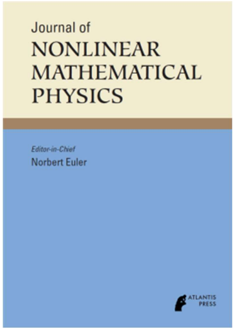

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A generalization of the Landau-Lifschitz equation: breathers and rogue waves

Ruomeng Li, Xianguo Geng[*] and Bo Xue

School of Mathematics and Statistics, Zhengzhou University, 100 Kexue Road, Zhengzhou, Henan 450001, People's Republic of China

liruomeng@zzu.edu.cn (RM Li); *xggeng@zzu.edu.cn* (XG Geng); *xuebo@zzu.edu.cn* (B Xue)

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A generalization of the Landau-Lifschitz equation with uniaxial anisotropy is proposed, which can also reduce to the derivative nonlinear Schrödinger equation under an infinitesimal parameter. Based on the gauge transformation between Lax pairs, an N-fold generalized Darboux transformation is constructed for the generalization of the Landau-Lifschitz equation with uniaxial anisotropy. As applications of the N-fold generalized Darboux transformations, several types of exact solutions of the generalization of the Landau-Lifschitz equation with uniaxial anisotropy are obtained, including soliton solutions, Akhmediev breather solutions and rogue-wave solutions.

Keywords: generalized uniaxial Landau-Lifschitz equation, N-fold generalized Darboux transformation, Akhmediev breathers, solitons, rogue waves

2000 Mathematics Subject Classification: 37K35, 35Q51, 35Q58

1. Introduction

The continuous classical Heisenberg ferromagnet equation (the Heisenberg equation for short) can be written as either

$$iq_t - wq_{xx} + w_{xx}q = 0, \quad |q|^2 + w^2 = 1, \tag{1.1}$$

or

$$Q_t + Q \times Q_{xx} = 0, \quad Q \cdot Q = 1, \tag{1.2}$$

where q is a complex potential, w is a real potential, $Q = (\mathrm{Re}\, q, \mathrm{Im}\, q, w)^T$ is a column vector, the symbols $\times$ and $\cdot$ denote the cross and the dot product, respectively. In Ref. [38], the inverse scattering method was applied to the Heisenberg equation. The gauge transformations between the Heisenberg equation (1.2) and the focusing or the defocusing nonlinear Schrödinger equations are found [28, 43]. An effective approach [11] is developed to obtain the generation of closed-form solitons solution of the Heisenberg equation (1.2). In Ref. [2], the higher-order soliton solutions of the Heisenberg equation (1.2) are constructed by using the Darboux transformation (see Refs. [13, 15, 17, 18, 22, 26, 27, 29, 32, 36, 39, 40, 44, 45]). The Heisenberg equation (1.2) is the

[*]Corresponding author (xggeng@zzu.edu.cn)

isotropic case of the Landau-Lifschitz (LL) equation,

$$Q_t + Q \times Q_{xx} + Q \times (JQ) = 0, \quad Q \cdot Q = 1, \tag{1.3}$$

where $J = \mathrm{diag}\{J_1, J_2, J_3\}$ is a real diagonal matrix, $J_1 \le J_2 \le J_3$, which can be reduced to the sine-Gordon equation and the nonlinear Schrödinger equation [12]. The Cauchy problem for the LL equation (1.3) is studied by using the inverse transform method, and the Cauchy problem is also changed into the matrix Riemann problem on a torus and a certain Fredholm integral equation, from which an exact N-soliton solution is constructed [30, 33, 34]. When $J_1 = J_2$ or $J_2 = J_3$, the LL equation is called an LL equation with uniaxial anisotropy (a uniaxial LL equation for short). Please see Refs. [3,4] and the references therein for more information on the names of this equation. Particularly, when $J = \mathrm{diag}\{0, 0, 4\beta\}$, the LL equation (1.3) can be rewritten as

$$iq_t - wq_{xx} + w_{xx}q + 4\beta wq = 0, \quad |q|^2 + w^2 = 1, \quad \beta \in \mathbb{R}. \tag{1.4}$$

In Ref. [20], a Darboux transformation of the LL equation (1.4) was constructed and its various exact solutions were obtained. The Heisenberg equation (1.2) is also called the isotropy LL equation [2, 20, 37].

In this paper, we propose a generalized uniaxial LL equation (a guLL equation for short),

$$iq_t - wq_{xx} + w_{xx}q + 2i\alpha(w^2 q)_x + 4\beta wq = 0, \quad |q|^2 + w^2 = 1, \quad \alpha, \beta \in \mathbb{R}, \tag{1.5}$$

or equivalently

$$Q_t + Q \times Q_{xx} + 4\beta Q \times (\hat{J}Q) + \alpha\{Q \times [Q \times (\hat{J}Q)] - \hat{J}Q\}_x = 0, \tag{1.6}$$

where $\hat{J} = \mathrm{diag}(0, 0, 1)$. Apparently, equation (1.5) can be reduced to the uniaxial LL equation (1.4) and the Heisenberg equation (1.1) when $\alpha = 0$ and $\alpha = \beta = 0$, respectively. Let $\varepsilon \in \mathbb{R}$ be a small parameter and $q = \varepsilon u$, $\alpha = \varepsilon^{-2}$ and $\beta = 0$. Then (1.5) can be rewritten as

$$iu_t - \sqrt{1 - \varepsilon^2 |u|^2} u_{xx} + (\sqrt{1 - \varepsilon^2 |u|^2})_{xx} u - 2i(|u|^2 u)_x = 0, \tag{1.7}$$

which turns out the derivative nonlinear Schrödinger equation

$$iu_t - u_{xx} - 2i(|u|^2 u)_x = 0, \tag{1.8}$$

when $\varepsilon \to 0$. Therefore, (1.5) is closely related to the derivative nonlinear Schrödinger equation (1.8).

Rogue waves are giant wave events in nonlinear deep water gravity waves that rise to surprising heights above the background wave field. Holes are deep troughs which occur before and/or after the largest crests [31]. Rogue periodic waves [7,8], rouge waves [5,6] and other exact solutions [35] are always interesting issues. Solitons and breathers are also important nonlinear phenomena that attract much attention. Our main aim in this paper is to construct N-fold Darboux transformations, by which soliton solutions, breather solutions and rogue-hole solutions of the guLL equation (1.5) are obtained. Some special limits of N-fold Darboux transformations are called generalized Darboux transformations [19], by which rogue wave solutions of the corresponding integrable nonlinear equation can be obtained. However, it is difficult to calculate the limits of N-fold Darboux transformations. To solve this problem, we give a formula for generalized Darboux transformations without taking limits. Because classical and generalized Darboux transformations are given by the same formula (2.33) in this paper, classical and generalized Darboux transformations are both called N-fold Darboux transformations for simplicity.

The major innovations of the present paper include the following aspects. First, we propose a generalized uniaxial anisotropy Landau-Lifschitz equation (1.5), which contains two arbitrary constants $\alpha, \beta \in \mathbb{R}$. Second, we give exact formulas for N-fold Darboux transformations without taking limits. Avoiding limits can greatly reduce computational complexity, especially when the seed solution is not trivial or when N is large. Third, as applications of the obtained Darboux transformations, several nonlinear phenomena (e.g., solitons, breathers and rogue holes) are revealed.

The outline of this paper is as follows. In Section 2, we present a Lax pair for the guLL equation and then construct its N-fold generalized Darboux transformations with the help of gauge transformations between Lax pairs. In Section 3, as applications of the multi-fold generalized Darboux transformations, we obtain various explicit solutions for the generalization of the guLL equation, including soliton solutions, Akhmediev breather solutions and rogue-hole solutions.

2. Lax pair and Darboux transformations

The spectral and auxiliary problems associated with the guLL equation (1.5) read

$$\Phi_x(\lambda) = U(\lambda)\Phi(\lambda), \quad \Phi_t(\lambda) = V(\lambda)\Phi(\lambda), \tag{2.1}$$

where $\Phi(\lambda) = \Phi(x,t,\lambda)$ is a 2×1 vector eigenfunction, $\lambda \in \mathbb{C}$ is a constant spectral parameter, and

$$\begin{aligned} U(\lambda) &= \begin{pmatrix} i\lambda w & (\lambda-\zeta_1)q \\ -(\lambda-\zeta_2)q^* & -i\lambda w \end{pmatrix}, \\ V(\lambda) &= \begin{pmatrix} i\lambda b + 2i(\lambda-\zeta_1)(\lambda-\zeta_2)w & (\lambda-\zeta_1)(a+2\lambda q) \\ -(\lambda-\zeta_2)(a^*+2\lambda q^*) & -i\lambda b - 2i(\lambda-\zeta_1)(\lambda-\zeta_2)w \end{pmatrix}, \\ a &= -i(wq_x - qw_x) - (\zeta_1+\zeta_2)w^2 q, \quad b = -\operatorname{Im}(q^* q_x) + (\zeta_1+\zeta_2)(2w - w^3), \end{aligned} \tag{2.2}$$

where ζ_1 and ζ_2 are the two roots of $\zeta^2 - 2\alpha\zeta + \beta = 0$. Here we assume that $\zeta_1\zeta_2 \in \mathbb{R}$ and $\zeta_1 + \zeta_2 \in \mathbb{R}$, which means: (i) if $\zeta_1 \in \mathbb{R}$, then $\zeta_2 \in \mathbb{R}$; otherwise (ii) if $\zeta_1 \in \mathbb{C}\setminus\mathbb{R}$, then $\zeta_2 = \zeta_1^*$.

Suppose that $\Psi(\lambda) = \Psi(x,t,\lambda)$ is the fundamental solution matrix of (2.1) with the initial condition

$$\Psi(0,0,\lambda) = I_2, \tag{2.3}$$

and expand $\Psi(\lambda+\varepsilon)$ into the Taylor series,

$$\Psi(\lambda+\varepsilon) = \sum_{j=0}^{\infty} \tfrac{1}{j!}\Psi^{(j)}(\lambda)\varepsilon^j = \tfrac{1}{0!}\Psi(\lambda) + \tfrac{1}{1!}\dot{\Psi}(\lambda)\varepsilon + \tfrac{1}{2!}\ddot{\Psi}(\lambda)\varepsilon^2 + \tfrac{1}{3!}\dddot{\Psi}(\lambda)\varepsilon^3 + \cdots, \tag{2.4}$$

where $\Psi^{(j)}(\lambda) = \frac{\partial^j}{\partial\lambda^j}\Psi(\lambda)$, $\dot{\Psi}(\lambda) = \Psi^{(1)}(\lambda)$, $\ddot{\Psi}(\lambda) = \Psi^{(2)}(\lambda)$ and $\dddot{\Psi}(\lambda) = \Psi^{(3)}(\lambda)$. Denote

$$\dot{U} = \tfrac{\partial}{\partial\lambda}U(\lambda), \quad \dot{V}(\lambda) = \tfrac{\partial}{\partial\lambda}V(\lambda), \quad \ddot{V} = \tfrac{\partial^2}{\partial\lambda^2}V(\lambda), \tag{2.5}$$

where $\dot{U}(\lambda)$ and $\ddot{V}(\lambda)$ are written as $\dot{U}$ and $\ddot{V}$, because they are independent of λ. Taking the derivatives of (2.1) with respect to λ to the jth order we have

$$\begin{aligned}&\dot{\Psi}_x(\lambda)=U(\lambda)\dot{\Psi}(\lambda)+\dot{U}\Psi(\lambda),\quad \dot{\Psi}_t(\lambda)=V(\lambda)\dot{\Psi}(\lambda)+\dot{V}(\lambda)\Psi(\lambda),\\&\Psi_x^{(j)}(\lambda)=U(\lambda)\Psi^{(j)}(\lambda)+j\dot{U}\Psi^{(j-1)}(\lambda),\\&\Psi_t^{(j)}(\lambda)=V(\lambda)\Psi^{(j)}(\lambda)+j\dot{V}(\lambda)\Psi^{(j-1)}(\lambda)+\tfrac{1}{2}j(j-1)\ddot{V}\Psi^{(j-2)}(\lambda),\quad (j\geq 2).\end{aligned}\tag{2.6}$$

Substituting

$$U(\lambda)=-\Psi(\lambda)[\Psi(\lambda)^{-1}]_x,\quad V(\lambda)=-\Psi(\lambda)[\Psi(\lambda)^{-1}]_t\tag{2.7}$$

into (2.6), we find

$$\begin{aligned}&[\Psi(\lambda)^{-1}\dot{\Psi}(\lambda)]_x=\Psi(\lambda)^{-1}\dot{U}\Psi(\lambda),\quad [\Psi(\lambda)^{-1}\dot{\Psi}(\lambda)]_t=\Psi(\lambda)^{-1}\dot{V}(\lambda)\Psi(\lambda),\\&[\Psi(\lambda)^{-1}\Psi^{(j)}(\lambda)]_x=j\Psi(\lambda)^{-1}\dot{U}\Psi^{(j)}(\lambda),\\&[\Psi(\lambda)^{-1}\dot{\Psi}^{(j)}(\lambda)]_t=j\Psi(\lambda)^{-1}\dot{V}(\lambda)\Psi^{(j-1)}(\lambda)+\tfrac{1}{2}j(j-1)\Psi(\lambda)^{-1}\ddot{V}\Psi^{(j-2)}(\lambda),\quad (j\geq 2).\end{aligned}\tag{2.8}$$

Using (2.3) we have

$$\begin{aligned}\dot{\Psi}(x,t,\lambda)&=\int_{(0,0)}^{(x,t)}\Psi(x,t,\lambda)\Psi(\hat{x},\hat{t},\lambda)^{-1}[\dot{U}(\hat{x},\hat{t})\mathrm{d}\hat{x}+\dot{V}(\hat{x},\hat{t},\lambda)\mathrm{d}\hat{t}]\Psi(\hat{x},\hat{t},\lambda),\\ \Psi^{(j)}(x,t,\lambda)&=\int_{(0,0)}^{(x,t)}\Psi(x,t,\lambda)\Psi(\hat{x},\hat{t},\lambda)^{-1}[j\dot{U}(\hat{x},\hat{t})\mathrm{d}\hat{x}+j\dot{V}(\hat{x},\hat{t},\lambda)\mathrm{d}\hat{t}\\&\qquad+\tfrac{1}{2}j(j-1)\ddot{V}(\hat{x},\hat{t},\lambda)\mathrm{d}\hat{t}]\Psi^{(j-1)}(\hat{x},\hat{t},\lambda),\quad (j\geq 2).\end{aligned}\tag{2.9}$$

In terms of $\Psi(\lambda)$, a general solution $\Phi(\lambda)$ of the Lax pair (2.1) can be given by

$$\Phi(x,t,\lambda)=\Psi(x,t,\lambda)Y(\lambda),\tag{2.10}$$

where $Y(\lambda)=\Phi(0,0,\lambda)$ denotes an intial condition. Resorting to

$$\Phi^{(j)}(\lambda)=\tfrac{\partial^j}{\partial\lambda^j}\Phi(\lambda)=\sum_{k=0}^{j}\binom{j}{k}\Psi^{(j-k)}(\lambda)Y^{(k)}(\lambda),\tag{2.11}$$

we can give $\Phi^{(j)}(\lambda)$ in terms of $\{\Psi^{(0)}(\lambda),\ldots,\Psi^{(j)}(\lambda)\}$.

Suppose $\lambda_1,\ldots,\lambda_K\in\mathbb{C}\setminus\mathbb{R}$, ($\lambda_j\neq\lambda_k$ if $j\neq k$), and $N_1,\ldots,N_K\in\mathbb{Z}_+$ are fixed. Set $N=N_1+\cdots+N_K$, and denote

$$\Phi_k=\Phi(\lambda_k),\quad \Phi_k^{(j)}=\Phi^{(j)}(\lambda_k),\quad (1\leq k\leq K, 0\leq j\leq N_k-1).\tag{2.12}$$

Remark 2.1. From (2.11) it is apparent that

$$\Phi_k^{(j)}=\sum_{l=0}^{j}\binom{j}{l}\Psi^{(j-l)}(\lambda_k)Y^{(l)}(\lambda_k),\quad (1\leq k\leq K, 0\leq j\leq N_k).\tag{2.13}$$

The quantities $\Psi^{(j)}(\lambda_k)$ can be obtained from two approaches. First, one can solve the Lax pair (2.1) at $\lambda=\lambda_k+\varepsilon$, and then calclculate the derivatives $\frac{\partial^j}{\partial\varepsilon^j}|_{\varepsilon=0}\Psi(\lambda_k+\varepsilon)$. However, when the seed solution is not trivial, even though it is easy to obtain $\Psi(\lambda_k)$, it is usually difficult or tedious to obatin the exact expression of $\Psi(\lambda_k+\varepsilon)$, however small ε is. Therefore, we develop a second approach, i.e., the equations (2.9) and (2.11). In this approach, the spectral constant λ_k is fixed. The derivatives with resect to λ are replaced with integrals with resect to (x,t). In practice, we find the integrals in (2.9) are always easy to calculate with the support of some mathematical software, like `Mathematica`. The quantites $Y^{(j)}(\lambda_k)$ denotes some arbitrary constants, because $Y(\lambda)$ is the initial condition of $\Phi(x,t,\lambda)$ that can be chosen freely.

Now we construct an N-fold generalized Darboux transformation directly in terms of N quantities

$$\{\Phi_k^{(j)} \mid 1 \le k \le K, 0 \le j \le N_k\} \tag{2.14}$$

without taking limits. In order to derive a Darboux transformation, we have to find a Darboux matrix $\mathscr{T}(\lambda)$ and two new potentials $\tilde{q}$ and $\tilde{w}$ such that $\tilde{\Phi}(\lambda) = \mathscr{T}(\lambda)\Phi(\lambda)$ satisfies the new Lax pair

$$\tilde{\Phi}_x(\lambda) = \tilde{U}(\lambda)\tilde{\Phi}(\lambda), \quad \tilde{\Phi}_t(\lambda) = \tilde{V}(\lambda)\tilde{\Phi}(\lambda), \tag{2.15}$$

where $\tilde{U}(\lambda) = U(\lambda)|_{q=\tilde{q},w=\tilde{w}}$ and $\tilde{V}(\lambda) = V(\lambda)|_{q=\tilde{q},w=\tilde{w}}$. A direct calculation shows that $\mathscr{T}(\lambda)$ satisfies

$$\tilde{U}(\lambda)\mathscr{T}(\lambda) = \mathscr{T}_x(\lambda) + \mathscr{T}(\lambda)U(\lambda), \quad \tilde{V}(\lambda)\mathscr{T}(\lambda) = \mathscr{T}_t(\lambda) + \mathscr{T}(\lambda)V(\lambda). \tag{2.16}$$

To construct an N-fold generalized Darboux transformation, we assume that $\mathscr{T}(\lambda)$ is a suitable polynomial of degree N. For the sake of simplicity, we first consider the one-fold case. In this case, we write $\tilde{q}, \tilde{w}$ and $\mathscr{T}(\lambda)$ as $\hat{q}, \hat{w}$ and $T(\lambda)$, respectively, to emphasize that $N = 1$.

Theorem 2.1. *Suppose that* (q,w) *is a known solution of the guLL equation* (1.5)*, and fix* $\lambda_1 \in \mathbb{C}\setminus\mathbb{R}$. *Assume that* $\Phi_1 = \Phi(\lambda_1) = (\phi_1, \psi_1)^T$ *is a solution of the Lax pair* (2.1) *when* $\lambda = \lambda_1$ *and* $\phi_1 \neq 0$. *Suppose* r *and* s *are defined by*

$$r = -\frac{1 + (\lambda_1 - \zeta_1)(\lambda_1 - \zeta_2)|\rho_1|^2}{\lambda_1 + \lambda_1^*(\lambda_1 - \zeta_1)(\lambda_1 - \zeta_2)|\rho_1|^2}, \quad s = \frac{(\lambda_1 - \lambda_1^*)\rho_1^*}{\lambda_1 + \lambda_1^*(\lambda_1 - \zeta_1)(\lambda_1 - \zeta_2)|\rho_1|^2}, \tag{2.17}$$

where $\rho_1 = (\lambda_1 - \zeta_2)^{-1}\psi_1\phi_1^{-1}$. *Then* $(\hat{q}, \hat{w})$ *determined by the Darboux transformation*

$$\hat{q} = \frac{ms}{m^*s^*}q^* - \frac{m_x}{\zeta_1\zeta_2 m^* s^*}, \quad \hat{w} = w + \frac{im_x}{\zeta_1\zeta_2 mr} \tag{2.18}$$

is a new solution of the guLL equation (1.5)*, and*

$$T(\lambda) = \begin{pmatrix} m(1+\lambda r) & (\lambda - \zeta_1)ms \\ -(\lambda - \zeta_2)m^*s^* & m^*(1+\lambda r^*) \end{pmatrix} \tag{2.19}$$

is the corresponding Darboux matrix, where

$$m = \begin{cases} [(1+\zeta_1 r)^{\zeta_2}(1+\zeta_2 r)^{-\zeta_1}]^{1/(\zeta_1-\zeta_2)}, & \text{if } \alpha^2 \neq \beta, \\ \frac{1}{1+\alpha r} e^{\frac{\alpha r}{1+\alpha r}}, & \text{if } \alpha^2 = \beta. \end{cases} \tag{2.20}$$

Moreover, $\hat{\Phi}(\lambda) = T(\lambda)\Phi(\lambda)$ *satisfies the new Lax pair*

$$\hat{\Phi}_x = \hat{U}\hat{\Phi}, \quad \hat{\Phi}_t = \hat{V}\hat{\Phi}, \quad \hat{U} = U|_{q=\hat{q},w=\hat{w}}, \quad \hat{V} = V|_{q=\hat{q},w=\hat{w}}. \tag{2.21}$$

Proof. We first prove that $(\hat{q},\hat{w})$ is a new solution of the guLL equation (1.5) by proving that $T(\lambda)$ satisfies (2.16). For notation convenience, we denote

$$\begin{aligned} D(\lambda) &= \begin{pmatrix} D_1(\lambda) & (\lambda-\zeta_1)D_2(\lambda) \\ -(\lambda-\zeta_2)D_2(\lambda^*)^* & D_2(\lambda^*)^* \end{pmatrix} = \hat{U}(\lambda)T(\lambda) - T_x(\lambda) - T(\lambda)U(\lambda), \\ \Delta(\lambda) &= \begin{pmatrix} \Delta_1(\lambda) & (\lambda-\zeta_1)\Delta_2(\lambda) \\ -(\lambda-\zeta_2)\Delta_2(\lambda^*)^* & \Delta_1(\lambda^*)^* \end{pmatrix} = \hat{V}(\lambda)T(\lambda) - T_t(\lambda) - T(\lambda)V(\lambda), \end{aligned} \tag{2.22}$$

and aim at showing $D(\lambda)\equiv 0$ and $\Delta(\lambda)\equiv 0$.

When $(\hat{w},\hat{q})$ are defined by (2.18), it is easy to see that $D_1(\lambda)$ is a linear polynomial in λ and $D_1(0)=0$. When (2.20) holds, we have

$$\frac{m_x}{m} = -\frac{\zeta_1\zeta_2 r r_x}{1+(\zeta_1+\zeta_2)r+\zeta_1\zeta_2 r^2}, \tag{2.23}$$

which implies $D_1(\lambda)\equiv 0$. From $T(\lambda_1)\Phi_1=0$ we can derive

$$D(\lambda_1)\Phi_1 = \hat{U}(\lambda_1)T(\lambda_1)\Phi_1 - T_{1,x}\Phi_1 - T(\lambda_1)U(\lambda_1)\Phi_1 = -(T_1\Phi_1)_x = 0. \tag{2.24}$$

Noting that $D_2(\lambda)$ is linear polynomials in λ, we finally arrive at $D_2(\lambda)\equiv 0$, and thus $D(\lambda)\equiv 0$.

By using $D(\lambda)\equiv 0$ and (2.20), we have

$$\frac{m_t}{m} = -\frac{\zeta_1\zeta_2 r r_t}{1+(\zeta_1+\zeta_2)r+\zeta_1\zeta_2 r^2}. \tag{2.25}$$

A straightforward calculation shows that $\Delta_1(\lambda)$ and $\Delta_2(\lambda)$ can written as

$$\Delta_1(\lambda) = \{[1+(\zeta_1+\zeta_2)r]\lambda+\zeta_1\zeta_2 r\}\delta_1, \quad \Delta_2(\lambda)=\delta_2, \tag{2.26}$$

where δ_1 and δ_2 are functions of (x,t). In view of

$$\Delta(\lambda_1)\Phi_1 = \hat{V}(\lambda_1)T(\lambda_1)\Phi_1 - T_{1,t}\Phi_1 - T(\lambda)V(\lambda_1)\Phi_1 = -(T_1\Phi_1)_t = 0, \tag{2.27}$$

we obtain $\Delta_1(\lambda)\equiv\Delta_2(\lambda)\equiv 0$, which means $\Delta(\lambda)\equiv 0$.

Since $D(\lambda)\equiv\Delta(\lambda)\equiv 0$, $(\hat{q},\hat{w})$ must be a new solution of the guLL equation (1.5). Note that the coefficients of λ^2 in $D(\lambda)=0$ implies

$$\begin{pmatrix} i\hat{w} & \hat{q} \\ -\hat{q}^* & -i\hat{w} \end{pmatrix}\begin{pmatrix} mr & ms \\ -m^*s^* & m^*r^* \end{pmatrix} = \begin{pmatrix} mr & ms \\ -m^*s^* & m^*r^* \end{pmatrix}\begin{pmatrix} iw & q \\ -q^* & -iw \end{pmatrix}, \tag{2.28}$$

and hence $|\hat{q}|^2+\hat{w}^2=1$ and $\hat{w}=\hat{w}^*$. □

Now we make full use of the N quantities (2.14) to construct and N-fold Darboux transformation. Note, how to obtain the quantites (2.14) is well introduced in Remark 2.1.

Theorem 2.2. *Suppose (q,w) is a known solution of the guLL equation* (1.5). *Fix $\lambda_1,\ldots,\lambda_K\in\mathbb{C}\setminus\mathbb{R}$, ($\lambda_j\neq\lambda_k$ if $j\neq k$) and choose $N_1,\ldots,N_K\in\mathbb{Z}_+$. Denote $N=N_1+\cdots+N_K$. Suppose Φ_k and $\Phi_k^{(j)}$*

are defined by (2.9), (2.11) *and* (2.12)*. Suppose* $\mathscr{T}(\lambda)$ *is a polynomial in* λ *of degree* N*,*

$$\mathscr{T}(\lambda)=\begin{pmatrix} M[1+\lambda R(\lambda)] & (\lambda-\zeta_1)MS(\lambda) \\ -(\lambda-\zeta_2)M^*S(\lambda^*)^* & M^*[1+\lambda R(\lambda^*)^*] \end{pmatrix}, \tag{2.29}$$

where $R(\lambda)=R(x,t,\lambda)$ *and* $S(\lambda)=S(x,t,\lambda)$ *are polynomials in* λ *of degree* $N-1$*,*

$$R(\lambda)=\sum_{j=1}^{N}\lambda^{j-1}R_j, \quad S(\lambda)=\sum_{j=1}^{N}\lambda^{j-1}S_j, \tag{2.30}$$

and $M=M(x,t)$ *is given by the values* $R(\zeta_1)=R(x,t,\zeta_1)$ *and* $R(\zeta_2)=R(x,t,\zeta_2)$*,*

$$M=\begin{cases} \{[1+\zeta_1R(\zeta_1)]^{\zeta_2}[1+\zeta_2R(\zeta_2)]^{-\zeta_1}\}^{1/(\zeta_1-\zeta_2)}, & \textit{if } \alpha^2\neq\beta, \\ \frac{1}{1+\alpha R(\alpha)}\exp\left[\frac{\alpha R(\alpha)+\alpha^2R'(\alpha)}{1+\alpha R(\alpha)}\right], & \textit{if } \alpha^2=\beta. \end{cases} \tag{2.31}$$

Suppose $\mathscr{T}(\lambda)$ *satisfies the equations*

$$\sum_{l=0}^{j}\binom{j}{l}\mathscr{T}^{(j-l)}(\lambda_k)\Phi_k^{(l)}=0, \quad (1\le k\le K, 0\le j\le N_k), \tag{2.32}$$

where $\mathscr{T}^{(j)}(\lambda)=\frac{\partial^j}{\partial\lambda^j}\mathscr{T}(\lambda)$*. Then* (i) R_j *and* S_j*,* $(1\le j\le N)$*, are uniquely determined by* (2.32)*;* (ii) $(\tilde{q},\tilde{w})$ *determined by the N-fold Darboux transformation*

$$\tilde{q}=\frac{M(R_N^2q+S_N^2q^*-2iR_NS_Nw)}{M^*(|R_N|^2+|S_N|^2)}, \quad \tilde{w}=\frac{-iR_NS_N^*q+iR_N^*S_Nq^*+(|R_N|^2-|S_N|^2)w}{|R_N|^2+|S_N|^2}, \tag{2.33}$$

is a new solution of the guLL equation (1.5)*; and* (iii) *the corresponding Darboux matrix is* $\mathscr{T}(\lambda)$*. Moreover,* R_j *and* S_j *are given explicitly in* (2.46) *for convenience of application because the their expressions are very long.*

Proof. Denote

$$(\hat{\lambda}_1,\dots,\hat{\lambda}_N)=(\underbrace{\lambda_1,\dots,\lambda_1}_{N_1},\dots,\underbrace{\lambda_K,\dots,\lambda_K}_{N_K}), \tag{2.34}$$

or equivalently

$$\begin{aligned} &\hat{\lambda}_1=\cdots=\hat{\lambda}_{N_1}=\lambda_1, \quad \hat{\lambda}_{N_1+1}=\cdots=\hat{\lambda}_{N_1+N_2}=\lambda_2, \\ &\hat{\lambda}_{(N_1+\cdots+N_{K-1})+1}=\cdots=\hat{\lambda}_{(N_1+\cdots+N_{K-1})+N_K}=\lambda_K. \end{aligned} \tag{2.35}$$

Then we introduce N iterated one-fold Darboux transformations $T_1(\lambda),\dots,T_N(\lambda)$ recursively by

$$T_k(\lambda)=\begin{pmatrix} m_k(1+\lambda r_k) & (\lambda-\zeta_1)m_ks_k \\ -(\lambda-\zeta_2)m_k^*s_k^* & m_k^*(1+\lambda r_k^*) \end{pmatrix}, \quad (1\le k\le N), \tag{2.36}$$

and

$$T_k(\hat{\lambda}_k)F_k(\hat{\lambda}_k)=0, \quad m_k=\begin{cases} [(1+\zeta_1r_k)^{\zeta_2}(1+\zeta_2r_k)^{\zeta_1}]^{1/(\zeta_1-\zeta_2)}, & \text{if } \alpha^2\neq\beta, \\ \frac{1}{1+\alpha r_k}e^{\frac{\alpha r_k}{1+\alpha r_k}}, & \text{if } \alpha^2=\beta. \end{cases} \tag{2.37}$$

where $F_1(\lambda) = \Phi(\lambda)$ and

$$F_{k+1}(\lambda) = \frac{1}{\lambda-\hat{\lambda}_k} T_k(\lambda) F_k(\lambda) = \frac{1}{\lambda-\hat{\lambda}_k}[T_k(\lambda)F_k(\lambda) - T_k(\hat{\lambda}_k)F_k(\hat{\lambda}_k)], \quad (1 \le k \le N). \tag{2.38}$$

Denote $\hat{\mathscr{T}}(\lambda) = T_N(\lambda)\cdots T_1(\lambda)$. Then we can calculate

$$\begin{aligned}\hat{\mathscr{T}}(\lambda)\Phi(\lambda) &= T_N(\lambda)\cdots T_1(\lambda)F_1(\lambda) = (\lambda-\hat{\lambda}_1)T_N(\lambda)\cdots T_2(\lambda)F_2(\lambda)\\ &= (\lambda-\hat{\lambda}_1)\cdots(\lambda-\hat{\lambda}_{N-1})T_N(\lambda)F_N(\lambda) = (\lambda-\hat{\lambda}_1)\cdots(\lambda-\hat{\lambda}_N)F_{N+1}(\lambda)\\ &= (\lambda-\lambda_1)^{N_1}\cdots(\lambda-\lambda_K)^{N_K}F_{N+1}(\lambda),\end{aligned} \tag{2.39}$$

and hence

$$\sum_{l=0}^{j}\binom{j}{l}\hat{\mathscr{T}}^{(j-l)}(\lambda_k)\Phi_{k,l} = \frac{\partial^j}{\partial\lambda^j}|_{\lambda=\lambda_k}\hat{\mathscr{T}}(\lambda)\Phi(\lambda) = 0, \quad (1 \le k \le K, 0 \le j \le N_{k-1}). \tag{2.40}$$

Some further but easy calculations show that $\hat{\mathscr{T}}(\lambda)$ satisfies the symmetric relation in (2.29) and (2.31). This means that $\mathscr{T}(\lambda)$ and $\hat{\mathscr{T}}(\lambda)$ satisfy the same symmetric relations and the same equations. In other words, $\mathscr{T}(\lambda) \equiv \hat{\mathscr{T}}(\lambda)$. Since $\mathscr{T}(\lambda) = T_N(\lambda)\cdots T_1(\lambda)$, $\mathscr{T}(\lambda)$ itself is an N-fold generalized Darboux matrix. Finally, comparing the coefficients of λ^{N+1} in $\tilde{U}(\lambda)\mathscr{T}(\lambda) = \mathscr{T}_x(\lambda) + \mathscr{T}(\lambda)U(\lambda)$ we get (2.33). □

At the end of this section, we introduce some notations to give neat formulas for R_N and S_N. Denote

$$\begin{aligned} &g_j(\lambda) = \lambda^j\phi(\lambda), && h_j(\lambda) = (\lambda^j - \zeta_1\lambda^{j-1})\psi(\lambda),\\ &\hat{g}_j(\lambda) = \lambda^j\psi(\lambda^*)^*, && \hat{h}_j(\lambda) = (\lambda^j - \zeta_2^*\lambda^{j-1})\phi(\lambda^*)^*.\end{aligned} \tag{2.41}$$

Then (2.32) can be written as

$$\begin{aligned}&\tfrac{\partial^j}{\partial\lambda^j}|_{\lambda=\lambda_j}\{\phi(\lambda) + [R_1g_1(\lambda)+\cdots+R_Ng_N(\lambda)] + [S_1h_1(\lambda)+\cdots+S_Nh_N(\lambda)]\} = 0,\\ &\tfrac{\partial^j}{\partial\lambda^j}|_{\lambda=\lambda_j^*}\{\psi(\lambda^*)^* - [S_1\hat{h}_1(\lambda)+\cdots+S_N\hat{h}_N(\lambda)] + [R_1\hat{g}_1(\lambda)+\cdots+R_N\hat{g}_N(\lambda)]\} = 0,\\ &\qquad\qquad\qquad\qquad\qquad\qquad (1 \le k \le K, 0 \le j \le N_k - 1),\end{aligned} \tag{2.42}$$

or more compactly as

$$\boldsymbol{B} + \boldsymbol{X}\boldsymbol{A} = 0, \tag{2.43}$$

where

$$\begin{aligned}&\boldsymbol{X} = (R_1,\ldots,R_N,S_1,\ldots,S_N), \quad \boldsymbol{B} = (B_1,\ldots,B_K,\hat{B}_1,\ldots,\hat{B}_K),\\ &B_k = (\phi_k^{(0)},\ldots,\phi_k^{(N_k-1)}), \quad \hat{B}_k = ((\psi_k^{(0)})^*,\ldots,(\psi_k^{(N_k-1)})^*),\\ &\boldsymbol{A} = (A_1,\ldots,A_K,\hat{A}_1,\ldots,\hat{A}_K),\\ &A_k = \begin{pmatrix} g_1^{(0)}(\lambda_k) & \cdots & g_1^{(N_k-1)}(\lambda_k)\\ \vdots & & \vdots\\ g_N^{(0)}(\lambda_k) & \cdots & g_N^{(N_k-1)}(\lambda_k)\\ h_1^{(0)}(\lambda_k) & \cdots & h_1^{(N_k-1)}(\lambda_k)\\ \vdots & & \vdots\\ h_N^{(0)}(\lambda_k) & \cdots & h_N^{(N_k-1)}(\lambda_k)\end{pmatrix}, \quad \hat{A}_k = \begin{pmatrix} \hat{g}_1^{(0)}(\lambda_k^*) & \cdots & \hat{g}_1^{(N_k-1)}(\lambda_k^*)\\ \vdots & & \vdots\\ \hat{g}_N^{(0)}(\lambda_k^*) & \cdots & \hat{g}_N^{(N_k-1)}(\lambda_k^*)\\ -\hat{h}_1^{(0)}(\lambda_k^*) & \cdots & -\hat{h}_1^{(N_k-1)}(\lambda_k^*)\\ \vdots & & \vdots\\ -\hat{h}_N^{(0)}(\lambda_k^*) & \cdots & -\hat{h}_N^{(N_k-1)}(\lambda_k^*)\end{pmatrix}.\end{aligned} \tag{2.44}$$

Noting that $g_l^{(j)}(\lambda_k)$, $h_l^{(j)}(\lambda_k)$, $\hat{g}_l^{(j)}(\lambda_k^*)$ and $\hat{h}_l^{(j)}(\lambda_k^*)$ are linear combinations of $\phi_k^{(j)}$ and $\psi_k^{(j)}$, where

$$(\phi_k^{(j)}, \psi_k^{(j)})^T = \Phi_k^{(j)} = \Phi^{(j)}(\lambda_k), \quad (1 \leq k \leq K, 0 \leq j \leq N_k - 1). \tag{2.45}$$

we obtain by Cramer's rule that

$$R_j = -\frac{\det \boldsymbol{E}_j}{\det \boldsymbol{A}}, \quad S_j = -\frac{\det \boldsymbol{E}_{N+j}}{\det \boldsymbol{A}}, \quad (1 \leq j \leq N), \tag{2.46}$$

where $\boldsymbol{E}_j$ is obtained from $\boldsymbol{A}$ by replacing its jth row with $\boldsymbol{B}$, $(1 \leq j \leq 2N)$.

3. Exact solutions

In this section, we construct some exact solutions of the guLL equation by using the Darboux transformations. For the sake of convenience, we consider only the special case of guLL equation (1.5) when $\alpha = \beta = 1$ $(\zeta_1 = \zeta_2 = 1)$:

$$iq_t - wq_{xx} + w_{xx}q + 2i(w^2q)_x + 4wq = 0, \quad |q|^2 + w^2 = 1. \tag{3.1}$$

In this case, the relation (2.31) is reduced to

$$M = \left(1 + \sum_{j=1}^{N} R_j\right)^{-1} \exp\left[\sum_{j=1}^{N} jR_j \left(1 + \sum_{j=1}^{N} R_j\right)^{-1}\right]. \tag{3.2}$$

Our aim is to find the typical phenomena: solitons, rogue waves or rogue holes and Akhmediev breathers. To this end, we study two seed solutions in the following two subsections.

3.1. *Seed solution 1*

The first seed solution of guLL equation (3.1) is taken as

$$q = 0, \quad w = 1. \tag{3.3}$$

Then the N-fold Darboux transformation (2.33) is reduced to

$$\tilde{q} = \frac{-2iMR_NS_N}{M^*(|R_N|^2 + |S_N|^2)}, \quad \tilde{w} = \frac{|R_N|^2 - |S_N|^2}{|R_N|^2 + |S_N|^2}. \tag{3.4}$$

Substituting (3.3) into Lax pair (2.1), we obtain its general solution

$$\Phi(\lambda) = (f_1(\lambda)e^{i[2(\lambda^2-\lambda+1)t+\lambda x]}, f_2(\lambda)e^{-i[2(\lambda^2-\lambda+1)t+\lambda x]})^T, \tag{3.5}$$

where $f_1(\lambda)$ and $f_2(\lambda)$ are constants of integration. In the following, we derive some exact solutions of guLL equation (3.1) by using N-fold Darboux transformations in Theorem 2.2.

Example 3.1. Choosing $K=1$, $N=N_1=1$, $\lambda_1=\xi_1+i\eta_1$, and $f_1(\lambda_1)=f_2(\lambda_1)=1$, (3.5) is reduced to

$$\Phi_1=\Phi(\lambda_1)=(\mathrm{e}^{-\frac{1}{2}(\theta_1+i\omega_1)},\mathrm{e}^{\frac{1}{2}(\theta_1+i\omega_1)})^T, \tag{3.6}$$

where $\theta_1=4\eta_1(2\xi_1-1)t+2\eta_1 x$ and $\omega_1=4t(\eta_1^2-\xi_1^2+\xi_1-1)-2\xi_1 x$. Based on (2.46) and (3.2), we arrive at a one-soliton solution of guLL equation (3.1)

$$\begin{aligned}\tilde{q}&=-\frac{2\eta_1(i\eta_1\tanh\theta_1+\xi_1-1)\operatorname{sech}\theta_1}{\eta_1^2+(\xi_1-1)^2}\exp\left[-\frac{2i\eta_1\tanh\theta_1}{\eta_1^2+(\xi_1-1)^2}-i\omega_1\right],\\ \tilde{w}&=1-\frac{2\eta_1^2\operatorname{sech}^2\theta_1}{\eta_1^2+(\xi_1-1)^2}.\end{aligned} \tag{3.7}$$

From

$$|\tilde{q}|=\sqrt{1-\left(1-\frac{2\eta_1^2\operatorname{sech}^2\theta_1}{\eta_1^2+(\xi_1-1)^2}\right)^2},\quad \tilde{w}=1-\frac{2\eta_1^2\operatorname{sech}^2\theta_1}{\eta_1^2+(\xi_1-1)^2}, \tag{3.8}$$

we can see that $|\tilde{q}|$ and $\tilde{w}$ are traveling waves. When $\xi_1=\eta_1=\frac{1}{2}$, the solution is illustrated in Fig. 1. Please note, only $|\tilde{q}|$ and $\tilde{w}$ are stationary, $\arg\tilde{q}$ depends on t.

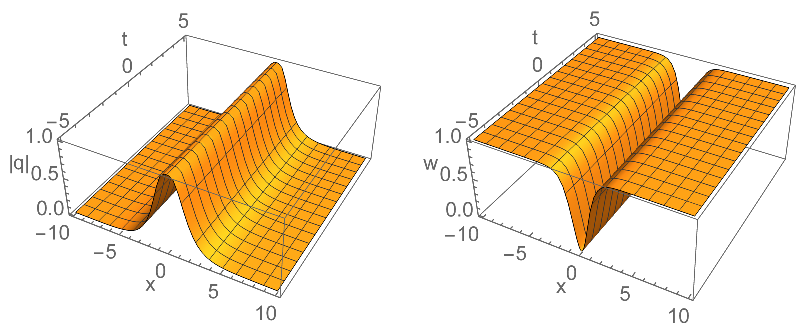

Fig. 1. A stationary solution $(K=1, N_1=N=1, \lambda_1=\frac{1}{2}+\frac{i}{2})$

Example 3.2. Let $K=2$, $N_1=N_2=1$, $N=2$, and $\lambda_1=\xi_1+i\eta_1$ and $\lambda_2=\xi_2+i\eta_2$. Choose $f_1(\lambda_1)=f_1(\lambda_2)=f_2(\lambda_1)=f_2(\lambda_2)=1$. Then we obtain from (3.5) that

$$\Phi_j=\Phi(\lambda_j)=(\mathrm{e}^{-\frac{1}{2}(\theta_j+i\omega_j)},\mathrm{e}^{\frac{1}{2}(\theta_j+i\omega_j)})^T,\quad (1\le j\le 2), \tag{3.9}$$

where $\theta_j=4\eta_j(2\xi_j-1)t+2\eta_j x$ and $\omega_j=4t(\eta_j^2-\xi_j^2+\xi_j-1)-2\xi_j x$. Resorting to the formulas (2.44), (2.46), (2.31) and (3.4) we can obtain a two-fold Darboux transformation $(\tilde{q},\tilde{w})$. The new solution $(\tilde{q},\tilde{w})$ has different properties when the constants ξ_1,ξ_2,η_1 and η_2 vary.

Case 1: when $\xi_1\neq\xi_2$, the new solution $(\tilde{q},\tilde{w})$ is a two-soliton (see Fig. 2).

Case 2: when $\xi_1=\xi_2$ and $\eta_1\neq\eta_2$, the new solution $(\tilde{q},\tilde{w})$ is a general breather (see Fig. 3).

Case 3: especially, when $\xi_1 = \xi_2 = \frac{1}{2}$ and $\eta_1 \neq \eta_2$, the new solution $(\tilde{q}, \tilde{w})$ is an Akhmediev breather, and the periodic in the t-direction is $\frac{1}{2}\pi(\eta_1^2 - \eta_2^2)^{-1}$. As $\eta_2 \to \eta_1$, the limits of $\tilde{q}$ and $\tilde{w}$ exists, e.g.,

$$\begin{aligned}
\lim_{\eta_1,\eta_2 \to 1/2} \tilde{q} &= \frac{2(8it^2 + 4t + 2ix^2 + 2x + \sinh 2x)[2i(2t+x)\sinh x + 2(-2t + x - i)\cosh x]}{(8t^2 + 2x^2 + \cosh 2x + 1)^2} \\
&\quad \times \exp\left(-\frac{4i(4t + \sinh 2x)}{8t^2 + 2x^2 + \cosh 2x + 1} + 2it + ix\right), \\
\lim_{\eta_1,\eta_2 \to 1/2} \tilde{w} &= \{-4(8t^2 + 2x^2 + 1)\cosh 2x + 8[16t^4 + t^2(8x^2 + 4) + 8tx + x^4 + x^2] \\
&\quad + 16(2t + x)\sinh 2x + \cosh 4x - 5\}\{2(8t^2 + 2x^2 + \cosh 2x + 1)^2\}^{-1},
\end{aligned} \tag{3.10}$$

which is a solution (see Fig. 4) of guLL equation (3.1).

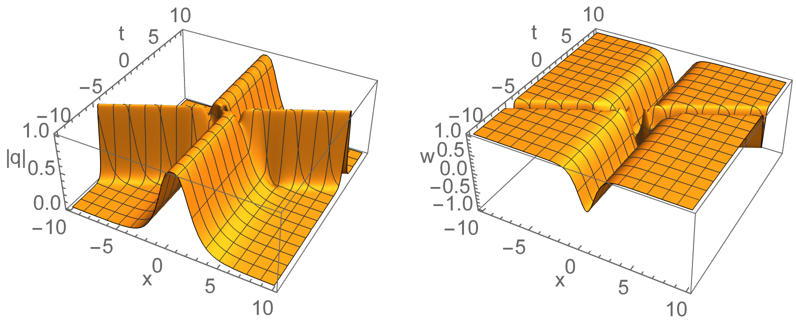

Fig. 2. A two-soliton solution ($K = 2, N_1 = N_2 = 1, N = 2, \lambda_1 = \frac{1}{2} + \frac{i}{2}, \lambda_2 = i$)

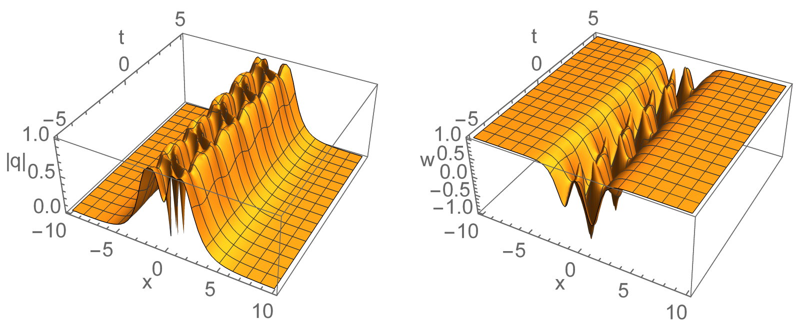

Fig. 3. An Akhmediev breather solution ($K = 2, N_1 = N_2 = 1, N = 2, \lambda_1 = \frac{1}{2} + \frac{i}{2}, \lambda_2 = \frac{1}{2} + i$)

Because taking limits always means a lot of calculations, we give an example to reduce the calculations.

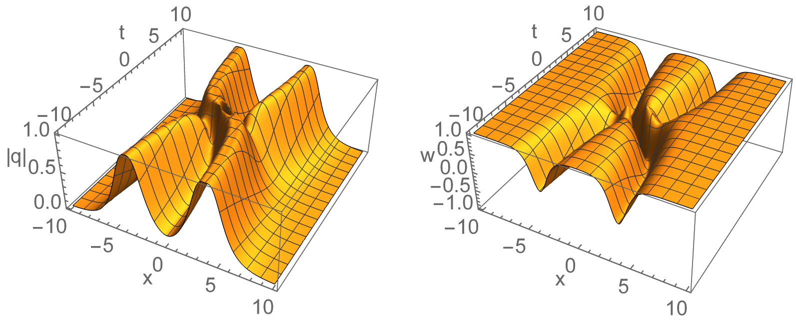

Fig. 4. An aperiodic solution ($K = 1, N_1 = N = 2, \lambda_1 = \frac{1}{2} + \frac{i}{2}$)

Example 3.3. Let $K = 1$, $N = N_1 = 2$ and $\lambda_1 = \xi_1 + i\eta_1$. Choose $f_1(\lambda_1) = f_2(\lambda_1) = 1$ and $f_1'(\lambda_1) = f_2'(\lambda_1) = 0$. We have from (3.5) that

$$\begin{aligned} \Phi_1 &= \Phi(\lambda_1) = (\mathrm{e}^{-\frac{1}{2}(\theta_1+i\omega_1)}, \mathrm{e}^{\frac{1}{2}(\theta_1+i\omega_1)})^T, \\ \Phi_1^{(1)} &= \Phi^{(1)}(\lambda_1) = [ix + it(4\lambda_1 - 2)](\mathrm{e}^{-\frac{1}{2}(\theta_1+i\omega_1)}, -\mathrm{e}^{\frac{1}{2}(\theta_1+i\omega_1)})^T, \end{aligned} \tag{3.11}$$

where $\theta_1 = 4\eta_1(2\xi_1 - 1)t + 2\eta_1 x$ and $\omega_1 = 4t(\eta_1^2 - \xi_1^2 + \xi_1 - 1) - 2\xi_1 x$. Finally, by using (2.44), (3.2) and (2.33), we can obtain a generalized Darboux transformation. Especially, when $\xi_1 = \eta_1 = \frac{1}{2}$, we arrive at the same result (3.10) without taking limits. Even though the results are the same as the limits of Case 3 in Example 3.2, the computational complexity is reduced by much. The computational complexity is usually a key issue whether an exact solution can be obtained. When K and $N_1, \ldots, N_K$ are larger, or when the seed solution is more complicated, the reduced computational complexity is more considerable.

3.2. *Seed Solution 2*

The second seed solution of guLL equation (3.1) is

$$q = \mathrm{e}^{2ix}, \quad w = 0. \tag{3.12}$$

And the N-fold Darboux transformation (2.33) is reduced to

$$\tilde{q} = \frac{M(R_N^2 \mathrm{e}^{2ix} + S_N^2 \mathrm{e}^{-2ix})}{M^*(|R_N|^2 + |S_N|^2)}, \quad \tilde{w} = \frac{-iR_N S_N^* \mathrm{e}^{2ix} + iR_N^* S_N \mathrm{e}^{-2ix}}{|R_N|^2 + |S_N|^2}. \tag{3.13}$$

In the following, we construct some explicit rouge-wave solutions.

Example 3.4. Let $K = 1$, $N = N_1 = 1$ and $\lambda_1 = 1 + i$. Then the Lax pair (2.1) has a solution

$$\Phi_1 = \Phi(\lambda_1) = ([(4-4i)t - 2ix + 1]\mathrm{e}^{ix}, [(4-4i)t - 2ix - 1]\mathrm{e}^{-ix})^T. \tag{3.14}$$

By using (2.44), (2.31) and (2.33), we have

$$
\begin{aligned}
\tilde{q} &= -\frac{1024t^4+512t^3(2x+i)+64t^2\left(8x^2+8ix-1\right)+16t(2x+i)^3+(2x+i)^4}{\left(32t^2+16tx+4x^2+1\right)^2} \\
&\quad \times \exp\left(\frac{2i\left(32t^2x+8t\left(2x^2+1\right)+4x^3+x\right)}{32t^2+16tx+4x^2+1}\right), \\
\tilde{w} &= -\frac{64t(2t+x)}{\left(32t^2+16tx+4x^2+1\right)^2}.
\end{aligned}
\tag{3.15}
$$

Because

$$
|\tilde{q}| = \sqrt{1-\left[\frac{64t(2t+x)}{\left(32t^2+16tx+4x^2+1\right)^2}\right]^2} \tag{3.16}
$$

is zero at four points,

$$
(x,t) = \pm(0, \tfrac{1}{4\sqrt{2}}), \quad (x,t) = \pm(-\tfrac{1}{\sqrt{2}}, \tfrac{1}{4\sqrt{2}}), \tag{3.17}
$$

and $|\tilde{q}| \to 1$ as $\max\{|x|,|t|\} \to +\infty$, $\tilde{q}$ has four holes (see Fig. 5, where the figure of $|\tilde{q}|$ is presented upside-down for clarity).

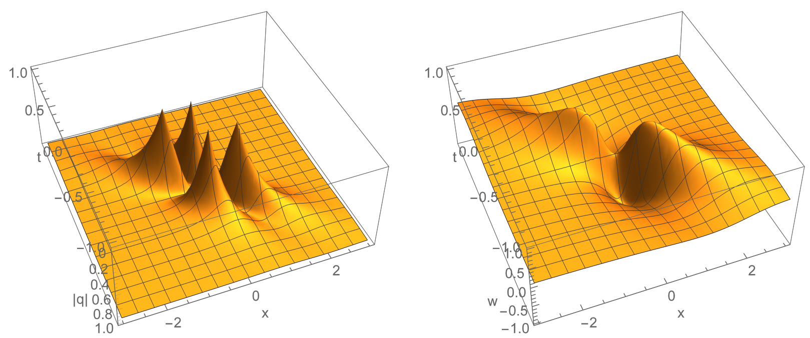

Fig. 5. A rouge-hole solution ($K = 1, N_1 = N = 1, \lambda_1 = 1+i$). The left figure is upside-down.

Example 3.5. Choose $K = 1$, $N = N_1 = 2$ and $\lambda_1 = 1+i$. Then the Lax pair (2.1) has a solution

$$
\Phi_1 = \Phi(\lambda_1) = ([(4-4i)t-2ix+1]\mathrm{e}^{ix}, [(4-4i)t-2ix-1]\mathrm{e}^{-ix})^T, \tag{3.18}
$$

which together with (2.9) implies ($\Phi_1^{(1)} = \Phi^{(1)}(\lambda_1)$)

$$
\begin{aligned}
\Phi_1^{(1)} = {} & (\mathrm{e}^{ix}\{(\tfrac{32}{3} - \tfrac{32i}{3})t^3 - [16it^2 - (4-4i)t + 1]x + 8t^2 - [i + (4+4i)t]x^2 \\
& - (2+6i)t - \tfrac{2}{3}x^3\}, \mathrm{e}^{-ix}\{(\tfrac{32}{3} - \tfrac{32i}{3})t^3 - [16it^2 + (4-4i)t + 1]x - 8t^2 \\
& + [i - (4+4i)t]x^2 - (2+6i)t - \tfrac{2}{3}x^3\})^T \\
& + f_1(\mathrm{e}^{ix}[(2-2i)t - ix + 1], \mathrm{e}^{-ix}[(2-2i)t - ix])^T \\
& + f_2(\mathrm{e}^{ix}[(-2+2i)t + ix], \mathrm{e}^{-ix}[-(2-2i)t + ix + 1])^T,
\end{aligned}
\tag{3.19}
$$

where f_1 and f_2 are two constants of integration. By using (2.44), (2.31) and (2.33), we can obtain a two-fold Darboux transformation. But because the results are too tedious to write down, we only present a graphical illustration for the solution when $f_1 = 0, f_2 = 20$ in Fig. 6 (the figure of $|\tilde{q}|$ is presented upside-down for clarity).

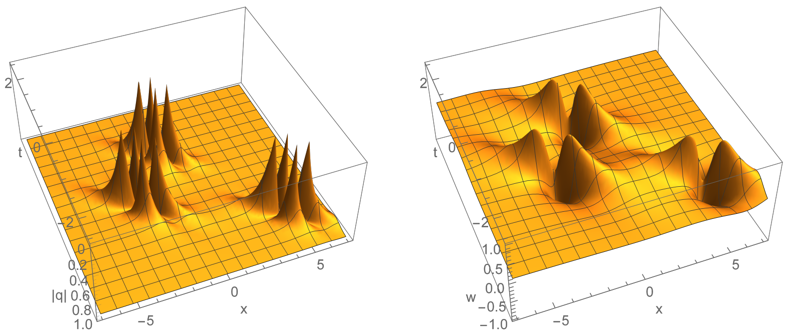

Fig. 6. A rouge-hole solution with more holes ($K = 1, N_1 = N = 1, \lambda_1 = 1+i$). The left figure is upside-down.

Remark 3.1. All the above explicit solutions have been verified by using `Mathematica`.

4. Conclusions and discussions

In this paper, a generalization of the uniaxial anisotropic Landau-Lifschitz equation is proposed. Based on the gauge transformation between Lax pairs, an N-fold generalized Darboux transformation is constructed. As applications of these Darboux transformation, several examples of exact solutions, including breather solutions and rogue-wave solutions, are given. The Landau-Lifschitz equation is important in the fields of both physics and mathematics. The generalization of the uniaxial anisotropic Landau-Lifschitz equation deserves further study.

Acknowledgments

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