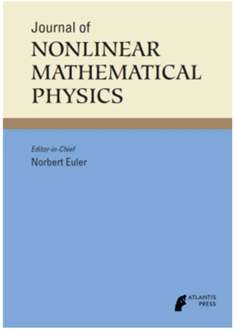

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Solutions of the constrained mKP hierarchy by Boson-Fermion correspondence

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In this paper, the Hirota bilinear equation of the constrained modified KP hierarchy is expressed as the vacuum expectation values of Clifford operators by using the free fermions method of mKP hierarchy. Then we mainly use the Boson-Fermion correspondence to solve the Hirota bilinear equation of the k-constrained mKP hierarchy. Further, by choosing special group elements in GL_∞, the corresponding rational and soliton solutions are given.

Keywords: Hirota bilinear equation; Boson-Fermion correspondence; k-constrained mKP hierarchy; rational and soliton solutions.

2010 MSC: 35Q53, 37K10, 37K40

1. Introduction

The modified KP hierarchy [14, 15] is introduced as a series of bilinear equations by using the Boson-Fermion correspondence, i.e.

$$\mathrm{res}_\lambda\Big(\tau_n(t-[\lambda^{-1}];g)\tau_{n'}(t+[\lambda^{-1}];g)\lambda^{n-n'}e^{\xi(t-t',\lambda)}\Big)=0,\quad n\geq n',\tag{1.1}$$

where $t=(t_1=x,t_2,t_3,\cdots)$ and $[\lambda]=(\lambda,\frac{\lambda^2}{2},\frac{\lambda^3}{3},\cdots)$. Here the tau functions are defined as follows

$$\tau_n(t;g)=\langle n|e^{H(t)}g|n\rangle,\quad g\in G,\tag{1.2}$$

where

$$H(t)=\sum_{l=1}^{\infty}\sum_{n\in\mathbb{Z}}\phi_n\phi^*_{n+l}t_l.$$

ϕ_n, $\phi^*_n(n\in\mathbb{Z})$ are generators of the Clifford algebra $\mathscr{A}$ which satisfy

$$\begin{aligned}&[\phi_n,\phi^*_m]_+=\phi_n\phi^*_m+\phi^*_m\phi_n=\delta_{m,n},\\&[\phi_n,\phi_m]_+=[\phi^*_n,\phi^*_m]_+=0.\end{aligned}\tag{1.3}$$

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$G=\{g\in\mathscr{A}|\exists g^{-1},\quad gVg^{-1}=V,\quad gV^*g^{-1}=V^*\}$, with $V=\sum_{n\in\mathbb{Z}}C\phi_n$ and $V^*=\sum_{n\in\mathbb{Z}}C\phi_n^*$. $\langle n|$ and $|n\rangle$ are the states of the charge n, which are defined as follows:

$$\langle n|=\begin{cases}\langle 0|\phi_{-1}\cdots\phi_n, & n<0\\ \langle 0|, & n=0\\ \langle 0|\phi_0^*\cdots\phi_{n-1}^*, & n>0.\end{cases},$$

$$|n\rangle=\begin{cases}\phi_{n+1}^*\cdots\phi_{-1}^*|0\rangle, & n<0\\ |0\rangle, & n=0\\ \phi_{-n+1}\cdots\phi_0|0\rangle, & n>0.\end{cases}.$$

And $\langle 0|$ and $|0\rangle$ are the vacuum states

$$\begin{aligned}&\phi_n|0\rangle=0,\quad (n<0);\quad \phi_n^*|0\rangle=0,\quad (n\geq 0),\\ &\langle 0|\phi_n=0,\quad (n\geq 0);\quad \langle 0|\phi_n=0,\quad (n<0),\\ &\langle 0|0\rangle=1.\end{aligned}\tag{1.4}$$

After the work above, people paid more attention to find the Lax equation forms. There are many versions of the mKP hierarchy [3,7,16–18,21,26]. All the forms are trying to generalize the Miura link between the mKdV and KdV equations to the KP cases. Here in this paper, we only consider the Kupershmidt-Kiso version [16–18,26]. In fact, the mKP hierarchy of Kupershmidt-Kiso version is corresponding to the bilinear equation [5]

$$\mathrm{res}_\lambda\left(\lambda^{-1}e^{\xi(t-t',\lambda)}\tau_0(t-[\lambda])\tau_1(t'+[\lambda^{-1}])\right)=\tau_1(t)\tau_0(t'),\tag{1.5}$$

which can be rewritten [22] from the original bilinear equation (1.1).

The wave function and the adjoint wave function [5, 14, 15] can be defined through the tau functions τ_1 and τ_0 in the following way

$$w(t,\lambda)=\frac{\tau_0(t-[\lambda^{-1}])}{\tau_1(t)}e^{\xi(t,\lambda)}=\frac{\langle 1|e^{H(t)}\phi(\lambda)g|0\rangle}{\langle 1|e^{H(t)}g|1\rangle},\tag{1.6}$$

$$w^*(t,\lambda)=\frac{\tau_1(t+[\lambda^{-1}])}{\tau_0(t)}\lambda^{-1}e^{-\xi(t,\lambda)}=\frac{\langle 0|e^{H(t)}\phi^*(\lambda)g|1\rangle}{\lambda\langle 0|e^{H(t)}g|0\rangle},\tag{1.7}$$

where $\xi(t,\lambda)=\sum_{j=1}^{\infty}t_j\lambda^j$. Thus the bilinear equations (1.5) are equivalent to

$$\mathrm{res}_\lambda w(t',\lambda)w^*(t,\lambda)=1.\tag{1.8}$$

Then one can introduce two pseudo-differential operators Z and W such that the relations below hold

$$w(t,\lambda)=Z(e^{\xi(t,\lambda)}),\quad w^*(t,\lambda)=W(-e^{(t,\lambda)}),\tag{1.9}$$

with

$$Z=z_0+z_1\partial^{-1}+z_2\partial^{-2}+\cdots,\quad W=w_1\partial^{-1}+w_2\partial^{-2}+\cdots.\tag{1.10}$$

Here $\partial = \partial_x$. The algebraic multiplication of ∂^i with the multiplication operator f is given by the usual Leibnitz rule [8]

$$\partial^i f = \sum_{j\geq 0} \binom{i}{j} f^{(j)} \partial^{i-j}, \quad i \in \mathbb{Z}, \tag{1.11}$$

where $f^{(j)} = \frac{\partial^j f}{\partial x^j}$. For $A = \sum_i a_i \partial^i$, $A_{\geq k} = \sum_{i\geq k} a_i \partial^i$, $A_{<k} = \sum_{i<k} a_i \partial^i$ and $A_{[k]} = a_k$. In this paper, for any pseudo differential operator A and a function f, the symbol $A(f)$ will indicate the action of A on f, whereas the symbol Af or $A\cdot f$ will denote the operator product of A and f, and $*$ stands for the conjugate operation: $(AB)^* = B^*A^*$, $\partial^* = -\partial$, $f^* = f$.

Lemma 1.1 ([12]). *If we let $A(x) = \sum_i a_i(x)\partial_x^i$ and $B(x') = \sum_j b_j(x')\partial_{x'}^j$ be two operators, then*

$$A(x)B^*(x)\partial_x(\Delta^0) = \mathrm{res}_\lambda A(x)(e^{x\lambda}) \cdot B(x')(e^{-x'\lambda}), \tag{1.12}$$

where $\Delta^0 = (x-x')^0$ and

$$\partial_x^{-a}(\Delta^0) = \begin{cases} 0, & a<0, \\ \dfrac{(x-x')^a}{a!}, & a\geq 0. \end{cases} \tag{1.13}$$

From the bilinear equation (1.8), one can find $W = (Z\partial^{-1})^*$ and the evolution equations of the operator Z as follows

$$\partial_{t_n} Z = -\left(Z\partial^n Z^{-1}\right)_{\leq 0} Z. \tag{1.14}$$

Here the operator Z is called the dressing operator or the wave operator. Then one can introduce the Lax operator of the mKP hierarchy [16–18, 26] in the way below

$$L = Z\partial Z^{-1} = \partial + u_0 + u_1\partial^{-1} + u_2\partial^{-2} + u_3\partial^{-3} + \cdots, \tag{1.15}$$

which satisfies the Lax equations

$$\partial_{t_n} L = [(L^n)_{\geq 1}, L],\ n = 1, 2, 3, \ldots. \tag{1.16}$$

The mKP hierarchy and its various extensions have many important integrable structures, such as tau function [5, 32], Hirota bilinear equation [2, 5], squared eigenfunction symmetries [5, 28, 29], additional symmetry [5, 33], Hamiltonian structures [3, 7, 17, 27], gauge transformation [1, 4, 6, 13, 23, 30, 34], and algebraic strucutre [21, 24] etc. In this paper, we mainly discuss the constrained mKP hierarchy, which can be viewed as the sub-hierarchy of the mKP hierarchy.

The k-constrained mKP hierarchy [2, 5, 29] is defined by imposing the following constraint on the Lax operator,

$$L^k = (L^k)_{\geq 1} + \sum_{i=1}^m q_i \partial^{-1} r_i \partial, \tag{1.17}$$

where q and r are the eigenfunction and the adjoint eigenfunction of the mKP hierarchy respectively, satisfying

$$\partial_{t_n} q_i = (L^n)_{\geq 1}(q_i), \quad \partial_{t_n} r_i = -\left(\partial (L^n)_{\geq 1}\partial^{-1}\right)^*(r_i). \tag{1.18}$$

Introduce two auxiliary functions $\rho(t)$ and $\sigma(t)$ such that

$$q_i(t) = \frac{\rho_i(t)}{\tau_1(t)}, \quad r_i(t) = \frac{\sigma_i(t)}{\tau_0(t)}, \tag{1.19}$$

Then we have the proposition below [2].

Proposition 1.1 ([2]). *The auxiliary functions $\sigma(t), \rho(t), \tau_1(t)$, and $\tau_0(t)$ satisfy the following bilinear equations:*

$$\sum_{i=1}^{m} \rho_i(t)\sigma_i(t') = \mathrm{res}_\lambda \left(\lambda^{k-1}\tau_0(t-[\lambda^{-1}])\tau_1(t'+[\lambda^{-1}])e^{\xi(t-t',\lambda)}\right), \tag{1.20}$$

$$\rho_i(t)\tau_0(t') = \mathrm{res}_\lambda \left(\lambda^{-1}\tau_0(t-[\lambda^{-1}])\rho_i(t'+[\lambda^{-1}])e^{\xi(t-t',\lambda)}\right), \tag{1.21}$$

$$\sigma_i(t')\tau_1(t) = \mathrm{res}_\lambda \left(\lambda^{-1}\tau_1(t'+[\lambda^{-1}])\sigma_i(t-[\lambda^{-1}])e^{\xi(t-t',\lambda)}\right). \tag{1.22}$$

Boson-Fermion correspondence [9–11, 14, 15, 19, 20, 25] describes the link between the free fermions and the Bosons, which is a very powerful tool to deal with the integrable hierarchy. By this method, the solutions for the constrained KP and BKP hierarchies are constructed from its bilinear representations [31,35]. In this paper, we will rewrite the Hirota bilinear equations of the constrained mKP hierarchy in the proposition above, in terms of the vacuum expectation values of Clifford operators by using Boson-Fermi correspondence [11, 25]. Then we mainly use the Boson-Fermion correspondence to solve the Hirota bilinear equation of the k-constrained mKP hierarchy. Further, by choosing special group elements in GL_∞, the corresponding rational and soliton solutions are given.

This paper is organized in the following way. Rational solutions for the vector k-constrained mKP hierarchy are derived in Section 2. Section 3 is devoted to the soliton solutions of k-constrained mKP hierarchy. At last, some conclusions and discussions are given in Section 4.

2. Rational solutions for the vector k-constrained mKP hierarchy

Lemma 2.1 (Wick's theorem [25]). *For $\phi_1, \cdots \phi_r \in V \oplus V^*$,*

$$\langle \phi_1 \cdots \phi_r \rangle = \begin{cases} 0, & \text{if } r \text{ is odd,} \\ \sum_\sigma \mathrm{sign}(\sigma)\langle \phi_{\sigma(1)}\phi_{\sigma(2)}\rangle \cdots \langle \phi_{\sigma(r-1)}\phi_{\sigma(r)}\rangle, & \text{if } r \text{ is even.} \end{cases} \tag{2.1}$$

where $\langle\cdot\rangle = \langle vac|\cdot|vac\rangle$ and $\mathrm{sign}(\sigma)$ *is the sign of a permutation; the sum runs over all permutations σ satisfying $\sigma(1) < \sigma(2), \cdots, \sigma(r-1) < \sigma(r)$ and $\sigma(1) < \sigma(3) < \cdots < \sigma(r-1)$, in other words, over all ways of grouping the ϕ_i into pairs.*

Denote $\phi(\lambda) = \sum_{n\in\mathbb{Z}} \phi_n \lambda^n$ and $\phi^*(\lambda) = \sum_{n\in\mathbb{Z}} \phi_n^* \lambda^{-n}$. Define the t-evolution of an operator a as $a(t) = e^{H(t)} a e^{-H(t)}$. Then

Lemma 2.2 ([11]). *The following relations hold*

$$H(t)|vac\rangle = 0, \tag{2.2}$$

$$e^{H(t)}\phi(\lambda)e^{-H(t)} = e^{\xi(t,\lambda)}\phi(\lambda), \tag{2.3}$$

$$e^{H(t)}\phi^*(\lambda)e^{-H(t)} = e^{-\xi(t,\lambda)}\phi^*(\lambda), \tag{2.4}$$

$$\langle vac|\phi_n^*(t)\phi_m(t)|vac\rangle = \sum_{i\geq 0} p_{i-n}(-t)p_{m-i}(t), \tag{2.5}$$

where $p_n(t)$ is the Schur polynomial, determined by $e^{\xi(t,\lambda)} = \sum_{n=0}^{\infty} p_n(t)\lambda^n$.

From this lemma, one can obtain

$$e^{H(t)}\phi_n e^{-H(t)} = \sum_{i=0}^{\infty} \phi_{n-i} p_i(t), \tag{2.6}$$

$$e^{H(t)}\phi_n^* e^{-H(t)} = \sum_{i=0}^{\infty} \phi_{n+i}^* p_i(-t). \tag{2.7}$$

The following differential operators of infinite order are called vertex operators.

$$X(\lambda) = e^{\xi(t,\lambda)} e^{-\xi(\widetilde{\partial},\lambda^{-1})} = \sum_{i\in\mathbb{Z}} X_i\Big(t, \frac{\partial}{\partial t}\Big)\lambda^i, \tag{2.8}$$

$$X^*(\lambda) = e^{-\xi(t,\lambda)} e^{\xi(\widetilde{\partial},\lambda^{-1})} = \sum_{i\in\mathbb{Z}} X_i^*\Big(t, \frac{\partial}{\partial t}\Big)\lambda^{-i}, \tag{2.9}$$

where X_i, X_i^* are given by

$$X_i = \sum_{n\geq 0} p_{n+i}(t) p_n(-\widetilde{\partial}), \quad X_i^* = \sum_{n\geq 0} p_{n-i}(-t) p_n(\widetilde{\partial}) \tag{2.10}$$

and $\widetilde{\partial} = (\partial_1, \frac{1}{2}\partial_2, \frac{1}{3}\partial_3, \dots)$.

Lemma 2.3 ([11]). *The vertex operators* (2.8) *and* (2.9) *have the following properties:*

$$\langle n|e^{H(t)}\phi(\lambda) = \lambda^{n-1} X(\lambda)\langle n-1|e^{H(t)}, \tag{2.11}$$

$$\langle n|e^{H(t)}\phi^*(\lambda) = \lambda^{-n} X^*(\lambda)\langle n+1|e^{H(t)}. \tag{2.12}$$

From Ref. [11], for any $g \in G$, it satisfies the following commutation relations with free fermion operators,

$$\phi_n g = \sum_{i\in\mathbb{Z}} g\phi_i (a^{-1})_{i,n}, \quad \phi_n^* g = \sum_{i\in\mathbb{Z}} g\phi_i^* a_{i,n}. \tag{2.13}$$

Proposition 2.1. *Let $\Gamma_k = \sum_{n\in\mathbb{Z}} \phi_n \phi_{n+k}^*$, if $g \in G$ satisfies the condition*

$$g^{-1}\Gamma_k g = \sum_{i,j\in\mathbb{Z}} f_{i,j}\phi_i\phi_j^*, \tag{2.14}$$

with $f_{i,j} = \sum_{l=1}^{m} d_i^{(l)} e_j^{(l)}$ for $i \geq 0$, $j \leq 0$, then

$$\tau_0(t) = \langle 0|e^{H(t)} g|0\rangle, \tag{2.15}$$

$$\tau_1(t) = \langle 1|e^{H(t)} g|1\rangle, \tag{2.16}$$

$$\rho_i(t) = \langle 1|e^{H(t)} g \sum_{n\geq 0} d_n^{(i)}\phi_n|0\rangle, \tag{2.17}$$

$$\sigma_i(t) = \langle 0|e^{H(t)} g \sum_{n\leq 0} e_n^{(i)}\phi_n^*|1\rangle. \tag{2.18}$$

$$i = 1, 2, \dots, m$$

satisfy the bilinear equations in Proposition 1.1.

Proof. We first prove that Eq. (2.14), (2.16), (2.17) and (2.18) satisfy (1.20). According to Lemma 2.3

$$\begin{aligned}
&\mathrm{res}_\lambda\left(\lambda^{k-1}\tau_0(t-[\lambda^{-1}])\tau_1(t'+[\lambda^{-1}])e^{\xi(t-t',\lambda)}\right)\\
&=\mathrm{res}_\lambda\left(\lambda^{k-1}X(\lambda)\langle 0|e^{H(t)}g|0\rangle X^*(\lambda)\langle 1|e^{H(t')}g|1\rangle\right)\\
&=\mathrm{res}_\lambda\left(\lambda^{k-1}\langle 1|e^{H(t)}\phi(\lambda)g|0\rangle\langle 0|e^{H(t')}\phi^*(\lambda)g|1\rangle\right)\\
&=\mathrm{res}_\lambda\left(\lambda^{k-1}\langle 1|e^{H(t)}\sum_{i\in\mathbb{Z}}\phi_i\lambda^i g|0\rangle\langle 0|e^{H(t')}\sum_{j\in\mathbb{Z}}\phi_j^*\lambda^{-j}g|1\rangle\right)\\
&=\sum_{i\in\mathbb{Z}}\langle 1|e^{H(t)}\phi_i g|0\rangle\langle 0|e^{H(t')}\phi_{i+k}^*g|1\rangle.
\end{aligned}\tag{2.19}$$

Due to $g\in G$, we have

$$\phi_i g=\sum_{m\in\mathbb{Z}}g\phi_m(a^{-1})_{m,i},\quad \phi_i^*g=\sum_{m\in\mathbb{Z}}g\phi_m^*a_{i,m},\tag{2.20}$$

further,

$$\begin{aligned}
g^{-1}\Gamma_k g=g^{-1}\sum_{l\in\mathbb{Z}}\phi_l\phi_{l+k}^*g&=\sum_{l\in\mathbb{Z}}g^{-1}\phi_l g g^{-1}\phi_{l+k}^*g\\
&=\sum_{l,i,j\in\mathbb{Z}}g^{-1}g\phi_i(a^{-1})_{i,l}g^{-1}g\phi_j^*a_{l+k,j}\\
&=\sum_{l,i,j\in\mathbb{Z}}\phi_i\phi_j^*(a^{-1})_{i,l}a_{l+k,j}.
\end{aligned}\tag{2.21}$$

According to the assumption of the theorem

$$\sum_{n\in\mathbb{Z}}(a^{-1})_{i,n}a_{n+k,j}=f_{i,j}=\sum_{l=1}^m d_i^{(l)}e_j^{(l)},\quad i\geq 0,\ j\leq 0,\tag{2.22}$$

by using (1.4) and (2.20), (2.19) can be further written as

$$\begin{aligned}
&\sum_{n,i,j\in\mathbb{Z}}\langle 1|e^{H(t)}g\phi_i|0\rangle(a^{-1})_{i,n}a_{n+k,j}\langle 0|e^{H(t')}g\phi_j^*|1\rangle\\
&=\sum_{l=1}^m\sum_{i\geq 0}\sum_{j\leq 0}\langle 1|e^{H(t)}g\phi_i|0\rangle d_i^{(l)}e_j^{(l)}\langle 0|e^{H(t')}g\phi_j^*|1\rangle\\
&=\sum_{l=1}^m\rho_l(t)\sigma_l(t').
\end{aligned}\tag{2.23}$$

So (1.20) is verified. And then we will prove (1.22). By using Lemma 2.3

$$\begin{aligned}
&\mathrm{res}_\lambda\left(\lambda^{-1}\tau_1(t'+[\lambda^{-1}])\sigma_i(t-[\lambda^{-1}])e^{\xi(t-t',\lambda)}\right)\\
&=\mathrm{res}_\lambda\left(\lambda^{-1}X^*(\lambda)\langle 1|e^{H(t')}g|1\rangle X(\lambda)\langle 0|e^{H(t)}g\sum_{n\leq 0}e_n^{(i)}\phi_n^*|1\rangle\right)\\
&=\mathrm{res}_\lambda\left(\lambda^{-1}\langle 0|e^{H(t')}\phi^*(\lambda)g|1\rangle\langle 1|e^{H(t)}\phi(\lambda)g\sum_{n\leq 0}e_n^{(i)}\phi_n^*|1\rangle\right)\\
&=\sum_{j\in\mathbb{Z}}\langle 0|e^{H(t')}\phi_j^*g|1\rangle\langle 1|e^{H(t)}\phi_j g\sum_{n\leq 0}e_n^{(i)}\phi_n^*|1\rangle.
\end{aligned}\tag{2.24}$$

(2.24) can be rewritten as

$$
\begin{aligned}
&\sum_{j,m,l\in\mathbb{Z}} \langle 0|e^{H(t')}g\phi_m^*|1\rangle a_{j,m}(a^{-1})_{l,j}\langle 1|e^{H(t)}g\phi_l \sum_{n\le 0} e_n^{(i)}\phi_n^*|1\rangle \\
&= \sum_{m,l\in\mathbb{Z}} \langle 0|e^{H(t')}g\phi_m^*|1\rangle \delta_{ml}\langle 1|e^{H(t)}g\phi_l \sum_{n\le 0} e_n^{(i)}\phi_n^*|1\rangle \\
&= \sum_{m\le 0} \langle 0|e^{H(t')}g\phi_m^*|1\rangle\langle 1|e^{H(t)}ge_m^{(i)}|1\rangle = \sigma_i(t')\tau_1(t),
\end{aligned}
\tag{2.25}
$$

where we haved used (1.3), (1.4) and (2.20). Therefore (1.22) is proven. In the same way, it is easy to prove (1.21). □

Remark 2.1. Note that these results are different from the ones in the KP case [35]. The major differences are as follows. 1) The subscript j in $e_j^{(l)}$ can be zero in the mKP case, while zero is forbidden in the KP case. 2) The expressions of σ_i are different. Since zero is allowed in subscript j of $e_j^{(l)}$, f_{00} will appear and it will bring much difficulty when discussing the rational solutions. So in the lemma below, in order to avoid these problems, we separate the zero parts from the nonzero parts in the expressions of the group elements in GL_∞.

In order to find rational solutions of the k-constrained mKP hierarchy more conveniently by using the proposition 2.1, the following lemma is needed.

Lemma 2.4. *Let $g = e^{a\phi_0^*\phi_0+\sum_{n=1}^N b_n\phi_{i_n}^*\phi_{j_n}} = e^Y$, where a and b_n are constant, $i_n<0$, $j_n>0$ for $n=1,2,\ldots,N$, then*

$$
\begin{aligned}
g^{-1}\Gamma_k g = \Gamma_k &+ \sum_{m,n=1}^N b_m b_n \delta_{i_n+k,j_n}\phi_{i_m}^*\phi_{j_n} + \sum_{n=1}^N b_n(\phi_{i_n}^*\phi_{j_n-k} - \phi_{i_m+k}^*\phi_{j_n}) \\
&+(1-e^a)(\phi_k^*\phi_0 - \sum_{n=1}^N b_n\delta_{j_n,k}\phi_{i_n}^*\phi_0) + (1-e^{-a})(\phi_0^*\phi_{-k} + \sum_{n=1}^N b_n\delta_{-i_n,k}\phi_0^*\phi_{j_n}).
\end{aligned}
\tag{2.26}
$$

Proof. Direct calculation shows

$$
\begin{aligned}
(adY)\Gamma_k &= a(\phi_k^*\phi_0 - \phi_0^*\phi_{-k}) + \sum_{n=1}^N b_n(\phi_{i_n+k}^*\phi_{j_n} - \phi_{i_n}^*\phi_{j_n-k}),\\
(adY)^2\Gamma_k &= a^2(-\phi_k^*\phi_0 - \phi_0^*\phi_{-k}) + 2a\sum_{n=1}^N b_n(\delta_{j_n,k}\phi_{i_n}^*\phi_0 + \delta_{k,-i_n}\phi_0^*\phi_{j_n})\\
&\quad +2\sum_{m,n=1}^N b_m b_n \delta_{i_n+k,j_m}\phi_{i_m}^*\phi_{j_n},\\
(adY)^3\Gamma_k &= a^3(\phi_k^*\phi_0 - \phi_0^*\phi_{-k}) + 3a^2\sum_{n=1}^N b_n(-\delta_{j_n,k}\phi_{i_n}^*\phi_0 + \delta_{k,-i_n}\phi_0^*\phi_{j_n}),\\
&\cdots
\end{aligned}
\tag{2.27}
$$

Further calculation by means of mathematical induction method,

$$
\begin{aligned}
(adY)^m\Gamma_k &= a^m\Big((-1)^{m-1}\phi_k^*\phi_0 - \phi_0^*\phi_{-k}\Big)\\
&\quad +ma^{m-1}\sum_{n=1}^N b_n\Big((-1)^m\delta_{j_n,k}\phi_{i_n}^*\phi_0 + \delta_{k,-i_n}\phi_0^*\phi_{j_n}\Big), \quad m\ge 3.
\end{aligned}
\tag{2.28}
$$

Then by Baker-Campbell-Hausdorff-formula

$$
\begin{aligned}
g^{-1}\Gamma_k g = e^{-Y}\Gamma_k e^{Y} &= \sum_{m=0}^{\infty}(-1)^m \frac{1}{m!}(adY)^m \Gamma_k \\
&= \Gamma_k + \sum_{m=1}^{\infty}(-1)^m \frac{a^m\Big((-1)^{m-1}\phi_k^*\phi_0 - \phi_0^*\phi_{-k}\Big)}{m!} \\
&\quad + \sum_{m=2}^{\infty}(-1)^m \frac{ma^{m-1}\sum_{n=1}^{N} b_n\Big((-1)^m\delta_{j_n,k}\phi_{i_n}^*\phi_0 + \delta_{k,-i_n}\phi_0^*\phi_{j_n}^*\Big)}{m!} \\
&\quad + \sum_{m,n=1}^{N} b_m b_n \delta_{i_n+k,j_m}\phi_{i_m}^*\phi_{j_n} - \sum_{n=1}^{N} b_n(\phi_{i_n+k}^*\phi_{j_n} - \phi_{i_n}^*\phi_{j_n-k}).
\end{aligned} \tag{2.29}
$$

Using the Taylor's formula to calculate the above formula, one can get (2.26). □

To better understand Proposition 2.1, let's use Lemma 2.4 to give an example.

Example 2.1. We consider the case where $m = 1$ in (1.20) and $N = 1$ in Lemma 2.4, then

$$
\begin{aligned}
g^{-1}\Gamma_k g = \Gamma_k &+ b_1^2\delta_{i_1+k,j_1}\phi_{i_1}^*\phi_{j_1} + (e^a-1)b_1\delta_{j_1,k}\phi_{i_1}^*\phi_0 + (1-e^{-a})b_1\delta_{-i_1,k}\phi_0^*\phi_{j_1} \\
&+ b_1\phi_{i_1}^*\phi_{j_1-k} - b_1\phi_{i_1+k}^*\phi_{j_1} + (1-e^a)\phi_k^*\phi_0 + (1-e^{-a})\phi_0^*\phi_{-k}.
\end{aligned} \tag{2.30}
$$

By direct calculation using the mathematical induction hypothesis one can obtain

$$
\begin{aligned}
e^Y &= e^{a\phi_0^*\phi_0 + b_1\phi_{i_1}^*\phi_{j_1}} \\
&= 1 + (e^a-1)\phi_0^*\phi_0 + b_1\phi_{i_1}^*\phi_{j_1} + (e^a-1)b_1\phi_0^*\phi_0\phi_{i_1}^*\phi_{j_1}.
\end{aligned} \tag{2.31}
$$

If $i_1 < 0$ and $j_1 > 0$, there are five cases to be discussed.
(1) When $i_1 + j_1 = k$, by using Proposition 2.1 we find $f_{j_1,i_1} = -b_1^2$. So we may choose $d_{j_1} = -1$, $e_{i_1} = b_1^2$. Then by Lemma 2.1-Lemma 2.3 and Eqs.(2.8)-(2.10), we obtain the solutions of (1.20)-(1.22) are

$$
\begin{aligned}
\tau_0(t) &= \langle 0|e^{H(t)}e^{a\phi_0^*\phi_0 + b_1\phi_{i_1}^*\phi_{j_1}}|0\rangle \\
&= \langle 0|e^{H(t)}\Big(1 + (e^a-1)\phi_0^*\phi_0 + b_1\phi_{i_1}^*\phi_{j_1} + (e^a-1)b_1\phi_0^*\phi_0\phi_{i_1}^*\phi_{j_1}\Big)|0\rangle \\
&= 1 + (e^a-1)\langle\phi_0^*(t)\phi_0(t)\rangle + b_1\phi_{i_1}^*(t)\phi_{j_1}(t) \\
&\quad + (e^a-1)b_1\Big(\langle\phi_0^*(t)\phi_0(t)\rangle\langle\phi_{i_1}^*(t)\phi_{j_1}(t)\rangle - \langle\phi_0^*(t)\phi_{j_1}(t)\rangle\langle\phi_{i_1}^*(t)\phi_0(t)\rangle\Big) \\
&= e^a + e^a b_1\sum_{n\geq 0} p_{n-i_1}(-t)p_{j_1-n}(t) - (e^a-1)b_1\sum_{n\geq 0} p_n(-t)p_{j_1-n}(t)p_{-i_1}(-t),
\end{aligned} \tag{2.32}
$$

$$\begin{aligned}
\tau_1(t) &= \langle 1|e^{H(t)}e^{a\phi_0^*\phi_0+b_1\phi_{i_1}^*\phi_{j_1}}|1\rangle \\
&= \langle 1|e^{H(t)}|1\rangle + (e^a-1)\langle 1|e^{H(t)}\phi_0^*\phi_0|1\rangle + b_1\langle 1|e^{H(t)}\phi_{i_1}^*\phi_{j_1}|1\rangle \\
&\quad +(e^a-1)b_1\langle 1|e^{H(t)}\phi_0^*\phi_0\phi_{i_1}^*\phi_{j_1}|1\rangle \\
&= X_0\langle 0|e^{H(t)}|0\rangle + (e^a-1)\langle 1|e^{H(t)}\phi_0^*\phi_0\phi_0|0\rangle - b_1X_{j_1}\langle 0|e^{H(t)}\phi_{i_1}^*|1\rangle \\
&\quad +(e^a-1)b_1\langle 1|e^{H(t)}\phi_0^*\phi_0\phi_{i_1}^*\phi_{j_1}\phi_0|0\rangle \\
&= 1-b_1X_{j_1}p_{-i_1}(-t),
\end{aligned} \tag{2.33}$$

$$\begin{aligned}
\rho_1(t) &= -\langle 1|e^{H(t)}e^{a\phi_0^*\phi_0+b_1\phi_{i_1}^*\phi_{j_1}}\phi_{j_1}|0\rangle \\
&= -\langle 1|e^{H(t)}\phi_{j_1}|0\rangle - (e^a-1)\langle 1|e^{H(t)}\phi_0^*\phi_0\phi_{j_1}|0\rangle \\
&= -X_{j_1}\langle 0e^{H(t)}|0\rangle - (e^a-1)\langle 1|e^{H(t)}(1-\phi_0\phi_0^*)\phi_{j_1}|0\rangle \\
&= -p_{j_1}(t) - (e^a-1)X_{j_1}\langle 0|e^{H(t)}|0\rangle + (e^a-1)X_{j_1}\langle 0|e^{H(t)}\phi_0\phi_0^*|0\rangle \\
&= -e^a p_{j_1}(t),
\end{aligned} \tag{2.34}$$

$$\begin{aligned}
\sigma_1(t) &= \langle 0|e^{H(t)}e^{a\phi_0^*\phi_0+b_1\phi_{i_1}^*\phi_{j_1}}b_1^2\phi_{i_1}^*|1\rangle \\
&= b_1^2\langle 0|e^{H(t)}\phi_{i_1}^*|1\rangle + b_1^2(e^a-1)\langle 0|e^{H(t)}\phi_0^*\phi_0\phi_{i_1}^*|1\rangle \\
&\quad +b_1^3\langle 0|e^{H(t)}\phi_{i_1}^*\phi_{j_1}\phi_{i_1}^*|1\rangle + b_1^3(e^a-1)\langle 0|e^{H(t)}\phi_0^*\phi_0\phi_{i_1}^*\phi_{j_1}\phi_{i_1}^*|1\rangle \\
&= b_1^2\langle 0|e^{H(t)}\phi_{i_1}^*|1\rangle = b_1^2 p_{-i_1}(-t).
\end{aligned} \tag{2.35}$$

(2) When $j_1 = k$ and $-i_1 \neq k$, we find $f_{0,i_1} = b_1(1-e^a)$. So one has $d_0 = b_1$, $e_{i_1} = -e^a$. Then the solutions of (1.20)-(1.22) are

$$\tau_0(t) = e^a + e^a b_1\sum_{n\geq 0} p_{n-i_1}(-t)p_{j_1-n}(t) - (e^a-1)b_1\sum_{n\geq 0} p_n(-t)p_{j_1-n}(t)p_{-i_1}(-t), \tag{2.36}$$

$$\tau_1(t) = 1-b_1X_{j_1}p_{-i_1}(-t), \tag{2.37}$$

$$\begin{aligned}
\rho_1(t) &= \langle 1|e^{H(t)}e^{a\phi_0^*\phi_0+b_1\phi_{i_1}^*\phi_{j_1}}b_1\phi_0|0\rangle \\
&= b_1\langle 1|e^{H(t)}\phi_0|0\rangle + b_1^2\langle 1|e^{H(t)}\phi_{i_1}^*\phi_{j_1}\phi_0|0\rangle \\
&= b_1X_0\langle 0|e^{H(t)}|0\rangle - b_1^2X_{j_1}\langle 0|e^{H(t)}\phi_{i_1}^*\phi_0|0\rangle \\
&= b_1 - b_1^2X_{j_1}p_{i_1}(-t),
\end{aligned} \tag{2.38}$$

$$\sigma_1(t) = \langle 0|e^{H(t)}e^{a\phi_0^*\phi_0+b_1\phi_{i_1}^*\phi_{j_1}}(1-e^a)\phi_{i_1}^*|1\rangle = (1-e^a)p_{-i_1}(-t). \tag{2.39}$$

(3) When $j_1 \neq k$ and $-i_1 = k$, one has $d_{j_1} = b_1$ and $e_0 = e^{-a}-1$. The solutions of (1.20)-(1.22) are

$$\tau_0(t) = e^a + e^a b_1\sum_{n\geq 0} p_{n-i_1}(-t)p_{j_1-n}(t) - (e^a-1)b_1\sum_{n\geq 0} p_n(-t)p_{j_1-n}(t)p_{-i_1}(-t), \tag{2.40}$$

$$\tau_1(t) = 1-b_1X_{j_1}p_{-i_1}(-t), \tag{2.41}$$

$$\rho_1(t) = \langle 1|e^{H(t)}e^{a\phi_0^*\phi_0+b_1\phi_{i_1}^*\phi_{j_1}}b_1\phi_{j_1}|0\rangle = b_1e^a p_{j_1}(t), \tag{2.42}$$

$$\begin{aligned}
\sigma_1(t) &= \langle 0|e^{H(t)}e^{a\phi_0^*\phi_0+b_1\phi_{i_1}^*\phi_{j_1}}(e^{-a}-1)\phi_0^*|1\rangle \\
&= (1-e^a)\langle\phi_0^*(t)\phi_0(t)\rangle \\
&\quad +(1-e^a)b_1\Big(\langle\phi_0^*(t)\phi_0(t)\rangle\langle\phi_{i_1}^*(t)\phi_{j_1}(t)\rangle - \langle\phi_0^*(t)\phi_{j_1}(t)\rangle\langle\phi_{i_1}^*(t)\phi_0(t)\rangle\Big) \\
&= 1-e^a + (1-e^a)b_1\Big(-p_{-i_1}(-t)\sum_{n\geq 0}p_n(-t)p_{j_1-n}(t) + \sum_{m\geq 0}p_{m-i_1}(-t)p_{j_1-m}(t)\Big).
\end{aligned} \tag{2.43}$$

(4) When $j_1 = k$ and $-i_1 = k$, one has $f_{0,i_1} = b_1(1-e^a)$, $f_{j_1,0} = b_1(1-e^{-a})$. So we may choose $d_0 = 1-e^a$, $\quad d_{j_1} = b_1$, $\quad e_{i_1} = b_1$, $\quad e_0 = 1-e^{-a}$. Then

$$\tau_0(t) = e^a + e^a b_1 \sum_{n\geq 0} p_{n-i_1}(-t)p_{j_1-n}(t) - (e^a-1)b_1 \sum_{n\geq 0} p_n(-t)p_{j_1-n}(t)p_{-i_1}(-t), \quad (2.44)$$

$$\tau_1(t) = 1 - b_1 X_{j_1} p_{-i_1}(-t), \quad (2.45)$$

$$\begin{aligned}\rho_1(t) &= \langle 1|e^{H(t)}e^{a\phi_0^*\phi_0 + b_1\phi_{i_1}^*\phi_{j_1}}\Big((1-e^a)\phi_0 + b_1\phi_{j_1}\Big)|0\rangle \\ &= (1-e^a)(1-b_1X_{j_1}p_{-i_1}(-t)) + b_1e^a p_{j_1}(t), \quad (2.46)\end{aligned}$$

$$\begin{aligned}\sigma_1(t) &= \langle 0|e^{H(t)}e^{a\phi_0^*\phi_0 + b_1\phi_{i_1}^*\phi_{j_1}}\Big((1-e^{-a})\phi_0^* + b_1\phi_{i_1}^*\Big)|0\rangle \\ &= (1-e^a)1 + b_1 p_{-i_1}(-t) \\ &\quad +(1-e^a)b_1\Big(\sum_{n\geq 0} p_{n-i_1}(-t)p_{j_1-n}(t) - p_{-i_1}(-t)\sum_{m\geq 0} p_m(-t)p_{j_1-m}(t)\Big). \quad (2.47)\end{aligned}$$

(5) When $j_1 < k$, $-i_1 < k$, $j_1 \neq -i_1 + k$, or $j_1 > k$, $-i_1 < k$, or $j_1 \neq k$, $i_1 < -k$, there are no corresponding solutions.

3. Soliton solutions of *k*-constrained mKP hierarchy

In this part, we mainly use the Boson-Fermion correspondence to find the soliton solution of mKP hierarchy.

Proposition 3.1 (Boson-Fermion correspondence, [11]). *There exists an isomorphism Φ from the Fermionic Fock $\mathscr{F} = \mathscr{A}|0\rangle$ to the Bosonic Fock space $\mathscr{B} = \mathbb{C}[t_1, t_2, \ldots, u, u^{-1}]$, which is defined in the way below,*

$$\Phi(a|vac\rangle) = \sum_{l\in\mathbb{Z}} \langle l|e^{H(t)}a|vac\rangle u^l, \quad (3.1)$$

where $a|vac\rangle \in \mathscr{F}$. Then the actions of $\phi(\lambda)$ and $\phi^*(\lambda)$ on $\mathscr{F}$ can be realised on $\mathscr{B}$ as follows,

$$\Phi(\phi(\lambda)a|0\rangle) = X(\lambda)S(\lambda)\Phi(a|0\rangle), \quad \Phi(\phi^*(\lambda)a|0\rangle) = X^*(\lambda)S^*(\lambda)\Phi(a|0\rangle). \quad (3.2)$$

Here

$$(S(k)f)(t,u) = uf(t,ku), \quad (S^*(k)f)(t,u) = ku^{-1}f(t,k^{-1}u). \quad (3.3)$$

Remark 3.1. Denote $A(m)$ as the subspace of $\mathscr{A}$ with the charge m operators. If $a \in A(m)$, then only the term with $l = m$ survives in the expression of $\Phi(a|vac\rangle)$.

Let

$$X(p,q) = e^{\xi(x,p)-\xi(x,q)}e^{-\xi(\tilde{\partial},p^{-1})+\xi(\tilde{\partial},q^{-1})}, \quad (3.4)$$

Then we have

$$X(p)X^*(q) = \frac{1}{1-q/p}X(p,q). \quad (3.5)$$

Corollary 3.1 ([11]). *For $a \in A(0)$,*

$$\Phi(\phi(p)a|0\rangle) = X(p)S(p)\langle 0|e^{H(t)}a|0\rangle = uX(p)\langle 0|e^{H(t)}a|0\rangle, \tag{3.6}$$
$$\Phi(\phi^*(q)a|0\rangle) = X^*(q)S^*(q)\langle 0|e^{H(t)}a|0\rangle = qu^{-1}X^*(q)\langle 0|e^{H(t)}a|0\rangle, \tag{3.7}$$
$$\Phi(\phi(p)\phi^*(q)a|0\rangle) = \frac{q}{p-q}X(p,q)\langle 0|e^{H(t)}a|0\rangle. \tag{3.8}$$

For $a \in A(1)$,

$$\Phi(\phi(p)a|0\rangle) = X(p)S(p)\langle 1|e^{H(t)}a|0\rangle u = u^2X(p)\langle 1|e^{H(t)}a|0\rangle, \tag{3.9}$$
$$\Phi(\phi^*(q)a|0\rangle) = X^*(q)S^*(q)\langle 1|e^{H(t)}a|0\rangle u = X^*(q)\langle 1|e^{H(t)}a|0\rangle, \tag{3.10}$$
$$\Phi(\phi(p)\phi^*(q)a|0\rangle) = \frac{up}{p-q}X(p,q)\langle 1|e^{H(t)}a|0\rangle. \tag{3.11}$$

After the preparation above, one can obtain the following conclusion.

Lemma 3.1 ([35]). *Let $g = e^{\sum_{i,j=1}^{N} b_{ij}\phi(p_i)\phi^*(q_j)}$, with b_{ij} has the form $\Pi_{l,s=1}^{N}(p_l - q_s)c_{ij}$, p_i, q_j, c_{ij} are given constant and $p_i \neq q_j$. Then*

$$g^{-1}\Gamma_k g = \Gamma_k + \sum_{l,m\in\mathbb{Z}} \sum_{i,j=1}^{N} b_{ij} p_i^l q_j^{-m}(p_i^k - q_j^k)\phi_l\phi_m^*. \tag{3.12}$$

Further if $b_{ij} = \sum_{l=1}^{m} \frac{d_i^{(l)} e_j^{(l)}}{p_i^k - q_j^k}$, $\quad i, j = 1, 2\cdots N$, by Lemma 3.1 we find easily $g^{-1}\Gamma_k g = \Gamma_k + \sum_{l=1}^{m}\sum_{i=1}^{N} d_i^{(l)}\phi(p_i)\sum_{j=1}^{N} e_j^{(l)}\phi^*(q_j)$.

Proposition 3.2. *The soliton solutions of* (1.20)-(1.22) *in Proposition* 1.1 *are listed as follows:*

$$\tau_0(t) = \langle 0|e^{H(t)}g|0\rangle, \tag{3.13}$$
$$\tau_1(t) = \langle 1|e^{H(t)}g|1\rangle, \tag{3.14}$$
$$\rho_i(t) = \langle 1|e^{H(t)}g\sum_{n=1}^{N} d_n^{(i)}\phi(p_n)|0\rangle, \tag{3.15}$$
$$\sigma_i(t) = \langle 0|e^{H(t)}g\sum_{n=1}^{N} e_n^{(i)}\phi^*(q_n)|1\rangle, \tag{3.16}$$
$$i = 1, 2, \ldots, m.$$

Proof. It is easy to prove by using the conclusion of proposition 2.1. □

Proposition 3.3. *The Eqs.* (3.13)-(3.16) *formulas in the proposition 3.2 can be expressed as the following forms respectively:*

$$\tau_0(t) = 1+\sum_{n=1}^{N}\sum_{\substack{1\le i_1<i_2<\cdots<i_n\le N,\\ 1\le j_u\le N, j_u\ne j_v, u\ne v}} \widetilde{b}_{i_1 j_1}\widetilde{b}_{i_2 j_2}\cdots\widetilde{b}_{i_n j_n}$$
$$\times\Pi_{s<l}\frac{(p_{i_s}-p_{i_l})(q_{j_s}-q_{j_l})}{(p_{i_s}-q_{j_l})(q_{j_s}-p_{i_l})}e^{\sum_{\alpha=1}^{n}(\xi(t,p_{i_\alpha})-(\xi(t,q_{j_\alpha})}, \tag{3.17}$$

$$\tau_1(t) = 1+\sum_{n=1}^{N}\sum_{\substack{1\le i_1<i_2<\cdots<i_n\le N,\\ 1\le j_u\le N, j_u\ne j_v, u\ne v}} \widetilde{b}_{i_1 j_1}\widetilde{b}_{i_2 j_2}\cdots\widetilde{b}_{i_n j_n}\frac{p_{i_1}\cdots p_{i_n}}{q_{j_1}\cdots q_{j_n}}$$
$$\times\Pi_{s<l}\frac{(p_{i_s}-p_{i_l})(q_{j_s}-q_{j_l})}{(p_{i_s}-q_{j_l})(q_{j_s}-p_{i_l})}e^{\sum_{\alpha=1}^{n}(\xi(t,p_{i_\alpha})-\xi(t,q_{j_\alpha}))}, \tag{3.18}$$

$$\rho_i(t) = \sum_{n=1}^{N} d_n^{(i)}e^{\xi(t,p_n)}+\sum_{j=1}^{N}\sum_{n=1}^{N-1}\sum_{\substack{1\le i_1<i_2<\cdots<i_n\le N\\ 1\le j_u\le N, j_u\ne j_v, u\ne v\\ j_u\ne j}} d_j^{(i)}\widetilde{b}_{i_1 j_1}\widetilde{b}_{i_2 j_2}\cdots\widetilde{b}_{i_n j_n}$$
$$\times\Pi_{s<l}\frac{(p_{i_s}-p_{i_l})(q_{j_s}-q_{j_l})}{(p_{i_s}-q_{j_l})(q_{j_s}-p_{i_l})}\Pi_{\alpha=1}^{n}\frac{(p_j-p_{i_\alpha})}{(p_j-q_{j_\alpha})}e^{\sum_{\beta=1}^{n}(\xi(t,p_{i_\beta})+\xi(t,p_j)-\xi(t,q_{j_\beta}))}, \tag{3.19}$$

$$\sigma_i(t) = \sum_{n=1}^{N} e_n^{(i)}e^{-\xi(t,q_n)}+\sum_{j=1}^{N}\sum_{n=1}^{N-1}\sum_{\substack{1\le i_1<i_2<\cdots<i_n\le N,\\ 1\le j_u\le N, j_u\ne j_v, u\ne v\\ j_u\ne j}} e_j^{(i)}\widetilde{b}_{i_1 j_1}\widetilde{b}_{i_2 j_2}\cdots\widetilde{b}_{i_n j_n}\frac{p_{i_1}\cdots p_{i_n}}{q_{j_1}\cdots q_{j_n}}$$
$$\times\Pi_{s<l}\frac{(p_{i_s}-p_{i_l})(q_{j_s}-q_{j_l})}{(p_{i_s}-q_{j_l})(q_{j_s}-p_{i_l})}\Pi_{\alpha=1}^{n}\frac{(p_j-q_{j_\alpha})}{(p_{i_\alpha}-q_j)}e^{\sum_{\beta=1}^{n}(\xi(t,p_{i_\beta})-\xi(t,q_j)-\xi(t,q_{j_\beta}))}, \tag{3.20}$$

where $\widetilde{b}_{ij}=\frac{b_{ij}q_j}{p_i-q_j}$, $i, j=1,2,\ldots,N$.

Proof. We first prove (3.18). Using Lemma 3.1, one can find

$$\begin{aligned}\tau_1(t) &= \langle 1|e^{H(t)}g|1\rangle\\ &= u^{-1}\Phi(g\phi_0|0\rangle)\\ &= u^{-1}\Phi\Big(\Pi_{i,j=1}^{N}(1+\widetilde{b}_{ij}\frac{p_i-q_j}{q_j}\phi(p_i)\phi^*(q_j)\cdot\phi_0|0\rangle)\Big).\end{aligned} \tag{3.21}$$

Then according to (3.11) of Corollary 3.1, the above equation can be written as

$$\begin{aligned}
&u^{-1}\Pi_{i,j=1}^{N}\Big(1+\widetilde{b}_{ij}\frac{p_i}{q_j}X(p_i,q_j)\Big)\langle 1|e^{H(t)}\phi_0|0\rangle u\\
&=\Pi_{i,j=1}^{N}\Big(1+\widetilde{b}_{ij}\frac{p_i}{q_j}X(p_i,q_j)\Big)X_0\langle 0|e^{H(t)}\phi_0|0\rangle\\
&=\Pi_{i,j=1}^{N}\Big(1+\widetilde{b}_{ij}\frac{p_i}{q_j}X(p_i,q_j)\Big)\cdot 1\\
&=1+\sum_{n=1}^{N}\sum_{\substack{1\le i_1<i_2<\cdots<i_n\le N,\\ 1\le j_u\le N, j_u\neq j_v, u\neq v}}\widetilde{b}_{i_1j_1}\widetilde{b}_{i_2j_2}\cdots\widetilde{b}_{i_nj_n}\frac{p_{i_1}\cdots p_{i_n}}{q_{j_1}\cdots q_{i_n}}\\
&\times\Pi_{s<l}\frac{(p_{i_s}-p_{i_l})(q_{j_s}-q_{j_l})}{(p_{i_s}-q_{j_l})(q_{j_s}-p_{i_l})}e^{\sum_{\alpha=1}^{n}(\xi(t,p_{i_\alpha})-\xi(t,q_{j_\alpha}))}.
\end{aligned}\tag{3.22}$$

Similarly, by lemma 3.1 and Corollary 3.1

$$\begin{aligned}
\sigma_m(t)&=\Phi\Big(g\sum_{j=1}^{N}e_j^{(m)}\phi^*(q_j)|1\rangle\Big)\\
&=\Pi_{i,j=1}^{N}\Big(1+\widetilde{b}_{ij}X(p_i,q_j)\Big)\Phi(\sum_{l=1}^{N}e_l^{(m)}\phi^*(q_l)\phi_0|0\rangle)\\
&=\Pi_{i,j=1}^{N}\Big(1+\widetilde{b}_{ij}X(p_i,q_j)\Big)\sum_{l=1}^{N}e_l^{(m)}X^*(q_l)S^*(q_l)\Phi(\phi_0|0\rangle\\
&=\Pi_{i,j=1}^{N}\Big(1+\widetilde{b}_{ij}X(p_i,q_j)\Big)\sum_{l=1}^{N}e_l^{(m)}X^*(q_l)S^*(q_l)\cdot X_0\langle 0|e^{H(t)}|0\rangle u\\
&=\Pi_{i,j=1}^{N}\Big(1+\widetilde{b}_{ij}X(p_i,q_j)\Big)\sum_{l=1}^{N}e_l^{(m)}e^{-\xi(t,q_l)}\cdot 1\\
&=\sum_{n=1}^{N}e_n^{(m)}e^{-\xi(t,q_n)}+\sum_{j=1}^{N}\sum_{n=1}^{N-1}\sum_{\substack{1\le i_1<i_2<\cdots<i_n\le N,\\ 1\le j_u\le N, j_u\neq j_v, u\neq v\\ j_u\neq j}}e_j^{(m)}\widetilde{b}_{i_1j_1}\widetilde{b}_{i_2j_2}\cdots\widetilde{b}_{i_nj_n}\frac{p_{j_1}\cdots p_{j_n}}{q_{j_1}\cdots q_{j_n}}\\
&\times\Pi_{s<l}\frac{(p_{i_s}-p_{i_l})(q_{j_s}-q_{j_l})}{(p_{i_s}-q_{j_l})(q_{j_s}-p_{i_l})}\Pi_{\alpha=1}^{n}\frac{(p_j-q_{j_\alpha})}{(p_{i_\alpha}-q_j)}e^{\sum_{\beta=1}^{n}(\xi(t,p_{i_\beta})-\xi(t,q_j)-\xi(t,q_{j_\beta}))}.
\end{aligned}\tag{3.23}$$

The proof method of (3.17) and (3.19) is the same as that of (3.18), and no more description is given here. □

Remark 3.2. If $N=m$ in Proposition 3.2 we may choose $d_i^{(l)}=b_{il}(p_i^k-q_l^k)$ and $e_l^{(j)}=\delta_{lj}$. Accordingly, the solutions of the Hirota bilinear equation can be simplified to

$$\tau_0(t)=\langle 0|e^{H(t)}g|0\rangle,\tag{3.24}$$
$$\tau_1(t)=\langle 1|e^{H(t)}g|1\rangle,\tag{3.25}$$
$$\rho_i(t)=\sum_{j=1}^{m}b_{ij}(p_i^k-q_j^k)\langle 1|e^{H(t)}g\phi(p_i)|0\rangle,\tag{3.26}$$
$$\sigma_i(t)=\langle 0|e^{H(t)}g\phi^*(q_i)|1\rangle,\tag{3.27}$$
$$i=1,2,\ldots,m.$$

Example 3.1. For $m = N = 2$, $b_{11} \neq 0$, $b_{22} \neq 0$, $b_{12} = b_{21} = 0$, we have two-soliton solutions of (1.20)-(1.22):

$$\tau_0(t) = 1 + \frac{b_{11}q_1}{p_1 - q_1} e^{\xi(t,p_1)-\xi(t,q_1)} + \frac{b_{22}q_2}{p_2 - q_2} e^{\xi(t,p_2)-\xi(t,q_2)}$$
$$+ b_{11}b_{22} \frac{q_1 q_2 (p_1 - p_2)(q_1 - q_2)}{(p_1 - q_1)(p_1 - q_2)(q_1 - p_2)(p_2 - q_2)} e^{\xi(t,p_1)+\xi(t,p_2)-\xi(t,q_1)-\xi(t,q_2)}, \quad (3.28)$$

$$\tau_1(t) = 1 + \frac{b_{11}p_1}{p_1 - q_1} e^{\xi(t,p_1)-\xi(t,q_1)} + \frac{b_{22}p_2}{p_2 - q_2} e^{\xi(t,p_2)-\xi(t,q_2)}$$
$$+ b_{11}b_{22} \frac{p_1 p_2 (p_1 - p_2)(q_1 - q_2)}{(p_1 - q_1)(p_1 - q_2)(q_1 - p_2)(p_2 - q_2)} e^{\xi(t,p_1)+\xi(t,p_2)-\xi(t,q_1)-\xi(t,q_2)}, \quad (3.29)$$

$$\rho_1(t) = b_{11}(p_1^k - q_1^k)\Big(e^{\xi(t,p_1)} + \frac{b_{22}q_2(p_1 - p_2)}{(p_1 - q_2)(p_2 - q_2)} e^{\xi(t,p_1)+\xi(t,p_2)-\xi(t,q_2)}\Big), \quad (3.30)$$

$$\rho_2(t) = b_{22}(p_2^k - q_2^k)\Big(e^{\xi(t,p_2)} - \frac{b_{11}q_1(p_1 - p_2)}{(p_1 - q_1)(p_2 - q_1)} e^{\xi(t,p_1)+\xi(t,p_2)-\xi(t,q_1)}\Big), \quad (3.31)$$

$$\sigma_1(t) = e^{-\xi(t,q_1)} - \frac{b_{22}p_2(q_2 - q_1)}{(p_2 - q_1)(p_2 - q_2)} e^{\xi(t,p_2)-\xi(t,q_1)-\xi(t,q_2)}, \quad (3.32)$$

$$\sigma_2(t) = e^{-\xi(t,q_2)} + \frac{b_{11}p_1(q_2 - q_1)}{(p_1 - q_1)(p_1 - q_2)} e^{\xi(t,p_1)-\xi(t,q_1)-\xi(t,q_2)}. \quad (3.33)$$

4. Conclusions and Discussions

In this paper, we construct the solutions of the constrained mKP hierarchy through the bilinear representation and the free fermion operators. The corresponding solutions are expressed in terms of the vacuum expectation value of the Clifford operators, which are presented in Proposition 2.1. Then by choosing $g = e^{a\phi_0^*\phi_0 + \sum_{n=1}^N b_n \phi_{i_n}^* \phi_{j_n}} = e^Y$ with $i_n < 0$ and $j_n > 0$, some examples of rational solutions are given. At last, by selecting $g = e^{\sum_{i,j=1}^N b_{ij}\phi(p_i)\phi^*(q_j)}$, the corresponding soliton solutions are obtained, which are summarized in Proposition 3.2 and Proposition 3.3.

Just as we know, the KdV hierarchy is corresponding to the affine Lie algebra $\hat{sl_2}$ [14, 25]. But the algebraic structures of the constrained mKP hierarchy are unknown. The results in this paper are expected to be helpful for the understanding the algebraic structures of the constrained mKP hierarchy.

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