarchies. Moreover, we give a few examples of some rational solutions for second and fourth order.

4.1. Bäcklund transformation of the qP_{II} hierarchy

To find a Bäcklund transformation for the qP_{II} hierarchy, we embed the $(n, 1)$ periodic reduction in three dimensions with a slight modification. A third direction in the $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ lattice is associated with parameters γ_1, γ_2 and variables v_i 's. Instead of imposing the $(n, 1)$ periodic reduction for the v 's variables, we use a twist reduction $v_{l+n, m+1} = v_{l, m}/d$ cf. [8]. This is because we want to create a Bäcklund transformation between two equations with different parameters. Parameters along the staircase of the v variables are as the same as the ones we use for the u variables. For example the $(2, 1)$ -reduction in three dimension corresponding to the first member in the qP_{II} hierarchy can be described in Figure 4.1.

We now illustrate a method of finding a Bäcklund transformation for the qP_{II} hierarchy by using the base case qP_{II} which is associated with $n = 2$.

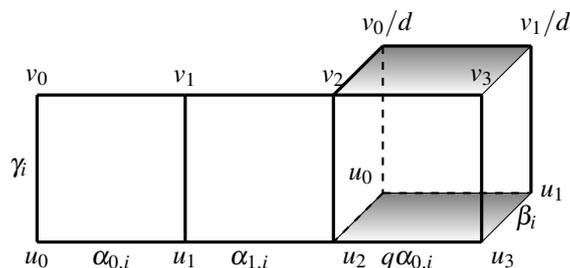


Fig. 4.1 Bäcklund transformation for qP_{II} .

The twisted reduction for variables v 's gives the top shaded equation which is given by

$$Q(v_2, v_3, v_0/d, v_1/d; q\alpha_{0,1}, q\alpha_{0,2}, \beta_1, \beta_2) = q\alpha_{0,1}v_2v_0/d - q\alpha_{0,2}v_3v_1/d - \beta_1v_2v_3 + \beta_2v_0v_1/d^2 = 0. \tag{4.2}$$

This gives us a shift map

$$S_v : (v_0, v_1, v_2, \alpha_{0,i}, \alpha_{1,i}, \beta_i) \mapsto (v_1, v_2, v_3, \alpha_{1,i}, q\alpha_{0,i}, \beta_i), \quad \text{with } i = 1, 2, \tag{4.3}$$

where v_3 can be solved from (4.2).

As we discussed above, a Bäcklund transformation can be inherited from CAC. Thus, a Bäcklund transformation between the shaded equations in Figure (4.1) is the following system

$$\begin{aligned} Q(u_0, u_1, v_0, v_1; \alpha_{0,1}, \alpha_{0,2}, \gamma_1, \gamma_2) &= 0, \\ Q(u_1, u_2, v_1, v_2; \alpha_{1,1}, \alpha_{1,2}, \gamma_1, \gamma_2) &= 0, \\ Q(u_2, u_0, v_2, v_0/d; \beta_1, \beta_2, \gamma_1, \gamma_2) &= 0. \end{aligned} \tag{4.4}$$

We consider this system as the system of variables v_0, v_1, v_2 . By writing v_2 and v_1 in terms of v_0 , we obtain a quadratic equation in v_0 which leads to non-rational solutions in general. However, if we take $\gamma_2 = 0$ then system (4.4) becomes a linear system and it can be solved uniquely with the following solution

$$\begin{aligned} v_0 &= \frac{\gamma_1 d (\alpha_{0,2} \alpha_{1,2} u_0 u_2 + \alpha_{0,2} \beta_1 u_1 u_2 + \alpha_{1,1} \beta_1 u_0 u_1)}{(d \alpha_{0,1} \alpha_{1,1} \beta_1 - \alpha_{0,2} \alpha_{1,2} \beta_2) u_0}, \\ v_1 &= \frac{\gamma_1 (d \alpha_{0,1} \alpha_{1,2} u_0 u_2 + d \alpha_{0,1} \beta_1 u_1 u_2 + \alpha_{1,2} \beta_2 u_0 u_1)}{(d \alpha_{0,1} \alpha_{1,1} \beta_1 - \alpha_{0,2} \alpha_{1,2} \beta_2) u_1}, \\ v_2 &= \frac{\gamma_1 (d \alpha_{0,1} \alpha_{1,1} u_0 u_2 + \alpha_{0,2} \beta_2 u_1 u_2 + \alpha_{1,1} \beta_2 u_0 u_1)}{u_2 (d \alpha_{0,1} \alpha_{1,1} \beta_1 - \alpha_{0,2} \alpha_{1,2} \beta_2)}. \end{aligned}$$

This defines the Bäcklund transformation $BT : (u_0, u_1, u_2) \mapsto (v_0, v_1, v_2)$. We note that the Bäcklund transformation should be compatible with the shift map S_v and S , i.e. we have

$$S_v \circ BT = BT \circ S.$$

This implies $d = q$. Similar to the qP_{II} equation, we introduce $g_i = \frac{v_{i+1}}{v_i}$ for $i = 0, 1$. Using the parameters given in (3.8), we obtain the equation

$$\tilde{g}_1 g_1 = \frac{\lambda^2 (b_0 q g_1 + 1)}{q g_1 (b_0 + q g_1)}, \quad \text{where } b_0 = t. \tag{4.5}$$

This suggests that we introduce $H_i = qg_i$ for $i = 0, 1, 2$, in which case equation (4.5) can be written as

$$\tilde{H}_1 \tilde{H}_1 = \frac{q^2 \lambda^2 (tH_1 + 1)}{H_1(t + H_1)}.$$

Therefore, H_1 satisfies qP_{II} , with parameter $\lambda^2 q^2$ instead of λ^2 . Hence, the transformation from h_i to H_i , where $h_i = u_{i+1}/u_i$, and $i = 0, 1, 2$, defines the Bäcklund transformation for qP_{II} . Thus, we can write the Bäcklund transformation of $qP_{II}^{(2)}$ as

$$H_1 = qg_1 = q \frac{v_2}{v_1} = \frac{q^{1/2}(q^{1/2} t \lambda^2 h_1 + q^{1/2} \lambda^2 + h_1 \tilde{h}_1)}{h_1 (q^{1/2} h_1 \tilde{h}_1 + t h_1 + 1)}, \quad \text{by taking } b_1 = q^{1/2} t, \quad (4.6)$$

where $H_1 = H(t)$, $h_1 = h_1(t)$ and $\tilde{h}_1 = h_1(q^{1/2}t)$. This is equivalent to the known Bäcklund transformation for qP_{II} [27].

We can see that this transformation produces a solution, H of qP_{II} with parameter $\lambda^2 q^2$ from a solution, h corresponding to λ^2 .

Proposition 4.1. *The simplest rational solution (seed solution) of the qP_{II} and qP_{III} hierarchies, equations (1.8) and (1.11), respectively is*

$$h_l = \pm 1, \quad l = 0, 1, \dots, n, \quad \text{and} \quad \lambda = \pm 1. \quad (4.7)$$

This can be checked by substituting the above values into (1.8) and (1.11), respectively. Applying the Bäcklund transformation (equation (4.6)) repeatedly on the seed solution will give us an infinite number of solutions of equation (2.1) for $\lambda^2 \mapsto q^2 \lambda^2$. In the next few examples we will use the notation $h(t; \lambda^2)$ to refer to a solution of equation corresponding to λ^2 .

Example 4.1. Let us look at few examples of rational solutions of $qP_{II}^{(2)}$. Starting with $\lambda^2 = 1$, we have

$$\begin{aligned} h_1(t; 1) &= 1, \\ h_1(t; p^4) &= p \frac{P_0(pt)}{P_0(t)}, \\ h_1(t; p^8) &= \frac{p^4 P_0(t) P_1(t)}{P_0(pt) P_2(t)}, \\ h_1(t; p^{12}) &= \frac{p^5 P_0(pt) P_2(t) \left(P_0(p^2 t) P_2(t) P_2(pt) + p^5 t P_0(pt) P_1(pt) P_2(t) + p^9 P_0(t) P_1(t) P_1(pt) \right)}{P_0(t) P_1(t) \left(p^9 P_0(p^2 t) P_2(t) P_2(pt) + p^4 t P_0(pt) P_1(pt) P_2(t) + p^8 P_0(t) P_1(t) P_1(pt) \right)}, \end{aligned}$$

where $p^2 = q$ and

$$\begin{aligned} P_0(t) &= 1 + p + t, \\ P_1(t) &= 1 + p^3 \frac{P_0(p^2 t)}{P_0(t)} + p^2 t \frac{P_0(p^2 t)}{P_0(pt)}, \\ P_2(t) &= p^5 + p^2 \frac{P_0(p^2 t)}{P_0(t)} + pt \frac{P_0(p^2 t)}{P_0(pt)}. \end{aligned}$$

We note that we can start with another seed solution $h_1 = -1$ so, we have another infinite solution for $qP_{II}^{(2)}$ which was already generated in [44].

Following the same method above, a Bäcklund transformation for the n^{th} member of the qP_{II} hierarchy (Figure 3.2) is given by the system below

$$\begin{aligned} Q(u_0, u_1, v_0, v_1, \alpha_{0,1}, \alpha_{0,2}, \gamma_1, \gamma_2) &= 0, \\ &\vdots \\ Q(u_{n-1}, u_n, v_{n-1}, v_n, \alpha_{n-1,1}, \alpha_{n-1,2}, \gamma_1, \gamma_2) &= 0, \\ Q(u_n, u_0, v_n, v_0/d, \beta_1, \beta_2, \gamma_1, \gamma_2) &= 0. \end{aligned}$$

Taking $\gamma_2 = 0$, we obtain a linear system of v_0, v_1, \dots, v_n . We also want that this Bäcklund transformation is compatible with the shift map; thus $d = q$.

Let $g_i = v_{i+1}/v_i$, and let the parameters be as given in (3.8). Then, we obtain a Bäcklund transformation of equation (1.8), which is

$$g_i h_i = \mathbf{N}/\mathbf{D}, \tag{4.8}$$

where

$$\begin{aligned} \mathbf{N} &= \frac{1}{q\lambda^{2n-2} - 1} \sum_{j=0}^{n-1} \frac{h_j \prod_{s=0}^{j-1} h_s^2}{b_j \lambda^{2j-2i-1}} + \sum_{j=i+1}^{n-1} \frac{h_j \prod_{s=0}^{j-1} h_s^2}{b_j \lambda^{2j-2i-1}} + \frac{\lambda^{2i+1} \prod_{s=0}^{n-1} h_s}{q\lambda^{2n-2} - 1}, \\ \mathbf{D} &= \frac{1}{q\lambda^{2n-2} - 1} \sum_{j=0}^{n-1} \frac{h_j \prod_{s=0}^{j-1} h_s^2}{b_j \lambda^{2j-2i+1}} + \sum_{j=i}^{n-1} \frac{h_j \prod_{s=0}^{j-1} h_s^2}{b_j \lambda^{2j-2i+1}} + \frac{\lambda^{2i-1} \prod_{s=0}^{n-1} h_s}{q\lambda^{2n-2} - 1}, \end{aligned}$$

and the following equation

$$g_n = \frac{\lambda^2 (qb_0 g_1 g_2 \dots g_{n-1} + 1)}{qg_0 g_1 \dots g_{n-1} (qg_1 g_2 \dots g_{n-1} + b_0)}. \tag{4.9}$$

To obtain the form of $qP_{\text{II}}^{(n)}$, we take $H_i = q^{\frac{1}{n-1}} g_i$ to cancel out each q present in the above equation. Thus, we obtain a Bäcklund transformation $h_i \mapsto H_i$ of (1.8) where $\lambda^2 \mapsto q^{\frac{2}{n-1}} \lambda^2$.

Proposition 4.2. *Let h_i be a solution of $qP_{\text{II}}^{(n)}$ (equation (1.8)) with parameter λ^2 . Then*

$$H_i = q^{\frac{1}{n-1}} g_i, \quad \text{where } g_i \text{ is given by equation (4.8), } (i = 0, 1, 2, \dots, n)$$

is also a solution of $qP_{\text{II}}^{(n)}$, with parameter $q^{\frac{2}{n-1}} \lambda^2$.

Corollary 4.1. *The Bäcklund transformation of $qP_{\text{II}}^{(4)}$ takes $h_i \mapsto H_i = q^{1/3} g_i$, where $i = 0, 1, 2, 3, 4$, $\lambda^2 \mapsto \lambda^2 q^{2/3}$ and*

$$H_2 = \frac{q^{1/12} \left(q^{1/2} \lambda^4 + q^{1/4} \lambda^2 \underline{h}_2 \underline{h}_2 + h_2 \underline{h}_2^2 \underline{h}_2 + q^{1/2} \lambda^4 t \underline{h}_2 \underline{h}_2 \tilde{h}_2 + q^{3/4} \lambda^4 \underline{h}_2 \underline{h}_2^2 \underline{h}_2^2 \tilde{h}_2 \right)}{h_2 \left(q^{1/4} \lambda^2 + \underline{h}_2 \underline{h}_2 + q^{3/4} \lambda^4 \underline{h}_2 \underline{h}_2^2 \underline{h}_2 + q^{1/4} \lambda^2 t \underline{h}_2 \underline{h}_2 \tilde{h}_2 + q^{1/2} \lambda^2 \underline{h}_2 \underline{h}_2^2 \underline{h}_2^2 \tilde{h}_2 \right)}, \tag{4.10}$$

where $H_2 = H_2(t), h_2 = h_2(t)$ and $\tilde{h}_2 = h_2(q^{1/4}t)$.

Example 4.2. We generate the first three rational solutions of the $qP_{\text{II}}^{(4)}$. Starting with $\lambda^2 = 1$, we have

$$\begin{aligned} h_2(t; 1) &= 1, \\ h_2(t; p^{\frac{8}{3}}) &= \frac{p^{1/3}P(p^2t)}{P(pt)}, \\ h_2(t; p^{\frac{16}{3}}) &= \frac{p^{2/3}P(pt)\left(p^3Q(t) + Q(pt) + p^9Q(p^2t) + p^5tP(p^4t)P(t) + p^6Q(p^3t)\right)}{P(p^2t)\left(Q(t) + p^9Q(pt) + p^6Q(p^2t) + p^2tP(p^4t)P(t) + p^3Q(p^3t)\right)}, \end{aligned}$$

where $p^4 = q$ and

$$\begin{aligned} P(t) &= 1 + p + p^2 + p^3 + t, \\ Q(t) &= P(t)P(pt). \end{aligned}$$

4.2. Bäcklund transformation of the qP_{III} hierarchy

We have the qP_{III} hierarchy given by equation (1.11). We can apply the method given in Subsection 4.1 to the qP_{III} hierarchy because Figure 3.3 is the same as Figure 3.2 with the square on the right extended one step. Hence, we can deduce the Bäcklund transformation of the qP_{III} hierarchy.

Proposition 4.3. Let h_i be a solution of $qP_{\text{III}}^{(n)}$ (equation (1.11)) with parameter λ^2 . Then

$$H_i = q^{\frac{1}{n-1}} g_i, \quad (i = 0, 1, 2, \dots, n),$$

is also a solution of $qP_{\text{III}}^{(n)}$, with parameter $q^{\frac{2}{n-1}} \lambda^2$ where

$$g_i = \begin{cases} \frac{1}{h_0} \frac{N_0}{D_0}, & i = 0, \\ \frac{1}{h_i} \frac{N_1}{D_1}, & i = 1, \dots, n-1, \\ \frac{1}{h_n} \frac{N_2}{D_2}, & i = n, \end{cases} \tag{4.11}$$

and

$$\begin{aligned} N_0 &= \frac{q}{(q\lambda^{2n-2} - 1)} \left(\sum_{j=1}^{n-1} \frac{h_j \prod_{s=0}^{j-1} h_s^2}{b_j \lambda^{2j-2n+1}} + \frac{h_0 \prod_{s=0}^n h_s}{q\lambda} + \frac{h_n \prod_{s=0}^{n-1} h_s^2}{qb_0 \lambda} \right), \\ D_0 &= \frac{q}{(q\lambda^{2n-2} - 1)} \left(\sum_{j=1}^{n-1} \frac{h_j \prod_{s=0}^{j-1} h_s^2}{b_j \lambda^{2j-2n+3}} + \frac{h_0 \prod_{s=0}^n h_s}{q\lambda^3} + \frac{h_n \prod_{s=0}^{n-1} h_s^2}{qb_0 \lambda^3} \right) + \frac{h_0}{b_0 \lambda}, \\ N_1 &= \frac{1}{q\lambda^{2n-2} - 1} \sum_{j=1}^{n-1} \frac{h_j \prod_{s=0}^{j-1} h_s^2}{b_j \lambda^{2j-2i-1}} + \sum_{j=i+1}^{n-1} \frac{h_j \prod_{s=0}^{j-1} h_s^2}{b_j \lambda^{2j-2i-1}} + \frac{h_n \prod_{s=0}^{n-1} h_s^2}{b_0 \lambda^{1-2i}} + \frac{h_0 \prod_{s=0}^n h_s}{\lambda^{1-2i}}, \\ D_1 &= \frac{1}{q\lambda^{2n-2} - 1} \sum_{j=1}^{n-1} \frac{h_j \prod_{s=0}^{j-1} h_s^2}{b_j \lambda^{2j-2i+1}} + \sum_{j=i}^{n-1} \frac{h_j \prod_{s=0}^{j-1} h_s^2}{b_j \lambda^{2j-2i+1}} + \frac{h_n \prod_{s=0}^{n-1} h_s^2}{b_0 \lambda^{3-2i}} + \frac{h_0 \prod_{s=0}^n h_s}{\lambda^{3-2i}}, \end{aligned}$$

$$N_2 = \frac{1}{(q\lambda^{2n-2} - 1)} \left(\sum_{j=1}^{n-1} \frac{h_j \prod_{s=0}^{j-1} h_s^2}{b_j \lambda^{2j-2n-1}} + \lambda^{2n-1} h_0 \prod_{s=0}^n h_s + \frac{\lambda h_n \prod_{s=0}^{n-1} h_s^2}{qb_0} \right),$$

$$D_2 = \frac{1}{q\lambda^{2n-2} - 1} \sum_{j=1}^{n-1} \frac{h_j \prod_{s=0}^{j-1} h_s^2}{b_j \lambda^{2j-2n+1}} + \frac{h_n \prod_{s=0}^{n-1} h_s^2}{b_0 \lambda^{3-2n}} + \frac{h_0 \prod_{s=0}^n h_s}{\lambda^{3-2n}}.$$

This proposition provides us with the following Bäcklund transformation for $qP_{III}^{(n)}$

$$h_i \mapsto H_i, \text{ and } \lambda^2 \mapsto q^{\frac{2}{n-1}} \lambda^2. \tag{4.12}$$

Corollary 4.2. The Bäcklund transformation of $qP_{III}^{(2)}$ is given by

$$\begin{aligned} H_0 &= \frac{\lambda^2 p (b t h_1 + b + q h_0 h_1)}{h_0 (b t h_1 + \lambda^2 b q + q h_0 h_1)}, \\ H_1 &= \frac{q (\lambda^2 b t h_1 + \lambda^2 b + h_0 h_1)}{h_1 (b t h_1 + b + q h_0 h_1)}, \end{aligned} \tag{4.13}$$

where $H_0 = H_0(t), H_1 = H_1(t), h_0 = h_0(t)$ and $h_1 = h_1(t)$. This system of two equations provide us a solution of qP_{III} with $\lambda^2 \mapsto q^2 \lambda^2$.

Example 4.3. We consider solutions of $qP_{III}^{(2)}$ which are given when $\lambda^2 \mapsto q^2 \lambda^2$. Starting with $\lambda^2 = 1$, we have

$$\begin{aligned} h_0(t; 1) &= 1 \quad \text{and} \quad h_1(t; 1) = 1, \\ h_0(t; q^2) &= \frac{qR_0(t)}{R_1(t)} \quad \text{and} \quad h_1(t; q^2) = \frac{q(1 + b + bt)}{R_0(t)}, \\ h_0(t; q^4) &= \frac{q^2 R_1(t)(bt + R_2(t) + qR_3(t))}{R_0(t)(bq^3 t + R_2(t) + qR_3(t))} \quad \text{and} \quad h_1(t; q^4) = \frac{R_0(t)(bq^2 t + q^2 R_2(t) + R_3(t))}{(1 + b + bt)(bt + R_2(t) + qR_3(t))}, \end{aligned}$$

where

$$\begin{aligned} R_0(t) &= bt + b + q \quad \text{and} \quad R_1(t) = bq + bt + q, \\ R_2(t) &= \frac{bqt^2(1 + b + bt)}{R_0(t)} \quad \text{and} \quad R_3(t) = \frac{q^2 t(1 + b + bt)}{R_1(t)}. \end{aligned}$$

In the next Corollary we give the explicit form of Bäcklund transformation for the fourth-order of the qP_{III} equation.

Corollary 4.3. The Bäcklund transformation of $qP_{III}^{(4)}$ is given by

$$\begin{aligned} H_2 &= \frac{\lambda^4 bq^{1/2} t h_2 h_3 h_3 + \lambda^4 bq^{1/2} + \lambda^4 q h_2^2 h_3 h_2 h_3^2 + \lambda^2 q^{1/2} h_2 h_3 + bh_2 h_2 h_3^2}{q^{1/6} h_2 (\lambda^4 bq^{1/2} h_2 h_2 h_3^2 + \lambda^2 b t h_2 h_3 h_3 + \lambda^2 b + \lambda^2 q^{1/2} h_2^2 h_3 h_2 h_3^2 + h_2 h_3)}, \\ H_3 &= \frac{q^{1/3} (\lambda^6 bq^{1/2} t h_2 h_3 h_3 + \lambda^6 bq^{1/2} + \lambda^4 q^{1/2} h_2 h_3 + \lambda^2 b h_2 h_2 h_3^2 + h_2^2 h_3 h_2 h_3^2)}{h_3 (\lambda^4 bq^{1/2} t h_2 h_3 h_3 + \lambda^4 bq^{1/2} + \lambda^4 q h_2^2 h_3 h_2 h_3^2 + \lambda^2 q^{1/2} h_2 h_3 + bh_2 h_2 h_3^2)}, \end{aligned}$$

where $H_2 = H_2(t), H_3 = H_3(t), h_2 = h_2(t), h_3 = h_3(t)$ and $h_2 = h_2(t/p), h_3 = h_3(t/p)$ with $p^2 = q$. This system of two equations give us solutions H_2 and H_3 from solutions h_2 and h_3 corresponding to $\lambda^2 \mapsto p^{4/3} \lambda^2$.

Example 4.4. Let us give a few solutions of $qP_{III}^{(4)}$. For $l = 0, 1, 2$, we have $\lambda^2 = 1, p^{4/3}, p^{8/3}$, and

$$\begin{aligned}
 h_2(t; 1) &= h_3(t; 1) = 1, \\
 h_2(t; p^{4/3}) &= \frac{F_0(pt)}{p^{1/3}F_1(t)} \quad \text{and} \quad h_3(t; p^{4/3}) = \frac{p^{2/3}F_1(pt)}{F_0(pt)}, \\
 h_2(t; p^{8/3}) &= \frac{p^{1/3}F_1(t) \left(p^2Q_0(t) + bp^4tF_1(pt)F_1(t/p) + Q_1(t) \right)}{F_0(pt) \left(Q_0(t) + bp^2tF_1(pt)F_1(t/p) + p^3Q_0(pt) \right)}, \quad \text{and} \\
 h_3(t; p^{8/3}) &= \frac{p^{1/3}F_0(pt) \left(p^3Q_0(t) + bp^5tF_1(pt)F_1(t/p) + Q_0(pt) \right)}{F_1(pt) \left(p^2Q_0(t) + bp^4tF_1(pt)F_1(t/p) + Q_1(t) \right)},
 \end{aligned}$$

where

$$\begin{aligned}
 F_0(t) &= bp + bt + b + p^2 + p, \\
 F_1(t) &= 1 + b + p + bp + bt, \\
 Q_0(t) &= F_0(t)(bpF_1(t/p) + F_1(t)), \\
 Q_1(t) &= F_0(pt)(bF_1(t) + p^5F_1(pt)).
 \end{aligned}$$

5. Exact solution of the Lax pairs for q -discrete second and third Painlevé equations

Because the general solutions of discrete Painlevé equations are new transcendental functions, the corresponding solutions of their associated linear problems are highly nontrivial. In this section, we consider special q -rational solutions of the qP_{II} and qP_{III} equations that exist for special values of parameters and deduce the solutions of their respective linear problems. The results we obtain are similar to those in [20, 31], where rational solutions of qP_{II} and qP_{III} and their associated linear problem were studied. In the latter paper, the monodromy matrix A was shown to be a product of diagonal matrices and we show that this is also true for the qP_{II} hierarchy as well as the hierarchies of qP_{III} . The main results of this section are stated in Propositions 5.1 and 5.2.

We use the notation $\Gamma_q(1 - z)$ to denote the product

$$\Gamma_q(1 - z) = \frac{1}{(z; q)_\infty} = \prod_{k=0}^{\infty} \frac{1}{(1 - q^k z)}, \tag{5.1}$$

to be consistent with the terminology used in [31]. Note that this convention differs from the definition of $\Gamma_q(z)$ in [11, Chapter 5].

The qP_{II} hierarchy equation (1.8) is solved by (4.7), and this allows its linear system to be diagonalized by a constant matrix. Hence, we can solve it in terms of q -Gamma functions.

Proposition 5.1. *Suppose h_i are solutions of the qP_{II} hierarchy given by Proposition 4.1. When B and \mathcal{T} are as given in (1.6) and (1.7), respectively, there exists a solution $\Phi(x, t)$ of Lax pair (1.1a, 1.1b) given by*

$$\Phi(x, t) = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_0(-i)^{\mu(n+1)+\bar{\mu}} \prod_{j=0}^n \Gamma_q(1 - b_j x) \\ c_1 i^{\mu(n+1)+\bar{\mu}} \prod_{j=0}^n \Gamma_q(1 + b_j x) \end{pmatrix},$$

where $\mu = \frac{\ln x}{\ln q}$, $\bar{\mu} = \frac{n \ln t}{\ln q}$, $b_0 = t$, $b_l = q^l t$ ($l = 1, \dots, n-1$), $b_n = q$ and c_0 and c_1 are constants.

Proof. Substituting the special solution (4.7) (take $\lambda = -1$) into the Lax pair (1.1a) and (1.1b), where A is given in (1.3), and B is given in (1.6) gives

$$\Phi(qx, t) = \mathbf{A} \Phi(x, t), \tag{5.2}$$

$$\mathcal{T}(\Phi(x, t)) = \mathbf{B} \Phi(x, t), \tag{5.3}$$

where

$$\mathbf{A} = \prod_{j=0}^n \begin{pmatrix} ib_{n-j}x & 1 \\ -1 & ib_{n-j}x \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} ib_0x & 1 \\ -1 & ib_0x \end{pmatrix}.$$

We denote the solution matrix of (5.2) and (5.3) as $\Phi(x, t) = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$.

We now diagonalise both \mathbf{A} and \mathbf{B} with constant matrix $P = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$. By taking $\Phi(x, t) = P\Psi(x, t)$ we obtain

$$\Psi(qx, t) = P^{-1}\mathbf{A}P\Psi(x, t). \tag{5.4}$$

where

$$P^{-1}\mathbf{A}P = \begin{pmatrix} (-i)^{n+1} \prod_{l=0}^n (1 - b_l x) & 0 \\ 0 & i^{n+1} \prod_{l=0}^n (1 + b_l x) \end{pmatrix}.$$

Let $\begin{pmatrix} u(x, t) \\ v(x, t) \end{pmatrix}$ be the matrix solution of equation (5.4), i.e. $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = P \begin{pmatrix} C_0(t)u(x, t) \\ C_1(t)v(x, t) \end{pmatrix}$ where $C_0(t)$ and $C_1(t)$ are constants, and

$$u(qx, t) = (-i)^{n+1} \prod_{l=0}^n (1 - b_l x) u(x, t), \tag{5.5a}$$

$$v(qx, t) = i^{n+1} \prod_{l=0}^n (1 + b_l x) v(x, t). \tag{5.5b}$$

Equation (5.5a) is solved by writing

$$u(x, t) = (-i)^{\mu(n+1)} \prod_{l=0}^n u_l(x, t) \tag{5.6}$$

where $\mu = \frac{n \ln x}{\ln q}$ and

$$u_l(qx, t) = (1 - b_l x) u_l(x, t), \quad \text{for } l = 0, 1, \dots, n. \tag{5.7}$$

These equations (5.7) can be solved in terms of q -Gamma function; therefore we get

$$u(x, t) = (-i)^{\mu(n+1)} \prod_{l=0}^n \Gamma_q(1 - b_l x). \tag{5.8}$$

Similarly, equation (5.5b) has a solution

$$v(x, t) = (i)^{\mu(n+1)} \prod_{l=0}^n \Gamma_q(1 + b_l x). \tag{5.9}$$

Therefore

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = P \begin{pmatrix} C_0(t) (-i)^{\mu(n+1)} \prod_{j=0}^n \Gamma_q(1 - b_j x) \\ C_1(t) i^{\mu(n+1)} \prod_{j=0}^n \Gamma_q(1 + b_j x) \end{pmatrix}.$$

To find $C_0(t)$ and $C_1(t)$, we use the deformation problem

$$\mathcal{F}(\Psi(x, t)) = P^{-1} \mathbf{B} P \Psi(x, t), \tag{5.10}$$

where

$$P^{-1} \mathbf{B} P = \begin{pmatrix} -i(1 - b_0 x) & 0 \\ 0 & i(1 + b_0 x) \end{pmatrix}.$$

This implies

$$\begin{aligned} \tilde{T}(u(x, t)) &= -i(1 - b_0 x)u(x, t), \\ \tilde{T}(v(x, t)) &= i(1 + b_0 x)v(x, t). \end{aligned}$$

Using (5.8) and (5.9), we obtain $C_0(t) = c_0(-i)^{\frac{n \ln t}{\ln q}}$ and $C_1(t) = c_1 i^{\frac{n \ln t}{\ln q}}$, where c_0 and c_1 are constants. Therefore, we get

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = P \begin{pmatrix} c_0 (-i)^{\mu(n+1) + \bar{\mu}} \prod_{j=0}^n \Gamma_q(1 - b_j x) \\ c_1 i^{\mu(n+1) + \bar{\mu}} \prod_{j=0}^n \Gamma_q(1 + b_j x) \end{pmatrix},$$

where $\mu = \frac{\ln x}{\ln q}$ and $\bar{\mu} = \frac{n \ln t}{\ln q}$. □

Remark 5.1. The simplest solution of the qP_{II} hierarchy given by Proposition 4.1 also happens to be a solution of the qP_{III} hierarchy under the condition $b = 1$. Since the qP_{II} and qP_{III} hierarchies share the same spectral linear problem, it follows that the solution of the linear problem given in Proposition 5.1 is also a solution of the corresponding linear problem for the qP_{III} hierarchy. The difference is only on the values of parameters b_i ($i = 0, 1, \dots, n$), in the case of qP_{III} the values take: $b_0 = b_1 = t$, $b_l = b_{l+1} = q^{\frac{l}{n}} t$ ($l = 2, 4, 6, \dots, n - 2$), and $b_n = q$.

Proposition 5.2. When B and \mathcal{T} were given in (1.9) and (1.10), respectively, and the qP_{III} hierarchy equation (1.11) is solved by (4.7), there exists a solution $\Phi(x, t)$ of Lax pair (1.1a, 1.1b) given by

$$\Phi(x, t) = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_0 (-1)^{\bar{\mu}} (-i)^{\mu(n+1)} \Gamma_q(1 - b_n x) \prod_{\substack{j=0 \\ \text{even}}}^{n-1} (\Gamma_q(1 - b_j x))^2 \\ c_1 (-1)^{\bar{\mu}} i^{\mu(n+1)} \Gamma_q(1 + b_n x) \prod_{\substack{j=0 \\ \text{even}}}^{n-1} (\Gamma_q(1 + b_j x))^2 \end{pmatrix},$$

where $\mu = \frac{\ln x}{\ln q}$, $\bar{\mu} = \frac{n \ln t}{2 \ln q}$ and c_0 and c_1 are constants.

6. Conclusion

In this paper, we have presented two hierarchies of Painlevé equations, one of them is a qP_{II} hierarchy which is found in [21] and another is a qP_{III} hierarchy which is new. Each of these hierarchies was obtained by reduction of the multi-parametric lattice mKdV equation, by using the staircase method. The explicit forms of these hierarchies are given in equations (1.8) and (1.11), with second members of each hierarchy provided by equations (2.8) and (2.10).

In addition to explicit construction of these hierarchies, we provided some properties which are deduced for the first time. One of these is a method to construct Bäcklund transformations for every member of the qP_{II} and qP_{III} hierarchies. We found so-called seed solutions for each member of these hierarchies. We then used these transformations on seed solutions to find rational solutions of second-order and fourth-order members of the qP_{II} and qP_{III} hierarchies. From the seed solution for the hierarchies, we also deduced the corresponding solutions of their Lax pair; see Propositions 5.1 and 5.2.

It is noteworthy that the spectral problem in each Lax pair in each hierarchy involves a 2×2 coefficient matrix, A from equation (1.1a), which satisfies the conditions of non-resonance in Birkhoff's theory of linear q -difference equations. To our knowledge, this is the first time such linear problems have been constructed for q -Painlevé hierarchies.

There still remain open questions. In the PDE setting, members of a hierarchy are related by recursion operators. However, such operators are not known in the difference equation setting. We have also not touched upon continuum limits, although there is reason to believe that the hierarchies we have provided have well known Painlevé hierarchies as continuum limits.

Finally, we note that the construction methods in this paper also lead to other hierarchies. These will be the subject of future publications.

A. Derivation of the Lax pair (3.2)

Here we provide how we derive the Lax pair (3.2). Using the CAC property, we obtain a Lax pair of equation (3.1)

$$L_1(w_0, w_1, \alpha_1, \alpha_2) = \begin{pmatrix} -\frac{\alpha_1 w_0}{w_1} & k_1 w_0 \\ \frac{k_2}{w_1} & -\alpha_2 \end{pmatrix} \quad \text{and} \quad M_1(w_0, w_2, \beta_1, \beta_2) = \begin{pmatrix} -\frac{\beta_1 w_0}{w_2} & k_1 w_0 \\ \frac{k_2}{w_2} & -\beta_2 \end{pmatrix}, \quad (A.1)$$

where k_1 and k_2 are spectral parameters. Using the gauge matrix

$$\mathcal{G}(z) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{z} \end{pmatrix},$$

we have a new-looking matrix

$$L(w_0, w_1, \alpha_1, \alpha_2) = \mathcal{G}^{-1}(w_1)L_1(w_0, w_1, \alpha_1, \alpha_2)\mathcal{G}(w_0) = \begin{pmatrix} -\frac{\alpha_1 w_0}{w_1} & k_1 \\ k_2 & -\frac{\alpha_2 w_1}{w_0} \end{pmatrix}, \quad (\text{A.2})$$

$$M(w_0, w_2, \beta_1, \beta_2) = \mathcal{G}^{-1}(w_2)M_1(w_0, w_2, \beta_1, \beta_2)\mathcal{G}(w_0) = \begin{pmatrix} -\frac{\beta_1 w_0}{w_2} & k_1 \\ k_2 & -\frac{\beta_2 w_2}{w_0} \end{pmatrix}. \quad (\text{A.3})$$

Now we replace k_1 and k_2 in (A.2) and (A.3) with $\frac{l_1}{x}$ and $\frac{l_2}{x}$, respectively. For simplicity, we multiply by x , which leads to the desired results in equations 3.2.

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